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REMARKS ON THE DEGENERATE RADON TRANSFORM IN $\mathbf{R}^2$

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ABSTRACT. The aim of this note is to prove endpoint boundedness of the generalized Radon transform which was introduced by Phong and Stein. M. Christ's combinatorial method is used to obtain restricted weak type at the endpoints. Also we show that the results of this note are essentially optimal.

1. INTRODUCTION

Phong and Stein [4] introduced the following integral operator as a model of degenerate Radon transform:

$$Rf(t,x) = \int f(t+S(x,y),y)\psi(t,x,y)dy.$$

Here $S(x,y) = \sum_{i=1}^{n-1} a_i x^i y^{n-i}$ and ψ is a compactly supported smooth function on $\mathbf{R}^3$. In their paper, they showed that if $a_1 a_{n-1} \neq 0$, R is bounded on the trapezoid with vertices $(0,0), (\frac{2}{n+1}, \frac{1}{n+1}), (\frac{n}{n+1}, \frac{n-1}{n+1}), (1,1)$ except on the boundary lines $\Big((0,0), (\frac{2}{n+1}, \frac{1}{n+1})\Big]$ and $\Big[(\frac{n}{n+1}, \frac{n-1}{n+1}), (1,1)\Big)$. More generally, they showed the following result.

Theorem 1. *If* $S(x,y) = a_j x^j y^{n-j} + \cdots + a_k x^k y^{n-k}$ *with* $1 \leq j < k \leq n-1$, $a_j a_k \neq 0$, *then* R *is bounded on the trapezoid with vertices* $(0,0)$,$(\frac{j+1}{n+1}, \frac{j}{n+1})$, $(\frac{k+1}{n+1}, \frac{k}{n+1})$,$(1,1)$ *except the boundary lines* $\Big((0,0), (\frac{j+1}{n+1}, \frac{j}{n+1})\Big]$ *and* $\Big[(\frac{k+1}{n+1}, \frac{k}{n+1}), (1,1)\Big)$.

It seems that the numbers are slightly misstated in [4] but a careful reading leads to the above theorem by their argument. In this note, we will prove the following result for the endpoints.

Theorem 2. *R is restricted weak type at* $(\frac{j+1}{n+1}, \frac{j}{n+1})$ *and* $(\frac{k+1}{n+1}, \frac{k}{n+1})$.

So via interpolation the undetermined boundary lines are included in the type set of R. Moreover this is sharp modulo two points $(\frac{j+1}{n+1}, \frac{j}{n+1})$ and $(\frac{k+1}{n+1}, \frac{k}{n+1})$. Recently Bak [1] obtained strong type at $(\frac{2}{n+1}, \frac{1}{n+1})$,$(\frac{n}{n+1}, \frac{n-1}{n+1})$ using a multilinear

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argument in the case $a_1a_{n-1} \neq 0$. Our method of proof is an adaptation of a method of M. Christ [3]. Finally we say $A \sim B$ if there is a constant $C > 0$ so that $\frac{1}{C}A \leq B \leq C$. Also C denotes positive constants which may vary from line to line.

2. Proof of Theorem 2

To begin with, observe that the adjoint operator R^* of R is given as follows:

$$R^*f(t,y) = \int f(t - S(x,y), x)\psi(t - s(x,y), x, y)dx.$$

Denote $[-1,1]$ by I. Since R is a positive operator, we may assume $\psi(t,x,y) = \chi_{I\times I}(x,y)$.

Theorem 2 ensues from the following result which is a little more general. Let P be a nonnegative function. Suppose S satisfies

$$\left|\partial_y^m \partial_x S(x,y)\right| \geq P(x)$$

for all (x,y) on the support of ψ. We will see that the type set of R is determined by the degeneracy of P. To describe the degeneracy, we say P has a zero at x_0 of order d if $P(x) \sim |x - x_0|^d$ for x in some neighborhood of x_0.

