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# DUNFORD–PETTIS SETS

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Abstract. Bibasic sequences are used to study relative weak compactness and relative norm compactness of Dunford–Pettis sets.

A Banach space $X$ has the Dunford–Pettis property provided that every weakly compact operator with domain $X$ and range an arbitrary Banach space $Y$ maps weakly compact sets in $X$ into norm compact sets in $Y$. Localizing this notion, a bounded subset $A$ of $X$ is said to be a Dunford–Pettis subset of $X$ if $T(A)$ is relatively norm compact in $Y$ whenever $T : X \to Y$ is a weakly compact operator. Consequently, a Banach space $X$ has the Dunford–Pettis property if and only if each of its weakly compact sets is a Dunford–Pettis set. The survey article by Diestel [5] is an excellent source of information about classical results in Banach spaces which relate to the Dunford–Pettis property.

Kevin Andrews utilized Dunford–Pettis sets in a study of the Bochner integral in [1]. In Theorem 1 of [1], Andrews showed that a subset $K$ of $X$ is a Dunford–Pettis subset of $X$ if and only if

$$lim_n\ (sup\{|x_n^*(x)| : x \in K\}) = 0$$

whenever $(x_n^*)$ is a weakly null sequence in $X^*$ ( = the continuous linear dual of $X$). In Corollary 4 of [1], Andrews used this characterization to show that if $X$ has the Dunford–Pettis property and $\ell^1$ does not embed in $X$, then the space $L^1(\mu,\mathrm{X})$ has the Dunford–Pettis property.

E. Bator showed in [2] that a dual space has the weak Radon–Nikodym property if and only if each Dunford–Pettis subset of $X$ is relatively compact. In addition to reproducing Bator's result, Emmanuele [8] established several other structure properties for Banach spaces in which all Dunford–Pettis sets are relatively compact. Since every bounded subset of a Banach space $X$ whose dual space $X^*$ has the Schur property is a Dunford–Pettis subset of $X$, it is clear that there are Dunford–Pettis sets which are not relatively weakly compact. However, we note that Odell [13, p. 377] showed that every sequence in a Dunford–Pettis set has a weakly Cauchy subsequence. In this paper we study Dunford–Pettis sets which fail to be relatively norm or weakly compact.

The following definitions and notation will be helpful. A sequence $(x_n, f_n^*)$ in $X \times \mathrm{X}^*$ is called bibasic [14, p. 85], [4] if $(x_n)$ is a basic sequence in $X$, $(f_n^*)$ is a basic sequence in $X^*$, and $f_i^*(x_j) = \delta_{ij}$. If $(x_n, f_n^*)$ is a bibasic sequence, $X_0 = [x_n]$,

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and $x_n^* = f^*_{n|X_0}$ (i.e., $(x_n^*)$ is the sequence of coefficient functionals corresponding to the basic sequence $(x_n)$), then certainly $f_n^*$ is a continuous linear extension of $x_n^*$ to all of $X$ for each $n$. The bibasic sequence $(x_n, f_n^*)$ is said to be seminormalized (or bounded) if there are positive numbers $p$ and $q$ so that $p \leq ||x_n|| \leq q$ and $p \leq ||f_n^*|| \leq q$ for all $n$. The unit vector basis of $c_0$ will be denoted by $(e_n)$, and the unit vector basis of $\ell^1$ will be denoted by $(e_n^*)$. If $K$ is a subset of $X$, then $K - K$ is defined to be $\{x - y : x, y \in K\}$. The reader is referred to Diestel [6] or Lindenstrauss and Tzafriri [9] for unexplained notation or terminology.

In Theorem 1 of [4], Davis, Dean, and Lin showed that every infinite dimensional Banach space contains a bounded bibasic sequence. Our first lemma will facilitate our discussion which shows that special bibasic sequences occur in $X \times X^*$ whenever $X$ contains a Dunford–Pettis set which fails to be relatively norm or weakly compact. Although versions of (ii) of the lemma are well known, a proof has been included for the convenience of the reader.

