

## THE CANONICAL SOLUTION OPERATOR TO $\bar{\partial}$ RESTRICTED TO BERGMAN SPACES

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ABSTRACT. We first show that the canonical solution operator to  $\bar{\partial}$  restricted to  $(0, 1)$ -forms with holomorphic coefficients can be expressed by an integral operator using the Bergman kernel. This result is used to prove that in the case of the unit disc in  $\mathbb{C}$  the canonical solution operator to  $\bar{\partial}$  restricted to  $(0, 1)$ -forms with holomorphic coefficients is a Hilbert-Schmidt operator. In the sequel we give a direct proof of the last statement using orthonormal bases and show that in the case of the polydisc and the unit ball in  $\mathbb{C}^n$ ,  $n > 1$ , the corresponding operator fails to be a Hilbert-Schmidt operator. We also indicate a connection with the theory of Hankel operators.

### 1. INTRODUCTION

Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$  and let  $A^2(\Omega)$  denote the Bergman space of all holomorphic functions  $f : \Omega \rightarrow \mathbb{C}$  such that

$$\int_{\Omega} |f(z)|^2 d\lambda(z) < \infty,$$

where  $\lambda$  denotes the Lebesgue measure in  $\mathbb{C}^n$ .

We solve the  $\bar{\partial}$ -equation  $\bar{\partial}u = g$ , where  $g = \sum_{j=1}^n g_j d\bar{z}_j$  is a  $(0, 1)$ -form with coefficients  $g_j \in A^2(\Omega)$ ,  $j = 1, \dots, n$ .

It is pointed out in [FS1] that in the proof that compactness of the solution operator for  $\bar{\partial}$  on  $(0, 1)$ -forms implies that the boundary of  $\Omega$  does not contain any analytic variety of dimension greater than or equal to 1, only the fact that there is a compact solution operator to  $\bar{\partial}$  on the  $(0, 1)$ -forms with holomorphic coefficients is used. In this case compactness of the solution operator restricted to  $(0, 1)$ -forms with holomorphic coefficients already implies compactness of the solution operator on general  $(0, 1)$ -forms.

The question of compactness is of interest for various reasons; see [FS2] for an excellent survey.

A similar situation appears in [SSU] where the Toeplitz  $C^*$ -algebra  $\mathcal{T}(\Omega)$  is considered and the relation between the structure of  $\mathcal{T}(\Omega)$  and the  $\bar{\partial}$ -Neumann problem is discussed (see [SSU], Corollary 4.6).

We first show that the canonical solution operator to  $\bar{\partial}$  restricted to  $(0, 1)$ -forms with holomorphic coefficients can be expressed by an integral operator using the

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Bergman kernel. This result is used to prove that in the case of the unit disc in  $\mathbb{C}$ , the canonical solution operator to  $\bar{\partial}$  restricted to  $(0, 1)$ -forms with holomorphic coefficients is a Hilbert-Schmidt operator.

In the sequel we give a direct proof of the last statement using orthonormal bases and show that in the case of the polydisc and the unit ball in  $\mathbb{C}^n$ ,  $n \geq 2$ , the corresponding operator fails to be a Hilbert-Schmidt operator.

The canonical solution operator to  $\bar{\partial}$  restricted to  $(0, 1)$ -forms with holomorphic coefficients can also be interpreted as the Hankel operator

$$H_{\bar{z}}(g) = (I - P)(\bar{z}g),$$

where  $P : L^2(\Omega) \rightarrow A^2(\Omega)$  denotes the Bergman projection. See [A], [AFP], [B], [J], [R], [W] and [Z] for details.

## 2. THE INTEGRAL REPRESENTATION

The canonical solution operator

$$S_1 : A^2_{(0,1)}(\Omega) \rightarrow L^2(\Omega)$$

has the properties  $\bar{\partial}S_1(g) = g$  and  $S_1(g) \perp A^2(\Omega)$ .

**Proposition 1.** *The canonical solution operator*

$$S_1 : A^2_{(0,1)}(\Omega) \rightarrow L^2(\Omega)$$

has the form

$$S_1(g)(z) = \int_{\Omega} B(z, w) \langle g(w), z - w \rangle d\lambda(w),$$

where  $B$  denotes the Bergman kernel of  $\Omega$  and

$$\langle g(w), z - w \rangle = \sum_{j=1}^n g_j(w)(\bar{z}_j - \bar{w}_j)$$

for  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$ .