Proposition 3. *If P has zeros of order at most l, then R is restricted weak type at $(\frac{l+2}{l+m+2}, \frac{l+1}{l+m+2})$.*

To prove Theorem 2, we only need to observe

$$\partial_y^{n-j}\partial_x S(x,y) = C_j x^{j-1}, \quad C_j \neq 0.$$

Using Proposition 3, we have restricted weak type at $(\frac{j+1}{n+1}, \frac{j}{n+1})$. Considering the dual operator R^*, we have

$$\partial_x^k \partial_y S(x,y) = C_k y^{n-k-1}, \quad C_k \neq 0.$$

Arguing as above with the roles of x and y exchanged, R^* is restricted weak type at $(\frac{n-k+1}{n+1}, \frac{n-k}{n+1})$. By duality, R is restricted weak type at $(\frac{k+1}{n+1}, \frac{k}{n+1})$. This completes the proof of Theorem 2.

Now we begin the proof of Proposition 3. Following M. Christ's method, we aim to show that for every positive number α and every Borel set $E \subset \mathbf{R}^2$, the set $F = \{x \in \mathbf{R}^2 : R(\chi_E)(x) \sim \alpha\}$ satisfies

$$|F| \leq C\alpha^{-q}|E|^{\frac{q}{p}} \tag{2-1}$$

which is equivalent to restricted weak type (p,q).

First we construct a map $\Phi : I \times I \to \mathbf{R}^2$ which carries a set $\Delta_0 \subset I \times I$ of known measure into E. Assuming $|E| > 0$, introduce

$$\beta = \frac{1}{|E|}\left\langle R(\chi_E), \chi_F \right\rangle. \tag{2-2}$$

Then there is a point (u,v) in E such that $R^*(\chi_F)(u,v) \geq \frac{1}{2}\beta$, because if there is no such point in E, then $\beta|E| = \langle R(\chi_E), \chi_F\rangle = \langle \chi_E, R^*(\chi_F)\rangle \leq \frac{1}{2}\beta|E|$, which is a contradiction.

Define $\Phi_1 : I \to \mathbf{R}^2$ by $\Phi_1(x) = (u - S(x,v), x)$. Since

$$R^*(\chi_F)(u,v) = \int \chi_F(u - S(x,v), x)\psi(x,y)dx \geq \frac{1}{2}\beta,$$

there is a set $X_0 \subset I$ such that $\Phi_1(X_0) \subset F$ and $|X_0| \geq C\beta$.

Define $\Phi_{2,x} : I \to \mathbf{R}^2$ by $\Phi_{2,x}(y) = (u - S(x,v) + S(x,y), y)$. By the definition of F for all $x \in X_0$,

$$R(\chi_E)(\Phi_1(x)) = \int \chi_E(u - S(x,v) + S(x,y), y)\psi(x,y)dy \sim \alpha.$$

So for each $x \in X_0$, there is a set $Y_x \subset I$ such that $\Phi_{2,x}(Y_x) \subset E$ and $|Y_x| \sim \alpha$. We should also note that since $\partial_y S$ is bounded on $I \times I$, for all $x \in X_0$ there is $C > 0$ independent of x such that

$$C^{-1}\alpha < |Y_x| < C\alpha.$$

From now on we express the last condition by saying that $|Y_x| \sim \alpha$ independently for all $x \in X_0$.

Set

$$\Delta_0 = \bigcup_{x \in X_0} (x, Y_x)$$

and note that $\Phi(\Delta_0) \subset E$ by the construction. Now define a map $\Phi : I \times I \to E$ by

$$\Phi(x,y) = (u - S(x,v) + S(x,y), y).$$

To prove (2-1), we want a lower bound for $|E|$ in terms of α and β. Let's denote by $J(\Phi)$ the Jacobian of Φ and observe

$$J(\Phi) = |S'_x(x,y) - S'_x(x,v)|.$$

Having constructed Φ and Δ_0, some part of Δ_0 will be traded for the lower bound of $J(\Phi)$ (see (2-4)). Actually we discard some portion of Δ_0 on which $J(\Phi)$ degenerates with no essential change to the measure of the remaining set. To achieve this, we will need the following elementary lemma.

Lemma 4. *Let u be a smooth function on $\mathbf{R}$ with $|u^{(k)}| \geq m$ for some $k \geq 1$. Then for a given $\lambda > 0$, there is a measurable set B such that*

$$|u| \geq m\lambda^k \text{ on } B^c \quad \text{and} \quad |B| \leq C\lambda;$$

here C is independent of u and λ.

