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UNBOUNDED CONVEX MAPPINGS OF THE BALL IN $\mathbb{C}^n$

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Abstract. In this paper, we study univalent holomorphic mappings of the unit ball in $\mathbb{C}^n$ that have the property that the image $F(B)$ contains a line $\{tu : t \in \mathbb{R}\}$ for some $u \in \mathbb{C}^n$, $u \neq 0$. We show that under certain rather reasonable conditions, up to composition with a holomorphic automorphism of the ball, the mapping F is an extension of the strip mapping in the plane to higher dimensions.

1. Introduction

Suppose F is a normalized ($F(0) = 0$, $DF(0) = I$) holomorphic univalent mapping of the ball $B^n = \{z : \|z\| < 1\}$ onto a convex domain $\Omega \subset \mathbb{C}^n$. Here $\|z\|$ is the Euclidean norm, $\|z\|^2 = \sum_{k=1}^n |z_k|^2$, and $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ is the usual inner product. We will write $z = (z_1, z')$, where $z' \in \mathbb{C}^{n-1}$ when $n \geq 2$ and use the symbol B for the ball rather than B^n except where we want to emphasize the dimension. Suppose that Ω is unbounded. Then there exists a unit vector u such that $L = L(u) = \{ru : r \geq 0\} \subset \Omega$. We are interested in particular in the case where both $L(u)$ and $L(-u)$ are subsets of Ω. That is, the entire line $\{tu : t \in \mathbb{R}\}$ is contained in Ω. We show that in this case, if $w \in F(B)$, then $w + tu \in F(B)$ for all $t \in \mathbb{R}$ and $\lim_{t\to\infty} F^{-1}(w + tu)$ and $\lim_{t\to-\infty} F^{-1}(w + tu)$ both exist (independent of w). If the two limits are different, then up to an automorphism of the ball, the mapping must have the form

$$z \mapsto \left(\frac{1}{2}\log\frac{1+z_1}{1-z_1}\right)e_1 + H\left(\frac{z'}{\sqrt{1-z_1^2}}\right), \tag{1.1}$$

where H is holomorphic on the ball B^{n-1} and $e_1 = (1, 0, \ldots, 0)$ is the standard basis vector. This is, of course, known to be true in the plane. (The mapping H does not appear in this case.)

The following notation will be used. The family of holomorphic automorphisms of B will be denoted by $\operatorname{Aut} B$. Also, the Koebe transform will be denoted by Λ_φ. That is, for locally biholomorphic mappings $F : B \to \mathbb{C}^n$ and $\varphi \in \operatorname{Aut} B$,

$$\Lambda_\varphi(F)(z) = D\varphi(0)^{-1}DF(\varphi(0))^{-1}[F(\varphi(z)) - F(\varphi(0))], \qquad z \in B.$$

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2. Characterization of the mappings

We begin with two geometrically obvious lemmas.

Lemma 2.1. *If $v \in \Omega$ and $L(u) \subset \Omega$, then $v + L \subset \Omega$.*

Proof. Let $\rho > 0$ be sufficiently small so that $B(v, \rho) \subset \Omega$, where $B(v, \rho)$ is the ball centered at v with radius ρ. Let $v' = v + Ru$ for some fixed $R > 0$, and consider the line $L_r = \{(1-t)v' + tru : t \in \mathbb{R}\}$ through v' and ru, as $r \to \infty$. Setting $t = \frac{R}{R-r}$, we see that the distance from v to L_r is at most $\frac{R\|v\|}{r-R}$. Thus, for sufficiently large r, L_r intersects $B(v, \rho)$ at a point p. Since v' lies on the segment connecting p to ru, convexity implies $v' \in F(B)$. □

Now consider the union of the set of all half lines emanating from the origin that are contained in Ω. This set is clearly convex. Finally, consider the intersection of this set with the unit sphere S (i.e. the set of all $u \in S$ such that $L(u) \subset \Omega$), which we denote by A.

Lemma 2.2. *If A is not connected, then $A = \{u, -u\}$ for some $u \in S$. That is, A consists of the endpoints of a diameter of the sphere.*

Proof. This follows from the convexity of the union of the half lines through the origin that are contained in Ω. □

Thus Lemma 2.2 says that $\{tu : t \in \mathbb{R}\} \subset F(B)$ for a fixed $u \in S$, and no other half-line through the origin is contained in $F(B)$.

Example 2.3. From a result in [2] (Corollary 1), if f is a univalent analytic mapping of the unit disk in the plane onto a convex domain, then the mapping $F : B \to \mathbb{C}^n$ given by $F(z) = (f(z_1), z'\sqrt{f'(z_1)})$ is a convex mapping. Thus, the mappings $z \mapsto (\frac{z_1}{1-z_1}, \frac{z'}{1-z_1})$ and $z \mapsto (\frac{1}{2}\log\frac{1+z_1}{1-z_1}, \frac{z'}{\sqrt{1-z_1^2}})$ are examples that yield images that are the type of domain being considered here.

Now assume the set A is not connected so that $A = \{u, -u\}$, $\|u\| = 1$. Define, for $t \in \mathbb{R}$,

$$\psi_t(z) = F^{-1}(F(z) + tu), \qquad z \in B. \tag{2.1}$$

Our characterization will be given by proving a sequence of relatively easy results.