Integral operators of similar type have been used to settle questions on compactness of the solution operator to  $\bar{\partial}$ ; see [CD] and [L].

*Proof.* Let  $v(z) = \sum_{j=1}^n \bar{z}_j g_j(z)$ . Then it follows that

$$\bar{\partial}v = \sum_{j=1}^n \frac{\partial v}{\partial \bar{z}_j} d\bar{z}_j = \sum_{j=1}^n g_j d\bar{z}_j = g.$$

Hence the canonical solution operator  $S_1$  can be written in the form  $S_1(g) = v - P(v)$ , where  $P : L^2(\Omega) \rightarrow A^2(\Omega)$  is the Bergman projection. If  $\tilde{v}$  is another solution to  $\bar{\partial}u = g$ , then  $v - \tilde{v} \in A^2(\Omega)$ ; hence  $v = \tilde{v} + h$ , where  $h \in A^2(\Omega)$ . Therefore

$$v - P(v) = \tilde{v} + h - P(\tilde{v}) - P(h) = \tilde{v} - P(\tilde{v}).$$

Since  $g_j \in A^2(\Omega)$ ,  $j = 1, \dots, n$ , we have

$$g_j(z) = \int_{\Omega} B(z, w) g_j(w) d\lambda(w).$$

Now we get

$$\begin{aligned} S_1(g)(z) &= \sum_{j=1}^n \bar{z}_j g_j(z) - \int_{\Omega} B(z, w) \left( \sum_{j=1}^n \bar{w}_j g_j(w) \right) d\lambda(w) \\ &= \int_{\Omega} \left[ \left( \sum_{j=1}^n \bar{z}_j g_j(w) \right) B(z, w) - \left( \sum_{j=1}^n \bar{w}_j g_j(w) \right) B(z, w) \right] d\lambda(w) \\ &= \int_{\Omega} B(z, w) \langle g(w), z - w \rangle d\lambda(w). \end{aligned}$$

□

*Remark.* It is pointed out that a  $(0, 1)$ -form  $g = \sum_{j=1}^n g_j d\bar{z}_j$  with holomorphic coefficients is not invariant under the pull back by a holomorphic map  $F = (F_1, \dots, F_n) : \Omega_1 \rightarrow \Omega$ . It can be shown that

$$F^*g = \sum_{j=1}^n \left( \sum_{l=1}^n g_l \frac{\partial \bar{F}_l}{\partial \bar{z}_j} \right) d\bar{z}_j$$

and the expressions  $\frac{\partial \bar{F}_l}{\partial \bar{z}_j}$  are not holomorphic.

Nevertheless it is true that  $\bar{\partial}u = g$  implies  $\bar{\partial}(u \circ F) = F^*g$ .

Now let  $\omega$  be a holomorphic  $(n, n)$ -form, i.e.

$$\omega = \tilde{\omega} dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n,$$

where  $\tilde{\omega} \in A^2(\Omega)$ . In this case we can express the canonical solution to  $\bar{\partial}u = \omega$  in the following form:

**Proposition 2.** *Let  $u$  be the  $(n, n - 1)$ -form*

$$u = \sum_{j=1}^n u_j dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge [d\bar{z}_j] \wedge \dots \wedge d\bar{z}_n,$$

where

$$u_j(z) = \frac{(-1)^{n+j-1}}{n} \int_{\Omega} (\bar{z}_j - \bar{w}_j) B(z, w) \tilde{\omega}(w) d\lambda(w).$$

Then  $u_j \perp A^2(\Omega)$ ,  $j = 1, \dots, n$ , and  $\bar{\partial}u = \omega$ .

