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THE CANONICAL SOLUTION OPERATOR TO $\overline{\partial}$ RESTRICTED TO BERGMAN SPACES

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ABSTRACT. We first show that the canonical solution operator to $\overline{\partial}$ restricted to $(0,1)$-forms with holomorphic coefficients can be expressed by an integral operator using the Bergman kernel. This result is used to prove that in the case of the unit disc in $\mathbb{C}$ the canonical solution operator to $\overline{\partial}$ restricted to $(0,1)$-forms with holomorphic coefficients is a Hilbert-Schmidt operator. In the sequel we give a direct proof of the last statement using orthonormal bases and show that in the case of the polydisc and the unit ball in $\mathbb{C}^n$, $n > 1$, the corresponding operator fails to be a Hilbert-Schmidt operator. We also indicate a connection with the theory of Hankel operators.

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$ and let $A^2(\Omega)$ denote the Bergman space of all holomorphic functions $f : \Omega \longrightarrow \mathbb{C}$ such that

$$\int_\Omega |f(z)|^2 \, d\lambda(z) < \infty,$$

where λ denotes the Lebesgue measure in $\mathbb{C}^n$.

We solve the $\overline{\partial}$-equation $\overline{\partial} u = g$, where $g = \sum_{j=1}^n g_j \, d\overline{z}_j$ is a (0,1)-form with coefficients $g_j \in A^2(\Omega)$, $j = 1, \dots, n$.

It is pointed out in [FS1] that in the proof that compactness of the solution operator for $\overline{\partial}$ on $(0,1)$-forms implies that the boundary of Ω does not contain any analytic variety of dimension greater than or equal to 1, only the fact that there is a compact solution operator to $\overline{\partial}$ on the $(0,1)$-forms with holomorphic coefficients is used. In this case compactness of the solution operator restricted to $(0,1)$-forms with holomorphic coefficients already implies compactness of the solution operator on general $(0,1)$-forms.

The question of compactness is of interest for various reasons; see [FS2] for an excellent survey.

A similar situation appears in [SSU] where the Toeplitz C^*-algebra $\mathcal{T}(\Omega)$ is considered and the relation between the structure of $\mathcal{T}(\Omega)$ and the $\overline{\partial}$-Neumann problem is discussed (see [SSU], Corollary 4.6).

We first show that the canonical solution operator to $\overline{\partial}$ restricted to $(0,1)$-forms with holomorphic coefficients can be expressed by an integral operator using the

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Bergman kernel. This result is used to prove that in the case of the unit disc in $\mathbb{C}$, the canonical solution operator to $\overline{\partial}$ restricted to $(0,1)$-forms with holomorphic coefficients is a Hilbert-Schmidt operator.

In the sequel we give a direct proof of the last statement using orthonormal bases and show that in the case of the polydisc and the unit ball in $\mathbb{C}^n,\ n \geq 2$, the corresponding operator fails to be a Hilbert-Schmidt operator.

The canonical solution operator to $\overline{\partial}$ restricted to $(0,1)$-forms with holomorphic coefficients can also be interpreted as the Hankel operator

$$H_{\overline{z}}(g) = (I-P)(\overline{z}g),$$

where $P : L^2(\Omega) \longrightarrow A^2(\Omega)$ denotes the Bergman projection. See [A], [AFP], [B], [J], [R], [W] and [Z] for details.

2. The integral representation

The canonical solution operator

$$S_1 : A^2_{(0,1)}(\Omega) \longrightarrow L^2(\Omega)$$

has the properties $\overline{\partial} S_1(g) = g$ and $S_1(g) \perp A^2(\Omega)$.

Proposition 1. *The canonical solution operator*

$$S_1 : A^2_{(0,1)}(\Omega) \longrightarrow L^2(\Omega)$$

has the form

$$S_1(g)(z) = \int_\Omega B(z,w)\langle g(w), z-w\rangle \, d\lambda(w),$$

where B denotes the Bergman kernel of Ω and

$$\langle g(w), z-w\rangle = \sum_{j=1}^n g_j(w)(\overline{z}_j - \overline{w}_j)$$

for $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_n)$.

Integral operators of similar type have been used to settle questions on compactness of the solution operator to $\overline{\partial}$; see [CD] and [L].