**Lemma 1.** (i) *If $K$ is a Dunford–Pettis subset of the Banach space $X$, $(x_n, f_n^*)$ is a biorthogonal sequence in $K \times X^*$, and $sup_n ||f_n^*|| < \infty$, then some subsequence of $(f_n^*)$ is equivalent to the unit vector basis of $\ell^1$.*

(ii) *If $(x_n)$ is an unconditional basis for $X$, $(x_n^*)$ is the sequence of coefficient functionals corresponding to $(x_n)$, and the subsequence $(x_{n_i}^*)$ of $(x_n^*)$ is equivalent to $(e_i^*)$, then $(x_{n_i}) \sim (e_i)$.*

*Proof.* (i) Suppose that $(x_n, f_n^*)$ is as in the statement of (i) and that $(f_{n_i}^*)$ is a weakly Cauchy subsequence of $(f_n^*)$. Let

$$z_i^* = f_{n_i}^* - f_{n_{i+1}}^*$$

for each $i$. Therefore, $(z_i^*) \to 0$ weakly and

$$lim_i(sup\{|z_i^*(u)| : u \in K\}) = 0.$$

However, $z_i^*(x_{n_i}) = 1$ for each $i$. Thus no subsequence of $(f_n^*)$ can be weakly Cauchy, and, by Rosenthal's $\ell^1$-theorem [12], [6], some subsequence of $(f_n^*)$ is equivalent to $(e_i^*)$.

(ii) Now suppose that $X$, $(x_n)$, and $(x_{n_i}^*)$ are as in the statement of (ii). Let $M$ be the unconditional basis constant for $(x_n)$, and let $J$ be a positive integer so that

$$(1/J)||\sum s_i x_{n_i}^*|| \leq \sum |s_i| \leq J||\sum s_i x_{n_i}^*||$$

for each finite sequence $(s_1, ..., s_n)$ of scalars. If $x \in X$ and $x^* \in X^*$, then

$$x^*(x) = lim_m \sum_{i=1}^m x_i^*(x) x^*(x_i)$$

$$= lim \langle \sum_1^m x^*(x_i) x_i^*, x \rangle$$

$$= lim_m \langle P_m^*(x^*), x \rangle,$$

where $(P_m)_{m=1}^\infty$ is the usual *sequence* of projection operators generated by a basic sequence. Thus

$$||x|| \leq sup\{|\langle \sum_{n=1}^k s_n x_n^*, x \rangle| : k \in \mathbf{N}, ||\sum s_n x_n^*|| \leq M\}.$$

If $x = \sum_{i=1}^p \alpha_i x_{n_i}$, then

$$\begin{aligned}(1/J)(\alpha_1, \dots, \alpha_p)||_\infty &= sup\{|(1/J)x^*_{n_i}(x)| : 1 \le i \le p\} \le ||x|| \\ &\le sup\{|\langle \sum_{n=1}^k s_n x^*_n, \sum_{i=1}^p \alpha_i x_{n_i}\rangle| : k \in \mathbf{N}, ||\sum_1^k s_n x^*_n|| \le M\} \\ &\le sup\{|\langle \sum_{u=1}^k t_u x^*_{n_u}, \sum_{i=1}^p \alpha_i x_{n_i}\rangle| : k \in \mathbf{N}, ||\sum_{u=1}^k t_u x^*_{n_u}|| \le M^2\} \\ &\le JM^2||(\alpha_1, ..., \alpha_p)||_\infty.\end{aligned}$$

Hence $(x_{n_i}) \sim (e_i)$. □

**Theorem 2.** *Suppose that $K$ is a subset of the Banach space $X$.*

(i) *If $K$ is a nonrelatively norm compact Dunford–Pettis set, then there is a seminormalized bibasic sequence $(x_n, f^*_n)$ in $(K - K) \times X^*$ such that $(f^*_n)$ is equivalent to $(e^*_n)$.*

(ii) *If $(x_n, f^*_n)$ is a biorthogonal sequence in $(K - K) \times X^*$ and $sup_n||f^*_n|| < \infty$, then $K$ is not relatively norm compact.*