*Proof.* It follows that

$$u_j(z) = \frac{(-1)^{n+j-1}}{n} (\bar{z}_j \tilde{\omega}(z) - P(\bar{w}_j \tilde{\omega})(z)).$$

From this we obtain

$$\frac{\partial u_j}{\partial \bar{z}_k} = \frac{(-1)^{n+j-1}}{n} \left( \frac{\partial \bar{z}_j}{\partial \bar{z}_k} \tilde{\omega} + \bar{z}_j \frac{\partial \tilde{\omega}}{\partial \bar{z}_k} \right) = \frac{(-1)^{n+j-1}}{n} \delta_{jk} \tilde{\omega},$$

where  $\delta_{jk}$  is the Kronecker delta symbol. Hence

$$\begin{aligned} \bar{\partial}u &= \sum_{k=1}^n \sum_{j=1}^n \frac{\partial u_j}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge [d\bar{z}_j] \wedge \cdots \wedge d\bar{z}_n \\ &= \sum_{k=1}^n \sum_{j=1}^n ((-1)^{n+j-1}/n) \delta_{jk} \tilde{\omega} d\bar{z}_k \\ &\quad \wedge dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge [d\bar{z}_j] \wedge \cdots \wedge d\bar{z}_n \\ &= \tilde{\omega} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n. \end{aligned}$$

□

*Remark.* The pull back by a holomorphic map  $F$  has in this case the form

$$F^*\omega = \left| \det \frac{\partial F_j}{\partial z_k} \right|^2 \tilde{\omega} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$

**Proposition 3.** *Suppose that  $\Omega$  is a smoothly bounded pseudoconvex domain of finite type in  $\mathbb{C}^n$ . Let  $T : L^2_{(0,1)}(\Omega) \rightarrow L^2(\Omega)$  be the operator defined by*

$$T(f)(z) = \int_{\Omega} B(z, w) \langle f(w), z - w \rangle d\lambda(w), \quad f \in L^2_{(0,1)}(\Omega).$$

*Then  $T$  is a compact operator.*

This follows from Theorem 1 in [CD].

The last result implies that the restriction of  $T$  to  $A^2_{(0,1)}(\Omega)$ , which is the canonical solution operator to  $\bar{\partial}$ , is also a compact operator. This fact also follows from [C], where it is shown that the  $\bar{\partial}$ -Neumann operator is compact.

Next we consider the integral kernel of the canonical solution operator  $S_1$  for the unit disc  $\mathbb{D}$  in  $\mathbb{C}$  and prove that this kernel is square integrable over  $\mathbb{D} \times \mathbb{D}$ .

**Proposition 4.**

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|\bar{z} - \bar{w}|^2}{|1 - z\bar{w}|^4} d\lambda(z) d\lambda(w) < \infty.$$

*Proof.* It is easily seen that  $|z - w| \leq |1 - z\bar{w}|$  for  $z, w \in \mathbb{D}$ . Hence we get

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|\bar{z} - \bar{w}|^2}{|1 - z\bar{w}|^4} d\lambda(z) d\lambda(w) \leq \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|1 - z\bar{w}|^2} d\lambda(z) d\lambda(w).$$

Using polar coordinates  $z = r e^{i\theta}$  and  $w = s e^{i\phi}$  we can write the last integral in the following form:

$$\begin{aligned} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|1 - z\bar{w}|^2} d\lambda(z) d\lambda(w) &= \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{r s d\theta d\phi dr ds}{1 - 2 r s \cos(\theta - \phi) + r^2 s^2} \\ &= \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{1 - r^2 s^2}{1 - 2 r s \cos(\theta - \phi) + r^2 s^2} \frac{r s}{1 - r^2 s^2} d\theta d\phi dr ds. \end{aligned}$$

Integration of the Poisson kernel with respect to  $\theta$  yields

$$\int_0^{2\pi} \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \phi) + \rho^2} d\theta = 2\pi, \quad 0 < \rho < 1.$$

Hence

$$\begin{aligned} & \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{1 - r^2 s^2}{1 - 2 r s \cos(\theta - \phi) + r^2 s^2} \frac{r s}{1 - r^2 s^2} d\theta d\phi dr ds \\ &= (2\pi)^2 \int_0^1 \int_0^1 \frac{r s}{1 - r^2 s^2} dr ds = - (2\pi)^2 \int_0^1 \frac{\log(1 - s^2)}{2s} ds < \infty. \end{aligned}$$

□

*Remark.* The last proposition implies that the operator  $T : L^2(\mathbb{D}) \rightarrow L^2(\mathbb{D})$  defined by

$$T(f)(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{\bar{z} - \bar{w}}{(1 - z\bar{w})^2} f(w) d\lambda(w),$$

for  $f \in L^2(\mathbb{D})$ , is a Hilbert-Schmidt operator; see [MV], 16.12.