Proof. Let $v(z) = \sum_{j=1}^n \overline{z}_j g_j(z)$. Then it follows that

$$\overline{\partial} v = \sum_{j=1}^n \frac{\partial v}{\partial \overline{z}_j} d\overline{z}_j = \sum_{j=1}^n g_j d\overline{z}_j = g.$$

Hence the canonical solution operator S_1 can be written in the form $S_1(g) = v - P(v)$, where $P : L^2(\Omega) \longrightarrow A^2(\Omega)$ is the Bergman projection. If $\tilde{v}$ is another solution to $\overline{\partial} u = g$, then $v - \tilde{v} \in A^2(\Omega)$; hence $v = \tilde{v} + h$, where $h \in A^2(\Omega)$. Therefore

$$v - P(v) = \tilde{v} + h - P(\tilde{v}) - P(h) = \tilde{v} - P(\tilde{v}).$$

Since $g_j \in A^2(\Omega),\ j = 1,\dots,n$, we have

$$g_j(z) = \int_\Omega B(z,w) g_j(w) \, d\lambda(w).$$

Now we get

$$\begin{aligned} S_1(g)(z) &= \sum_{j=1}^n \overline{z}_j g_j(z) - \int_\Omega B(z,w)\left(\sum_{j=1}^n \overline{w}_j g_j(w)\right) d\lambda(w) \\ &= \int_\Omega \left[\left(\sum_{j=1}^n \overline{z}_j g_j(w)\right) B(z,w) - \left(\sum_{j=1}^n \overline{w}_j g_j(w)\right) B(z,w)\right] d\lambda(w) \\ &= \int_\Omega B(z,w)\langle g(w), z-w\rangle \, d\lambda(w). \end{aligned}$$

□

Remark. It is pointed out that a $(0,1)$-form $g = \sum_{j=1}^n g_j \, d\overline{z}_j$ with holomorphic coefficients is not invariant under the pull back by a holomorphic map $F = (F_1, \dots, F_n) : \Omega_1 \longrightarrow \Omega$. It can be shown that

$$F^* g = \sum_{j=1}^n \left(\sum_{l=1}^n g_l \frac{\partial \overline{F}_l}{\partial \overline{z}_j}\right) d\overline{z}_j$$

and the expressions $\frac{\partial \overline{F}_l}{\partial \overline{z}_j}$ are not holomorphic.

Nevertheless it is true that $\overline{\partial} u = g$ implies $\overline{\partial}(u \circ F) = F^* g$.

Now let ω be a holomorphic (n,n)-form, i.e.

$$\omega = \tilde{\omega} \; dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n,$$

where $\tilde{\omega} \in A^2(\Omega)$. In this case we can express the canonical solution to $\overline{\partial} u = \omega$ in the following form:

Proposition 2. *Let u be the $(n, n-1)$-form*

$$u = \sum_{j=1}^n u_j \; dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge [d\overline{z}_j] \wedge \cdots \wedge d\overline{z}_n,$$

where

$$u_j(z) = \frac{(-1)^{n+j-1}}{n} \int_\Omega (\overline{z}_j - \overline{w}_j) B(z,w) \tilde{\omega}(w) \, d\lambda(w).$$

Then $u_j \perp A^2(\Omega)$, $j = 1, \dots, n$, and $\overline{\partial} u = \omega$.

Proof. It follows that

$$u_j(z) = \frac{(-1)^{n+j-1}}{n} \left(\overline{z}_j \tilde{\omega}(z) - P(\overline{w}_j \tilde{\omega})(z)\right).$$

From this we obtain

$$\frac{\partial u_j}{\partial \overline{z}_k} = \frac{(-1)^{n+j-1}}{n} \left(\frac{\partial \overline{z}_j}{\partial \overline{z}_k} \tilde{\omega} + \overline{z}_j \frac{\partial \tilde{\omega}}{\partial \overline{z}_k}\right) = \frac{(-1)^{n+j-1}}{n} \; \delta_{jk} \; \tilde{\omega},$$

where δ_{jk} is the Kronecker delta symbol. Hence

$$\begin{aligned}\overline{\partial} u &= \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial u_j}{\partial \overline{z}_k} \; d\overline{z}_k \wedge dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge [d\overline{z}_j] \wedge \cdots \wedge d\overline{z}_n \\ &= \sum_{k=1}^{n} \sum_{j=1}^{n} \left((-1)^{n+j-1}/n\right) \; \delta_{jk} \; \tilde{\omega} \; d\overline{z}_k \\ &\wedge dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge [d\overline{z}_j] \wedge \cdots \wedge d\overline{z}_n \\ &= \tilde{\omega} \; dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n.\end{aligned}$$