Lemma 4 can be proved by an inductive argument on k. But in fact, it is a mere restatement of the sublevel set estimate. The following is borrowed from [2].

Lemma 5. *Let v be the smooth function on $\mathbf{R}$. If $v^{(k)} \geq 1$ for some $k \geq 1$, then*

$$|\{t \in \mathbf{R} : |v(t)| \leq \lambda\}| \leq C\lambda^{\frac{1}{k}};$$

here C depends only on k.

Proof of Lemma 4. We apply Lemma 5 to the function $\frac{1}{m} \cdot u$ to get $|\{x \in \mathbf{R} : |u(x)| < m\lambda^k\}| \leq C\lambda$. So we only need to set $B = \{x \in \mathbf{R} : |u(x)| < m\lambda^k\}$. For the details of sublevel set estimates, see [2]. ☐

Since the measure of X_0 is $\sim \beta$, we can delete small $(\varepsilon\beta)$-neighborhoods of zeros of P which are in X_0 so that the measure of the remaining set $X \subset X_0$ is $\sim \beta$. Since P has zeros of order at most l and we deleted $(\varepsilon\beta)$-neighborhoods of zeros, for all $x \in X$

$$\left|\partial_y^m \partial_x S(x,y)\right| \geq C\beta^l.$$

Fixing $x \in X$, we apply Lemma 4 to the function $y : I \to S'_x(x,y) - S'_x(x,v)$. Since $|\partial_y^m (S'_x(x,y) - S'_x(x,v))| = |\partial_y^m \partial_x S(x,y)| \geq C\beta^l$, there exists a set B_x of measure $\varepsilon\alpha$ such that for all $y \in (I \setminus B_x)$

$$J(\Phi)(x,y) = |S'_x(x,y) - S'_x(x,v)| \geq C\beta^l \alpha^m. \tag{2-3}$$

Since $|Y_x| \sim \alpha$ independently for all $x \in X$, choosing sufficiently small ε, we can assume that $|(Y_x \setminus B_x)| \sim \alpha$ independently for all $x \in X$.

Now the proof may be completed in a few strokes. Define Δ by

$$\Delta = \bigcup_{x \in X} (x, (Y_x \setminus B_x))$$

and note that (2-3) holds for all $(x,y) \in \Delta$. It is not so clear whether the set Δ is measurable. But for simplicity, we assume momentarily that Δ is measurable. Since S is a polynomial, the multiplicity of the map Φ cannot be greater than the degree of S. So we have

$$|\Phi(\Delta)| \geq C \iint_\Delta J(\Phi) dxdy. \tag{2-4}$$

Since $\phi(\Delta) \subset E$ and $\beta|E| \sim \alpha|F|$ by (2-2),

$$|E| \geq C \int_X \int_{Y_x \setminus B_x} \beta^l \alpha^m dydx \sim \alpha^{m+1}\beta^{l+1} \geq C\alpha^{l+m+2} \left(\frac{|F|}{|E|}\right)^{l+1}.$$

So we have $|E| \geq C\alpha^{m+1}\beta^{l+1}$ and

$$|\{x \in R^2 : R(\chi_E)(x) \sim \alpha\}| \leq C \frac{|E|^{\frac{l+2}{l+1}}}{\alpha^{\frac{l+m+2}{l+1}}}.$$

Finally we drop the measurablity assumption on the set Δ. We will use outer measure to get around measurability. Denote by $|H|_e$ the outer measure of any set H. Let G be an open set containing E. By the argument used above we can write

$$|G| \geq C \int_{\Phi^{-1}(G)} J(\Phi)(x,y) dxdy \geq C \inf_{(x,y)\in\Delta} J(\Phi)(x,y) \cdot |\Delta|_e$$

because $\Phi^{-1}(G) \supset \Delta$. By a simple argument we can see that

$$|\Delta|_e \geq \inf_{x \in X} |(Y_x \setminus B_x)| \cdot |X| \geq C\alpha\beta.$$

Recalling that $J(\Phi) \geq C\beta^l\alpha^m$ on Δ, we again arrive at $|E| \geq C\alpha^{m+1}\beta^{l+1}$. This completes the proof of Proposition 3.