If we restrict this operator to the closed subspace  $A^2(\mathbb{D})$  we obtain

**Proposition 5.** *The canonical solution operator to  $\bar{\partial}$ ,*

$$S_1 : A^2(\mathbb{D}) \rightarrow L^2(\mathbb{D}),$$

*is a Hilbert-Schmidt operator.*

*Proof.* By [MV], 16.8, we have to show that there exists a complete orthonormal system  $(\phi_k)_{k=0}^\infty$  of  $A^2(\mathbb{D})$  such that

$$\sum_{k=0}^\infty \|S_1(\phi_k)\|^2 < \infty.$$

For this purpose we take a complete orthonormal system  $(\phi_k)_{k=0}^\infty$  of  $A^2(\mathbb{D})$  and extend it to a complete orthonormal system  $(\psi_j)_{j=0}^\infty$  of  $L^2(\mathbb{D})$ . Again by [MV], 16.8, and Proposition 3, it follows that

$$\sum_{j=0}^\infty \|T(\psi_j)\|^2 < \infty,$$

which implies that

$$\sum_{k=0}^\infty \|S_1(\phi_k)\|^2 < \infty.$$

□

### 3. HILBERT-SCHMIDT OPERATORS

Now we show directly that the canonical solution operator to  $\bar{\partial}$ ,

$$S_1 : A^2_{(0,1)}(\mathbb{D}) \rightarrow L^2(\mathbb{D}),$$

is a Hilbert-Schmidt operator if  $\mathbb{D}$  is the open unit disc in  $\mathbb{C}$ , and is not Hilbert-Schmidt if  $\mathbb{B}$  is the open unit ball in  $\mathbb{C}^n$  for  $n > 1$ .

Let  $\mathbb{D} \subset \mathbb{C}$  and let  $\|\cdot\|$  denote the norm in  $A^2(\mathbb{D})$  and consider the orthonormal basis

$$\{u_n(z) = [(n + 1)/\pi]^{1/2} z^n : n \in \mathbb{N}_0\}$$

of  $A^2(\mathbb{D})$ .

**Proposition 6.** *The canonical solution operator  $S_1$  for the unit disc  $\mathbb{D}$  in  $\mathbb{C}$  has the following property:*

$$\sum_{n=0}^{\infty} \|S_1(u_n d\bar{z})\| < \infty,$$

which implies that  $S_1 : A^2_{(0,1)}(\mathbb{D}) \rightarrow L^2(\mathbb{D})$  is a Hilbert-Schmidt operator (see [MV]).

*Proof.* Using calculations in [J] we can show that

$$S_1(u_n d\bar{z})(z) = [(n + 1)/\pi]^{1/2} z^n \bar{z} - [n^2/((n + 1)\pi)]^{1/2} z^{n-1}, \quad n \in \mathbb{N}_0.$$

The Bergman kernel  $B$  of  $\mathbb{D}$  has the form

$$B(z, \zeta) = \frac{1}{\pi} \frac{1}{(1 - z\bar{\zeta})^2};$$

hence by Proposition 1 we can express  $\|S_1(u_n d\bar{z})\|^2$  in the form

$$\int_{\mathbb{D}} \left| \bar{z} u_n(z) - \frac{1}{\pi} \int_{\mathbb{D}} \frac{\bar{\zeta} u_n(\zeta)}{(1 - z\bar{\zeta})^2} d\lambda(\zeta) \right|^2 d\lambda(z).$$

Therefore we get

$$\begin{aligned} \|S_1(u_n d\bar{z})\|^2 &= \int_{\mathbb{D}} \left| \left( \frac{n + 1}{\pi} \right)^{1/2} z^n \bar{z} - \frac{n z^{n-1}}{[(n + 1)\pi]^{1/2}} \right|^2 d\lambda(z) \\ &= \int_{\mathbb{D}} \left( \frac{(n + 1) |z|^{2n+2}}{\pi} - \frac{2n |z|^{2n}}{\pi} + \frac{n^2 |z|^{2n-2}}{(n + 1)\pi} \right) d\lambda(z) \\ &= 2\pi \int_0^1 \left( \frac{(n + 1) r^{2n+3}}{\pi} - \frac{2n r^{2n+1}}{\pi} + \frac{n^2 r^{2n-1}}{(n + 1)\pi} \right) dr \\ &= \frac{1}{(n + 1)(n + 2)}. \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} \|S_1(u_n d\bar{z})\|^2 < \infty.$$

□

*Remark.* It can be shown that the set  $\{S_1(u_n d\bar{z}) : n \in \mathbb{N}_0\}$  consists of pairwise orthogonal elements of  $L^2(\mathbb{D})$ .