□

Remark. The pull back by a holomorphic map F has in this case the form

$$F^*\omega = \left| \det \frac{\partial F_j}{\partial z_k} \right|^2 \; \tilde{\omega} \; dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n.$$

Proposition 3. *Suppose that Ω is a smoothly bounded pseudoconvex domain of finite type in $\mathbb{C}^n$. Let $T : L^2_{(0,1)}(\Omega) \longrightarrow L^2(\Omega)$ be the operator defined by*

$$T(f)(z) = \int_\Omega B(z,w) \langle f(w), z - w \rangle \, d\lambda(w), \quad f \in L^2_{(0,1)}(\Omega).$$

Then T is a compact operator.

This follows from Theorem 1 in [CD].

The last result implies that the restriction of T to $A^2_{(0,1)}(\Omega)$, which is the canonical solution operator to $\overline{\partial}$, is also a compact operator. This fact also follows from [C], where it is shown that the $\overline{\partial}$-Neumann operator is compact.

Next we consider the integral kernel of the canonical solution operator S_1 for the unit disc $\mathbb{D}$ in $\mathbb{C}$ and prove that this kernel is square integrable over $\mathbb{D} \times \mathbb{D}$.

Proposition 4.

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|\overline{z} - \overline{w}|^2}{|1 - z\overline{w}|^4} \, d\lambda(z) \, d\lambda(w) < \infty.$$

Proof. It is easily seen that $|z - w| \leq |1 - z\overline{w}|$ for $z, w \in \mathbb{D}$. Hence we get

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|\overline{z} - \overline{w}|^2}{|1 - z\overline{w}|^4} \, d\lambda(z) \, d\lambda(w) \leq \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|1 - z\overline{w}|^2} \, d\lambda(z) \, d\lambda(w).$$

Using polar coordinates $z = r\, e^{i\theta}$ and $w = s\, e^{i\phi}$ we can write the last integral in the following form:

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|1 - z\overline{w}|^2} \, d\lambda(z) \, d\lambda(w) = \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{r\, s\, d\theta\, d\phi\, dr\, ds}{1 - 2\, r\, s\, \cos(\theta - \phi) + r^2\, s^2}$$

$$= \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{1 - r^2\, s^2}{1 - 2\, r\, s\, \cos(\theta - \phi) + r^2\, s^2} \; \frac{r\, s}{1 - r^2\, s^2} \, d\theta\, d\phi\, dr\, ds.$$

Integration of the Poisson kernel with respect to θ yields

$$\int_0^{2\pi} \frac{1 - \rho^2}{1 - 2\rho\, \cos(\theta - \phi) + \rho^2} \, d\theta = 2\pi, \;\; 0 < \rho < 1.$$

Hence

$$\int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{1 - r^2\, s^2}{1 - 2\, r\, s\, \cos(\theta - \phi) + r^2\, s^2}\ \frac{r\, s}{1 - r^2\, s^2}\, d\theta\, d\phi\, dr\, ds$$
$$= (2\pi)^2 \int_0^1 \int_0^1 \frac{r\, s}{1 - r^2\, s^2}\, dr\, ds = -\,(2\pi)^2 \int_0^1 \frac{\log(1 - s^2)}{2s}\, ds < \infty.$$

□

Remark. The last proposition implies that the operator $T : L^2(\mathbb{D}) \longrightarrow L^2(\mathbb{D})$ defined by

$$T(f)(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{\overline{z} - \overline{w}}{(1 - z\overline{w})^2}\, f(w)\, d\lambda(w),$$

for $f \in L^2(\mathbb{D})$, is a Hilbert-Schmidt operator; see [MV], 16.12.

If we restrict this operator to the closed subspace $A^2(\mathbb{D})$ we obtain

Proposition 5. *The canonical solution operator to $\overline{\partial}$,*

$$S_1 : A^2(\mathbb{D}) \longrightarrow L^2(\mathbb{D}),$$

is a Hilbert-Schmidt operator.