(iii) *If $K$ is a nonrelatively weakly compact Dunford–Pettis set, then there is a bibasic sequence $(x_n, f^*_n)$ in $K \times X^*$ and an element $x^* \in X^*$ so that $(f^*_n)$ is equivalent to $(e^*_n)$ and $lim_n x^*(x_n) > 0$.*

(iv) *If $(x_n, f^*_n)$ is a biorthogonal sequence in $K \times X^*$, $(x_n)$ is basic, and $x^* \in X^*$ so that $lim\ x^*(x_n) > 0$, then $K$ is not relatively weakly compact.*

*Proof.* (i) Suppose that $K$ is a nonrelatively norm compact Dunford–Pettis set. Let $(y_n)$ be a sequence in $K$ and $\epsilon$ be a positive number so that if

$$x_n = y_n - y_{n+1}$$

for each $n$, then $||x_n|| > \epsilon$ for each $n$ and $(x_n) \to 0$ weakly. Apply a classical construction of Bessaga and Pelczynski [3], [9, p. 5], and suppose (without loss of generality) that $(x_n)$ is a seminormalized basic sequence. Let $(x^*_n)$ be the associated sequence of coefficient functionals, and for each $n$ let $f^*_n$ be a Hahn–Banach extension of $x^*_n$ to all of $X$. Apply (i) of Lemma 1 to the Dunford–Pettis set $K - K$ to complete the proof of (i).

(ii) If the hypotheses of (ii) are satisfied, then there is an $\epsilon > 0$ so that $||x_n - x_m|| > \epsilon$ for $n \ne m$. Thus $K - K$ is not relatively norm compact; hence $K$ is not relatively norm compact.

(iii) Suppose that $K$ is a nonrelatively weakly compact Dunford–Pettis set, and let $(x_n)$ be a sequence in $K$ with no weakly convergent subsequence. By Pelczynski's version of the Eberlein–Smulian theorem [10], [6, p. 41], we may suppose that $(x_n)$ is basic and $lim_n\ x^*(x_n) > 0$ for some $x^* \in X^*$. Note that $(x_n)$ is a seminormalized sequence, and let $(x^*_n)$ and $(f^*_n)$ be defined as in (i) above. Again apply Lemma 1 to finish the proof of (iii).

(iv) If $(x_n, f^*_n)$ is a biorthogonal sequence in $K \times X^*$, $(x_n)$ is basic, and $lim\ x^*(x_n) > 0$, then $(x_n)$ is a sequence in $K$ without a weakly convergent subsequence. (The fact that $(x^*_i(x_n))_{n=1}^\infty \to 0$ for each $i$ certainly implies that the only candidate for a weak limit is 0.) Therefore, $K$ is not relatively weakly compact. □

Using a fundamental result of Pelczynski and Singer [11] dealing with the existence of *conditional* basic sequences, Davis, Dean, and Lin [4, Proposition 1] showed that if $X$ is an infinite dimensional space, then there is a bounded bibasic sequence $(x_n, f_n^*)$ in $X \times X^*$ so that $(f_n^*) \not\sim (x_n^*)$. In the following result, we see that Dunford–Pettis sets which are not relatively weakly compact naturally generate such sequences.

**Theorem 3.** *If $K$ is a nonrelatively weakly compact Dunford–Pettis subset of $X$, then there is a seminormalized bibasic sequence $(x_n, f_n^*)$ in $K \times X^*$ so that $(f_n^*)$ is not equivalent to $(x_n^*)$. Further, if $(x_n)$ is unconditional, then there is no sequence $(g_n^*)$ in $X^*$ so that $g_n^*|_{[x_j]} = x_n^*$ for each $n$ and $(g_n^*) \sim (x_n^*)$.*

*Proof.* Suppose that $K$ is a Dunford–Pettis subset of $X$ and $K$ fails to be relatively weakly compact. Use Theorem 2, and let $(x_n, f_n^*)$ be a bounded bibasic sequence so that $\{x_n\} \subseteq K$, no subsequence of $(x_n)$ converges weakly to a point of $X$, and $(f_n^*) \sim (e_n^*)$. Suppose that $(f_n^*) \sim (x_n^*)$. Consequently, $(x_n) \sim (e_n)$, and $(x_n)$ converges weakly to 0. This contradiction finishes the proof of the first assertion.