In the following part we consider the case of the polydisc. For the sake of simplicity we concentrate on  $\mathbb{C}^2$ ; let

$$\mathbb{D}^2 = \{z = (z_1, z_2) : |z_1| < 1, |z_2| < 1\}.$$

Now  $\{z_1^{n_1} z_2^{n_2} : n_1, n_2 \in \mathbb{N}_0\}$  is an orthogonal basis in  $A^2(\mathbb{D}^2)$ . It is easily seen that the norms of the functions  $z_1^{n_1} z_2^{n_2}$  are  $\pi[1/((n_1 + 1)(n_2 + 1))]^{1/2}$ . The functions

$$u_{n_1, n_2}(z_1, z_2) = \frac{[(n_1 + 1)(n_2 + 1)]^{1/2}}{\pi} z_1^{n_1} z_2^{n_2}, \quad n_1, n_2 \in \mathbb{N}_0,$$

form an orthonormal basis of  $A^2(\mathbb{D}^2)$ , and the system

$$\{u_{n_1, n_2} d\bar{z}_1, u_{n_1, n_2} d\bar{z}_2 : n_1, n_2 \in \mathbb{N}_0\}$$

constitutes an orthonormal basis for  $A^2_{(0,1)}(\mathbb{D}^2)$ .

Next we compute the Bergman projections of the functions

$$(z_1, z_2) \mapsto \bar{z}_1 u_{n_1, n_2}(z_1, z_2) \quad \text{and} \quad (z_1, z_2) \mapsto \bar{z}_2 u_{n_1, n_2}(z_1, z_2)$$

and obtain

$$P(\bar{\zeta}_1 u_{n_1, n_2}(\zeta_1, \zeta_2))(z_1, z_2) = \frac{[(n_1 + 1)(n_2 + 1)]^{1/2}}{\pi} \frac{n_1}{n_1 + 1} z_1^{n_1 - 1} z_2^{n_2},$$

where we used similar computations as in Proposition 6.

The Bergman projection of the second function is

$$P(\bar{\zeta}_2 u_{n_1, n_2}(\zeta_1, \zeta_2))(z_1, z_2) = \frac{[(n_1 + 1)(n_2 + 1)]^{1/2}}{\pi} \frac{n_2}{n_2 + 1} z_1^{n_1} z_2^{n_2 - 1}.$$

Now we can compute the norms of the images under the canonical solution operator of the elements of our orthonormal basis of  $A^2_{(0,1)}(\mathbb{D}^2)$ :

$$\frac{(n_1 + 1)(n_2 + 1)}{\pi^2} \int_{\mathbb{D}^2} \left| \bar{z}_1 z_1^{n_1} z_2^{n_2} - \frac{n_1}{n_1 + 1} z_1^{n_1 - 1} z_2^{n_2} \right|^2 d\lambda(z) = \frac{1}{(n_1 + 2)(n_1 + 1)},$$

where we used the corresponding computation of Proposition 6 for the integral with respect to  $z_1$ .

In a similar way we obtain

$$\frac{(n_1 + 1)(n_2 + 1)}{\pi^2} \int_{\mathbb{D}^2} \left| \bar{z}_2 z_1^{n_1} z_2^{n_2} - \frac{n_2}{n_2 + 1} z_1^{n_1} z_2^{n_2 - 1} \right|^2 d\lambda(z) = \frac{1}{(n_2 + 2)(n_2 + 1)}.$$

Since

$$\sum_{n_1, n_2=1}^{\infty} \left( \frac{1}{(n_1 + 2)(n_1 + 1)} + \frac{1}{(n_2 + 2)(n_2 + 1)} \right) = \infty,$$

the canonical solution operator

$$S_1 : A^2_{(0,1)}(\mathbb{D}^2) \longrightarrow L^2(\mathbb{D}^2)$$

is not Hilbert-Schmidt.