Proof. By [MV], 16.8, we have to show that there exists a complete orthonormal system $(\phi_k)_{k=0}^{\infty}$ of $A^2(\mathbb{D})$ such that

$$\sum_{k=0}^{\infty} \|S_1(\phi_k)\|^2 < \infty.$$

For this purpose we take a complete orthonormal system $(\phi_k)_{k=0}^{\infty}$ of $A^2(\mathbb{D})$ and extend it to a complete orthonormal system $(\psi_j)_{j=0}^{\infty}$ of $L^2(\mathbb{D})$. Again by [MV], 16.8, and Proposition 3, it follows that

$$\sum_{j=0}^{\infty} \|T(\psi_j)\|^2 < \infty,$$

which implies that

$$\sum_{k=0}^{\infty} \|S_1(\phi_k)\|^2 < \infty.$$

□

3. Hilbert-Schmidt operators

Now we show directly that the canonical solution operator to $\overline{\partial}$,

$$S_1 : A^2_{(0,1)}(\mathbb{D}) \longrightarrow L^2(\mathbb{D}),$$

is a Hilbert-Schmidt operator if $\mathbb{D}$ is the open unit disc in $\mathbb{C}$, and is not Hilbert-Schmidt if $\mathbb{B}$ is the open unit ball in $\mathbb{C}^n$ for $n > 1$.

Let $\mathbb{D} \subset \mathbb{C}$ and let $\|.\|$ denote the norm in $A^2(\mathbb{D})$ and consider the orthonormal basis

$$\{u_n(z) = [(n+1)/\pi]^{1/2}\ \ z^n\ \ : n \in \mathbb{N}_0\}$$

of $A^2(\mathbb{D})$.

Proposition 6. *The canonical solution operator S_1 for the unit disc $\mathbb{D}$ in $\mathbb{C}$ has the following property:*

$$\sum_{n=0}^{\infty} \|S_1(u_n \ d\overline{z})\| < \infty,$$

which implies that $S_1 : A^2_{(0,1)}(\mathbb{D}) \longrightarrow L^2(\mathbb{D})$ is a Hilbert-Schmidt operator (see [MV]*).*

Proof. Using calculations in [J] we can show that

$$S_1(u_n \ d\overline{z})(z) = [(n+1)/\pi]^{1/2} \ \ z^n \ \overline{z} \ - \left[n^2/((n+1)\pi)\right]^{1/2} \ \ z^{n-1}, \ n \in \mathbb{N}_0.$$

The Bergman kernel B of $\mathbb{D}$ has the form

$$B(z,\zeta) = \frac{1}{\pi} \ \frac{1}{(1 - z\overline{\zeta})^2};$$

hence by Proposition 1 we can express $\|S_1(u_n d\overline{z})\|^2$ in the form

$$\int_{\mathbb{D}} \left| \overline{z} \ u_n(z) - \frac{1}{\pi} \int_{\mathbb{D}} \frac{\overline{\zeta} \ u_n(\zeta)}{(1 - z\overline{\zeta})^2} \, d\lambda(\zeta) \right|^2 \, d\lambda(z).$$

Therefore we get

$$\begin{aligned}
\|S_1(u_n d\overline{z})\|^2 &= \int_{\mathbb{D}} \left| \left(\frac{n+1}{\pi}\right)^{1/2} \ z^n \ \overline{z} - \frac{n \ z^{n-1}}{[(n+1)\pi]^{1/2}} \right|^2 \, d\lambda(z) \\
&= \int_{\mathbb{D}} \left(\frac{(n+1)\,|z|^{2n+2}}{\pi} - \frac{2n\,|z|^{2n}}{\pi} + \frac{n^2\,|z|^{2n-2}}{(n+1)\pi} \right) d\lambda(z) \\
&= 2\pi \ \int_0^1 \left(\frac{(n+1)\, r^{2n+3}}{\pi} - \frac{2n\, r^{2n+1}}{\pi} + \frac{n^2\, r^{2n-1}}{(n+1)\pi} \right) dr \\
&= \frac{1}{(n+1)(n+2)}.
\end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} \|S_1(u_n \, d\overline{z})\|^2 < \infty.$$

□

Remark. It can be shown that the set $\{S_1(u_n \, d\overline{z}) \ : \ n \in \mathbb{N}_0\}$ consists of pairwise orthogonal elements of $L^2(\mathbb{D})$.