Now suppose that $(x_n)$ is unconditional and $(g_n^*)$ is a sequence in $X^*$ so that $g^*_{n|_{[x_j]}} = x_n^*$ for each $n$ and $(g_n^*) \sim (x_n^*)$. Use Lemma 1(i) again, and let $(g_{n_i}^*)$ be a subsequence so that $(g_{n_i}^*) \sim (e_i^*)$. Therefore, $(x_{n_i}^*) \sim (e_i^*)$, and Lemma 1(ii) shows that $(x_{n_i}) \sim (e_i)$. Thus $(x_{n_i}) \to 0$ weakly, and we have a contradiction. □

The following result studies the structure of Dunford–Pettis sets which contain *unconditional* basic sequences $(x_n)$ so that $(x_n^*) \sim (f_n^*)$. As the theorem demonstrates, these are the sets which also contain weakly null basic sequences which are hereditarily Dunford–Pettis.

**Theorem 4.** *If $K$ is a Dunford–Pettis subset of the Banach space $X$, then the following are equivalent:*

(i) *There is an isomorphic embedding $T : c_0 \to X$ so that $\{T(e_n)\} \subseteq K$.*
(ii) *There is a seminormalized bibasic sequence $(x_n, f_n^*)$ in $K \times X^*$ so that $(x_n)$ is unconditional and $(x_n^*) \sim (f_n^*)$.*
(iii) *There is a seminormalized bibasic sequence $(x_n, f_n^*)$ in $K \times X^*$ so that $(x_n)$ is unconditional, $(f_n^*) \sim (x_n^*)$, and $[f_n^*]$ is complemented in $X^*$.*
(iv) *There is a seminormalized bibasic sequence $(x_n, f_n^*)$ in $K \times X^*$ so that $(x_n)$ is weakly null and $\{y_k\}$ is Dunford–Pettis in $[y_k]$ for each subsequence $(y_k)$ of $(x_n)$.*
(v) *There is a seminormalized bibasic sequence $(x_n, f_n^*)$ in $K \times X^*$ so that $(x_n)$ is unconditional and $\{y_k\}$ is a Dunford–Pettis subset of $[y_k]$ for each subsequence $(y_k)$ of $(x_n)$.*
(vi) *There is a seminormalized bibasic sequence $(x_n, f_n^*)$ in $K \times X^*$ so that $(x_n)$ is shrinking and $\{y_k\}$ is a Dunford–Pettis subset of $[y_k]$ for each subsequence $(y_k)$ of $(x_n)$.*
(vii) *There is a seminormalized bibasic sequence $(x_n, f_n^*)$ in $K \times X^*$ so that $(x_n)$ is unconditional and $\sum f^*(x_n) f_n^*$ converges for all $f^* \in X^*$.*

*Proof.* Suppose that (i) holds, and let $x_n = T(e_n)$ for each $n$. Let $(x_n^*)$ be the sequence of coefficient functionals corresponding to $(x_n)$, and for each $n$ let $f_n^*$ be a Hahn–Banach extension of $x_n^*$ to all of $X$. Let $A$ be a positive number so that

$$A \sum |a_i| \leq || \sum a_i x_i^* ||$$

for each finite sequence $(a_1, ..., a_m)$ of real numbers. Therefore,

$$(*) \qquad A\sum |a_i| \le ||\sum a_i f_i^*|_{[x_n]}|| \le ||\sum a_i f_i^*|| \le C\sum |a_i|,$$

where $C$ is a bound for $(||f_i^*||)$. Then $(x_n^*) \sim (e_n^*) \sim (f_n^*)$, and $(x_n, f_n^*)$ satisfies the conclusion of (ii).