*Remark.* With results from [K2] it can be shown that the canonical solution operator

$$S_1 : A^2_{(0,1)}(\mathbb{D}^2) \longrightarrow L^2(\mathbb{D}^2)$$

is not even compact.

We now consider the case of the unit ball  $\mathbb{B}^2$  in  $\mathbb{C}^2$ . Here we can use calculations from the proof of Proposition 1 in [W] .

The norms of the functions  $z_1^{n_1} z_2^{n_2}$  are now  $\pi[n_1! n_2! / (n_1 + n_2 + 2)!]^{1/2}$  (see [K1]). The functions

$$U_{n_1, n_2}(z_1, z_2) = \frac{[(n_1 + n_2 + 2)!]^{1/2}}{\pi(n_1! n_2!)^{1/2}} z_1^{n_1} z_2^{n_2}, \quad n_1, n_2 \in \mathbb{N}_0,$$

form an orthonormal basis of  $A^2(\mathbb{B}^2)$ , and the system

$$\{U_{n_1, n_2} d\bar{z}_1, U_{n_1, n_2} d\bar{z}_2 : n_1, n_2 \in \mathbb{N}_0\}$$

constitutes an orthonormal basis for  $A^2_{(0,1)}(\mathbb{B}^2)$ .

We compute the Bergman projections of the functions

$$(z_1, z_2) \mapsto \bar{z}_1 U_{n_1, n_2}(z_1, z_2) \quad \text{and} \quad (z_1, z_2) \mapsto \bar{z}_2 U_{n_1, n_2}(z_1, z_2)$$

and obtain

$$P(\bar{\zeta}_1 U_{n_1, n_2}(\zeta_1, \zeta_2))(z_1, z_2) = \frac{[(n_1 + n_2 + 2)!]^{1/2}}{\pi (n_1! n_2!)^{1/2}} \frac{n_1}{n_1 + n_2 + 2} z_1^{n_1-1} z_2^{n_2},$$

$$P(\bar{\zeta}_2 U_{n_1, n_2}(\zeta_1, \zeta_2))(z_1, z_2) = \frac{[(n_1 + n_2 + 2)!]^{1/2}}{\pi (n_1! n_2!)^{1/2}} \frac{n_2}{n_1 + n_2 + 2} z_1^{n_1} z_2^{n_2-1}.$$

Finally we compute the norms of the images of the basis elements under the canonical solution operator  $S_1$ , and obtain

$$\begin{aligned} & \frac{(n_1 + n_2 + 2)!}{\pi^2 n_1! n_2!} \int_{\mathbb{B}^2} \left| \bar{z}_1 z_1^{n_1} z_2^{n_2} - \frac{n_1}{n_1 + n_2 + 2} z_1^{n_1-1} z_2^{n_2} \right|^2 d\lambda(z) \\ &= \frac{n_2 + 2}{(n_1 + n_2 + 2)(n_1 + n_2 + 3)} \end{aligned}$$

and

$$\begin{aligned} & \frac{(n_1 + n_2 + 2)!}{\pi^2 n_1! n_2!} \int_{\mathbb{B}^2} \left| \bar{z}_2 z_1^{n_1} z_2^{n_2} - \frac{n_2}{n_1 + n_2 + 2} z_1^{n_1} z_2^{n_2-1} \right|^2 d\lambda(z) \\ &= \frac{n_1 + 2}{(n_1 + n_2 + 2)(n_1 + n_2 + 3)}. \end{aligned}$$

Since

$$\sum_{n_1, n_2=1}^{\infty} \left( \frac{n_2 + 2}{(n_1 + n_2 + 2)(n_1 + n_2 + 3)} + \frac{n_1 + 2}{(n_1 + n_2 + 2)(n_1 + n_2 + 3)} \right) = \infty,$$

the canonical solution operator

$$S_1 : A^2_{(0,1)}(\mathbb{B}^2) \longrightarrow L^2(\mathbb{B}^2)$$

is also not Hilbert-Schmidt.

*Remark.* In [Z] it is shown that there are no nonzero Hilbert-Schmidt Hankel operators on the Bergman space of the unit ball in  $\mathbb{C}^n$  with antiholomorphic symbol when  $n \geq 2$ .