3. Necessary conditions

In this section we will show that R is unbounded outside the closed trapezoid with vertices $(0,0), (\frac{j+1}{n+1}, \frac{j}{n+1}), (\frac{k+1}{n+1}, \frac{k}{n+1}), (1,1)$. We may assume $a_j = a_k = 1$. Let $f(s,y)$ be the function $s^{-\alpha}\chi_{[0,1]}(s)\chi_{[0,1]}(y)$. We have

$$\begin{aligned} Rf(t,x) &\geq \int_0^1 \left|t - (x^j y^{n-j} + \cdots + x^k y^{n-k})\right|^{-\alpha} dy \\ &= x \int_0^{\frac{1}{x}} \left|t - x^n(Y^{n-j} + \cdots + Y^{n-k})\right|^{-\alpha} dY \quad (\text{set } y = xY) \\ &\geq x \int_1^{\frac{1}{x}} \left|t - x^n(Y^{n-j} + \cdots + Y^{n-k})\right|^{-\alpha} dY. \end{aligned}$$

We changed the variable $y = xY$ in the second line. Set $S = x^n(Y^{n-j} + \cdots + Y^{n-k})$. Then $S \sim x^n Y^{n-j}$ for large $Y \gg 1$. So $dY \sim \frac{dS}{x^n Y^{n-j-1}}$ for $Y \gg 1$. Then the above is bounded below by

$$\begin{aligned} Cx \int_{x^n}^{Cx^j} |t - S|^{-\alpha} x^{-\frac{n}{n-j}} S^{-1+\frac{1}{n-j}} dS &\geq Cx^{-\frac{j}{n-j}} \int_{\frac{x^j}{2}}^{x^j} |t - S|^{-\alpha} S^{-1+\frac{1}{n-j}} dS \\ &\geq Cx^{-\frac{j}{n-j}} x^{j(1-\alpha)} x^{-j+\frac{j}{n-j}} = Cx^{-j\alpha} \end{aligned}$$

if $0 < x \ll 1$ and $t \sim x^j$. Overall we have

$$Rf(t, x) \geq Cx^{-j\alpha} \qquad \text{if } 0 < x \ll 1 \text{ and } t \sim x^j.$$

Therefore if $-j\alpha q + j < -1$,

$$\iint |Rf(t, x)|^q dx dt \geq \iint_{\{t \sim x^j\}} x^{-j\alpha q} dt dx = \int_0^C x^{-j\alpha q + j} dx = +\infty.$$

On the other hand,

$$\iint |f(t, x)|^p dx dt \sim \int_0^1 |t|^{-\alpha p} dt < +\infty \qquad \text{if } -\alpha p > -1.$$

To have $\|Rf\|_q \leq C\|f\|_p$, there cannot be α such that $\frac{j+1}{jq} < \alpha < \frac{1}{p}$. So (p, q) must satisfy that

$$\frac{1}{p} \leq \frac{j+1}{j} \frac{1}{q}. \tag{3-1}$$

By a similar method, R^* can be bounded from L^p to L^q only if $\frac{1}{p} \leq \frac{n-k+1}{n-k} \frac{1}{q}$. By duality R can be bounded from L^p to L^q only if

$$1 - \frac{1}{q} \leq \frac{n-k+1}{n-k}\left(1 - \frac{1}{p}\right). \tag{3-2}$$

Choosing $f(s, y) = X_{[0,\delta^n] \times [0,\delta]}(s, y)$, we easily see that R can be bounded from L^p to L^q only if

$$\frac{1}{p} - \frac{1}{q} \leq \frac{1}{n+1}. \tag{3-3}$$

Simple algebra shows that intersection of the above three regions (3-1), (3-2), (3-3) and $\{(\frac{1}{p}, \frac{1}{q}) \in I \times I : \frac{1}{p} \leq \frac{1}{q}\}$ is the closed trapezoid with vertices $(0, 0), (\frac{j+1}{n+1}, \frac{j}{n+1}), (\frac{k+1}{n+1}, \frac{k}{n+1}), (1, 1)$.

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