Now let $(x_n, f_n^*)$ satisfy the conclusions of (ii). Use Lemma 1, and let $(f_{n_i}^*)$ be a subsequence of $(f_n^*)$ so that $(f_{n_i}^*) \sim (e_i^*)$. Since the restriction of an isomorphism is an isomorphism, $(x_{n_i}^*) \sim (e_i^*)$. Therefore, $(x_{n_i}) \sim (e_i)$ by Lemma 1(ii), and $\sum x_{n_i}$ is weakly unconditionally convergent. Thus

$$\{f^* \in X^* : \sum f^*(x_{n_i}) f_{n_i}^* \text{converges}\}$$

is all of $X^*$. Consequently, by [14, Cor. 12b, p. 93], the space $[f_{n_i}^*]$ is complemented in $X^*$. The bibasic sequence $(x_{n_i}, f_{n_i}^*)$ satisfies the requirements of (iii).

The preceding paragraph shows that (iii) implies (i). Therefore, (i), (ii), and (iii) are equivalent. Certainly a seminormalized shrinking basic sequence is weakly null, and (vi) implies (iv). Also, (i) and the inequality $(*)$ above guarantee the existence of a bibasic sequence which satisfies (iv), (v), (vi), and (vii). We complete the proof by showing that (vii) implies (ii), (iv) implies (i) and (v) implies (iv).

If $(x_n, f_n^*)$ is a bibasic sequence which satisfies (vii), then, by Proposition 1.14 on p. 91 of [14], $[x_n]^\perp + [f_n^*] = X^*$. An application of Lemma 1.3 of [14] or Proposition 3 of [4] ensures that $(x_n^*) \sim (f_n^*)$, and (vii) implies (ii).

Now suppose that (iv) holds. Set $(y_n) = ((1/||x_n||)x_n)$, $(g_n^*) = (||x_n||f_n^*)$, and note that $(y_n, g_n^*)$ is bibasic and $(y_n)$ is weakly null. Following the lead of Diestel [5, p. 28] and Elton [7], we suppose that no subsequence of $(y_n)$ is equivalent to $(e_n)$, and directly apply a result of Elton [7], [5, p. 27] to select a subsequence $(z_n)$ of $(y_n)$ so that if $(w_n)$ is any subsequence of $(z_n)$, then

$$sup_n || \sum_{i=1}^{n} t_i w_i || = \infty$$

if $(t_i) \notin c_0$. If $(P_n)$ is the sequence of projections associated with $(z_n)$ and $M$ is the basis constant, then

$$||P_n^{**}(z^{**})|| = ||\sum_{i=1}^{n} z^{**}(z_i^*) z_i|| \le M||z^{**}||$$

for $z^{**} \in [z_n]^{**}$. Therefore $(z_i^*)$ is weakly null in $[z_n]^*$. Since $\{z_i\}$ is Dunford–Pettis in $[z_i]$, $z_i^*(z_i) \to 0$. This clear contradiction ensures that some subsequence of $(y_n)$ is equivalent to $(e_n)$. Thus some subsequence of $(x_n)$ is equivalent to $(e_n)$, and (iv) implies (i).

Next we suppose that $(x_n, f_n^*)$ satisfies the conditions of (v). We claim that $(x_n)$ has a weakly null subsequence. If not, there is an $x^* \in X^*$, a subsequence $(y_k)$ of $(x_n)$, and an $\epsilon > 0$ so that $x^*(y_k) > \epsilon$ for each $k$. Since $(y_k)$ is seminormalized and unconditional, $(y_k) \sim (e_k^*)$. However, this is not possible since $\{e_k^*\}$ is certainly not a Dunford–Pettis subset of $\ell^1$. Therefore, $(x_n)$ has a weakly null subsequence, and (v) implies (iv). □

We note that Problem 2.iii on p. 52 of [6] asserts that if $(x_n)$ is a normalized basic sequence, $\epsilon > 0$, and $x^*$ is an element of $X^*$ so that $x^*(x_n) > \epsilon$ for each $n$, then $(x_n)$ is equivalent to $(e_n^*)$. The sequence $(\sigma_n) = (\sum_{i=1}^{n} e_i)$ in $c_0$ clearly demonstrates that unconditionality (or some other property) has been omitted from the description of the basic sequence in the problem.

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