In the following part we consider the case of the polydisc. For the sake of simplicity we concentrate on $\mathbb{C}^2$; let

$$\mathbb{D}^2 = \{z = (z_1, z_2) \ : \ |z_1| < 1 \ , \ |z_2| < 1\}.$$

Now $\{z_1^{n_1} \, z_2^{n_2} \ : \ n_1, n_2 \in \mathbb{N}_0\}$ is an orthogonal basis in $A^2(\mathbb{D}^2)$. It is easily seen that the norms of the functions $z_1^{n_1} \, z_2^{n_2}$ are $\pi[1/((n_1+1)(n_2+1))]^{1/2}$. The functions

$$u_{n_1,n_2}(z_1, z_2) = \frac{[(n_1+1)(n_2+1)]^{1/2}}{\pi} \, z_1^{n_1} \, z_2^{n_2}, \ n_1, n_2 \in \mathbb{N}_0,$$

form an orthonormal basis of $A^2(\mathbb{D}^2)$, and the system

$$\{u_{n_1,n_2}\, d\overline{z}_1 \ ,\ u_{n_1,n_2}\, d\overline{z}_2 \ :\ n_1, n_2 \in \mathbb{N}_0\}$$

constitutes an orthonormal basis for $A^2_{(0,1)}(\mathbb{D}^2)$.

Next we compute the Bergman projections of the functions

$$(z_1, z_2) \mapsto \overline{z}_1\, u_{n_1,n_2}(z_1, z_2) \quad \text{and} \quad (z_1, z_2) \mapsto \overline{z}_2\, u_{n_1,n_2}(z_1, z_2)$$

and obtain

$$P(\overline{\zeta}_1\, u_{n_1,n_2}(\zeta_1, \zeta_2))(z_1, z_2) = \frac{[(n_1+1)(n_2+1)]^{1/2}}{\pi} \frac{n_1}{n_1+1}\ z_1^{n_1-1}\, z_2^{n_2},$$

where we used similar computations as in Proposition 6.

The Bergman projection of the second function is

$$P(\overline{\zeta}_2\, u_{n_1,n_2}(\zeta_1, \zeta_2))(z_1, z_2) = \frac{[(n_1+1)(n_2+1)]^{1/2}}{\pi} \frac{n_2}{n_2+1}\ z_1^{n_1}\, z_2^{n_2-1}.$$

Now we can compute the norms of the images under the canonical solution operator of the elements of our orthonormal basis of $A^2_{(0,1)}(\mathbb{D}^2)$:

$$\frac{(n_1+1)(n_2+1)}{\pi^2} \int_{\mathbb{D}^2} \left| \overline{z}_1\, z_1^{n_1}\, z_2^{n_2} - \frac{n_1}{n_1+1}\, z_1^{n_1-1}\, z_2^{n_2} \right|^2 d\lambda(z) = \frac{1}{(n_1+2)(n_1+1)},$$

where we used the corresponding computation of Proposition 6 for the integral with respect to z_1.

In a similar way we obtain

$$\frac{(n_1+1)(n_2+1)}{\pi^2} \int_{\mathbb{D}^2} \left| \overline{z}_2\, z_1^{n_1}\, z_2^{n_2} - \frac{n_2}{n_2+1}\, z_1^{n_1}\, z_2^{n_2-1} \right|^2 d\lambda(z) = \frac{1}{(n_2+2)(n_2+1)}.$$

Since

$$\sum_{n_1,n_2=1}^{\infty} \left(\frac{1}{(n_1+2)(n_1+1)} + \frac{1}{(n_2+2)(n_2+1)} \right) = \infty,$$

the canonical solution operator

$$S_1 : A^2_{(0,1)}(\mathbb{D}^2) \longrightarrow L^2(\mathbb{D}^2)$$

is not Hilbert-Schmidt.

Remark. With results from [K2] it can be shown that the canonical solution operator

$$S_1 : A^2_{(0,1)}(\mathbb{D}^2) \longrightarrow L^2(\mathbb{D}^2)$$

is not even compact.

We now consider the case of the unit ball $\mathbb{B}^2$ in $\mathbb{C}^2$. Here we can use calculations from the proof of Proposition 1 in [W] .

The norms of the functions $z_1^{n_1}\, z_2^{n_2}$ are now $\pi[n_1!\, n_2!/(n_1+n_2+2)!]^{1/2}$ (see [K1]). The functions

$$U_{n_1,n_2}(z_1, z_2) = \frac{[(n_1+n_2+2)!]^{1/2}}{\pi(n_1!\, n_2!)^{1/2}} z_1^{n_1}\, z_2^{n_2},\ \ n_1, n_2 \in \mathbb{N}_0,$$

form an orthonormal basis of $A^2(\mathbb{B}^2)$, and the system

$$\{U_{n_1,n_2}\, d\overline{z}_1 \ , \ U_{n_1,n_2}\, d\overline{z}_2 \ : \ n_1, n_2 \in \mathbb{N}_0\}$$

constitutes an orthonormal basis for $A^2_{(0,1)}(\mathbb{B}^2)$.