**Proposition 7.** *The integral kernel*

$$\frac{|z_1 - w_1|^2 + |z_2 - w_2|^2}{|1 - z_1 \bar{w}_1|^4 |1 - z_2 \bar{w}_2|^4}$$

does not belong to  $L^2(\mathbb{D}^2 \times \mathbb{D}^2)$  and the integral kernel

$$\frac{|z_1 - w_1|^2 + |z_2 - w_2|^2}{|1 - z_1 \bar{w}_1 - z_2 \bar{w}_2|^6}$$

does not belong to  $L^2(\mathbb{B}^2 \times \mathbb{B}^2)$ .



*Proof.* Suppose the first kernel belongs to  $L^2(\mathbb{D}^2 \times \mathbb{D}^2)$ . Then the corresponding integral operator from  $L^2_{(0,1)}(\mathbb{D}^2)$  to  $L^2(\mathbb{D}^2)$  is a Hilbert-Schmidt operator, which would imply that the restriction to  $A^2_{(0,1)}(\mathbb{D}^2)$  is also Hilbert-Schmidt. But this restriction coincides with the canonical solution operator  $S_1$ , from which we already know that it is not Hilbert-Schmidt. The proof for the second integral is analogous to the first.

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## REFERENCES

- [A] S. Axler, *The Bergman space, the Bloch space, and commutators of multiplication operators*, Duke Math. J. **53** (1986), 315–332. MR **87m**:47064
- [AFP] J. Arazy, S. Fisher and J. Peetre, *Hankel operators on weighted Bergman spaces*, Amer. J. of Math. **110** (1988), 989–1054. MR **90a**:47067
- [B] F.F. Bonsall, *Hankel operators on the Bergman space for the disc*, J. London Math. Soc. (2) **33** (1986), 355–364. MR **88e**:47045
- [C] D. Catlin, *Subelliptic estimates for the  $\bar{\partial}$ -Neumann problem on pseudoconvex domains*, Ann. of Math. **126** (1987), 131–191. MR **88i**:32025
- [CD] D. Catlin and J. D’Angelo, *Positivity conditions for bihomogeneous polynomials*, Math. Res. Lett. **4** (1997), 555–567. MR **98e**:32023
- [FS1] S. Fu and E.J. Straube, *Compactness of the  $\bar{\partial}$ -Neumann problem on convex domains*, J. of Functional Analysis **159** (1998), 629–641. MR **99h**:32019
- [FS2] S. Fu and E.J. Straube, *Compactness in the  $\bar{\partial}$ -Neumann problem*, preprint, 1999.
- [J] S. Janson, *Hankel operators between weighted Bergman spaces*, Ark. Mat. **26** (1988), 205–219. MR **91j**:47027
- [K1] St. Krantz, *Function theory of several complex variables*, Wadsworth & Brooks/Cole, 1992 (2nd edition). MR **93c**:32001
- [K2] St. Krantz, *Compactness of the  $\bar{\partial}$ -Neumann operator*, Proc. Amer. Math. Soc. **103**(1988), 1136–1138. MR **89f**:32032
- [L] Ewa Ligocka, “The regularity of the weighted Bergman projections”, in *Seminar on deformations, Proceedings, Lodz-Warsaw, 1982/84*, Lecture Notes in Math. **1165**, Springer-Verlag, Berlin, 1985, 197–203. MR **87d**:32044
- [MV] R. Meise und D. Vogt, *Einführung in die Funktionalanalysis*, Vieweg Studium **62**, Vieweg-Verlag, 1992. MR **94f**:46003
- [R] R. Rochberg, *Trace ideal criteria for Hankel operators and commutators*, Indiana Univ. Math. J. **31** (1982), 913–925. MR **84d**:47036
- [SSU] N. Salinas, A. Sheu and H. Upmeyer, *Toeplitz operators on pseudoconvex domains and foliation  $C^*$ -algebras*, Ann. of Math. **130** (1989), 531–565. MR **91e**:47026
- [W] R. Wallsten, *Hankel operators between weighted Bergman spaces in the ball*, Ark. Mat. **28** (1990), 183–192. MR **91i**:47041
- [Z] K.H. Zhu, *Hilbert-Schmidt Hankel operators on the Bergman space*, Proc. Amer. Math. Soc. **109** (1990), 721–730. MR **90k**:47060

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