We compute the Bergman projections of the functions

$$(z_1, z_2) \mapsto \overline{z}_1\, U_{n_1,n_2}(z_1, z_2) \quad \text{and} \quad (z_1, z_2) \mapsto \overline{z}_2\, U_{n_1,n_2}(z_1, z_2)$$

and obtain

$$P(\overline{\zeta}_1\, U_{n_1,n_2}(\zeta_1, \zeta_2))(z_1, z_2) = \frac{[(n_1+n_2+2)!]^{1/2}}{\pi\, (n_1!\, n_2!)^{1/2}} \frac{n_1}{n_1+n_2+2}\, z_1^{n_1-1} z_2^{n_2},$$

$$P(\overline{\zeta}_2\, U_{n_1,n_2}(\zeta_1, \zeta_2))(z_1, z_2) = \frac{[(n_1+n_2+2)!]^{1/2}}{\pi\, (n_1!\, n_2!)^{1/2}} \frac{n_2}{n_1+n_2+2}\, z_1^{n_1} z_2^{n_2-1}.$$

Finally we compute the norms of the images of the basis elements under the canonical solution operator S_1, and obtain

$$\frac{(n_1+n_2+2)!}{\pi^2\, n_1!\, n_2!} \int_{\mathbb{B}^2} \left| \overline{z}_1\, z_1^{n_1}\, z_2^{n_2} - \frac{n_1}{n_1+n_2+2}\, z_1^{n_1-1}\, z_2^{n_2} \right|^2 \, d\lambda(z)$$

$$= \frac{n_2+2}{(n_1+n_2+2)(n_1+n_2+3)}$$

and

$$\frac{(n_1+n_2+2)!}{\pi^2\, n_1!\, n_2!} \int_{\mathbb{B}^2} \left| \overline{z}_2\, z_1^{n_1}\, z_2^{n_2} - \frac{n_2}{n_1+n_2+2}\, z_1^{n_1}\, z_2^{n_2-1} \right|^2 \, d\lambda(z)$$
$$= \frac{n_1+2}{(n_1+n_2+2)(n_1+n_2+3)}.$$

Since

$$\sum_{n_1,n_2=1}^{\infty} \left(\frac{n_2+2}{(n_1+n_2+2)(n_1+n_2+3)} + \frac{n_1+2}{(n_1+n_2+2)(n_1+n_2+3)} \right) = \infty,$$

the canonical solution operator

$$S_1 : A^2_{(0,1)}(\mathbb{B}^2) \longrightarrow L^2(\mathbb{B}^2)$$

is also not Hilbert-Schmidt.

Remark. In [Z] it is shown that there are no nonzero Hilbert-Schmidt Hankel operators on the Bergman space of the unit ball in $\mathbb{C}^n$ with antiholomorphic symbol when $n \geq 2$.

Proposition 7. *The integral kernel*

$$\frac{|z_1 - w_1|^2 + |z_2 - w_2|^2}{|1 - z_1\overline{w}_1|^4\, |1 - z_2\overline{w}_2|^4}$$

does not belong to $L^2(\mathbb{D}^2 \times \mathbb{D}^2)$ *and the integral kernel*

$$\frac{|z_1 - w_1|^2 + |z_2 - w_2|^2}{|1 - z_1\overline{w}_1 - z_2\overline{w}_2|^6}$$

does not belong to $L^2(\mathbb{B}^2 \times \mathbb{B}^2)$.

Proof. Suppose the first kernel belongs to $L^2(\mathbb{D}^2 \times \mathbb{D}^2)$. Then the corresponding integral operator from $L^2_{(0,1)}(\mathbb{D}^2)$ to $L^2(\mathbb{D}^2)$ is a Hilbert-Schmidt operator, which would imply that the restriction to $A^2_{(0,1)}(\mathbb{D}^2)$ is also Hilbert-Schmidt. But this restriction coincides with the canonical solution operator S_1, from which we already know that it is not Hilbert-Schmidt. The proof for the second integral is analogous to the first.

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