Lemma 2.4. *The family $\Gamma = \{\psi_t : t \in \mathbb{R}\}$ is a one-parameter subgroup of $\operatorname{Aut} B$ under the operation of composition. That is, Γ is abelian and the homomorphism $t \mapsto \psi_t$ is continuous in the topology of uniform convergence on compact sets.*

Proof. The lemma follows from the formula

$$\psi_s \circ \psi_t = \psi_{s+t}$$

and the continuity of F and F^{-1}. □

Lemma 2.5. *Let K be a compact subset of B. Then $\lim_{t\to\infty}(\|\psi_t(\cdot)\|) \equiv 1$ and $\lim_{t\to-\infty}(\|\psi_t(\cdot)\|) \equiv 1$ uniformly on K.*

Proof. The image of the closed ball of radius $r < 1$ is a compact set, say K_r. Since K_r is bounded, for sufficiently large $|t|$, $F(z) + tu$ is in $\mathbb{C}^n \setminus K_r$ for all $z \in K$. This means that for every $r < 1$ and every $z \in K$, $\psi_t(z)$ cannot be in the ball of radius r for sufficiently large $|t|$. □

Observe that for $t \in \mathbb{R}$, $\psi_{nt} = \psi_t \circ \psi_{(n-1)t}$, $n = 2, 3, \dots$, and therefore the functions $\{\psi_{nt}\}$ are the iterates of ψ_t.

Theorem 2.6. *For all $t \in \mathbb{R} \setminus \{0\}$, the iterates $\{\psi_{nt}\}$ converge uniformly on compact sets to a boundary point $a_t \in S$. If $st > 0$, then $a_t = a_s$.*

Proof. Since ψ_t does not have a fixed point in B when $t \neq 0$, we may use a result of MacCluer, Theorem A, p. 99 of [1], which says the iterates of ψ_1 converge to a boundary point, say a, that is a fixed point. Now let t be an arbitrary positive number, and for each positive integer n such that $nt \geq 1$, let m be a positive integer such that $m \leq nt < m + 1$. Then $\psi_{nt}(z) = \psi_m \circ \psi_{nt-m}(z) \to a$ as $n \to \infty$. This concludes the proof for positive t. Now assume that $\psi_{-n}(z) \to b$. As in the above proof, it follows that $\psi_{ns}(z) \to b$ for all $z \in B$ when $s < 0$. □

Corollary 2.7. *We have $\lim_{t\to\infty} \psi_t \equiv a \in S$ and $\lim_{t\to-\infty} \psi_t \equiv b \in S$ uniformly on compact sets.*

Remark 2.8. We assume that $a \neq b$ in Theorem 2.6. The example $F(z) = \frac{z}{1-z_1}$, with $u = (i, 0)$, shows that a and b need not be different. However, the domain obtained in this case does not have the property that no other half line through the origin is contained in $F(B)$. It is probably true that $a \neq b$ if no other half line is contained in $F(B)$, but we do not have a proof.

Lemma 2.9. *For all $t \in \mathbb{R}$, a and b defined in Theorem 2.6 are fixed points of ψ_t.*

Proof. For $t > 0$, a is a fixed point of ψ_t and b is a fixed point of ψ_{-t}. However, $\psi_t \circ \psi_{-t} = \psi_0$, which is the identity. That is, ψ_t and ψ_{-t} are inverses so that a and b are fixed points of both ψ_t and ψ_{-t}. □

Since holomorphic automorphisms of the ball preserve affine spaces, we now know that $E = \{(1-\lambda)a + \lambda b : \lambda \in \mathbb{C}\} \cap B$ is invariant under the mappings ψ_t. We wish to show that by replacing F by $F \circ \Phi$ for an appropriately chosen $\Phi \in \operatorname{Aut} B$, renormalizing to form

$$G = \Lambda_\Phi(F),$$

renaming the mappings ψ_t, and changing the definition of u in an obvious way, we may conclude that the family $\psi_t(z) = G^{-1}(G(z) + tu)$, $t \in \mathbb{R}$, leaves the set $E_1 = \{\lambda e_1 : |\lambda| < 1\}$ invariant and fixes $\pm e_1$.

Lemma 2.10. *If $\Psi \in \operatorname{Aut} B$ leaves $\{\lambda e_1 : |\lambda| < 1\}$ invariant, then Ψ has the form*

$$\Psi(z) = \left(\gamma \frac{z_1 + \beta}{1 + \overline{\beta} z_1}, \frac{\sqrt{1 - |\beta|^2}\, U z'}{1 + \overline{\beta} z_1} \right), \qquad z \in B. \tag{2.2}$$

Here, $|\gamma| = 1$ and U is unitary on $\mathbb{C}^{n-1}$.

Proof. By hypothesis, $z' = 0$ implies $\Psi(z_1, z') = \Psi_1(z_1)e_1$ where Ψ_1 is a holomorphic automorphism of the unit disk. The lemma now follows from Theorem 2.2.5, p. 28 of [3]. □

Now let $\varphi = \varphi_{(a+b)/2}$, where $\varphi_{(a+b)/2}$ is given by (2) p. 25 of [3]. That is, with $(a + b)/2 = c$, φ is the identity if $c = 0$. Otherwise,

$$\varphi(z) = \frac{c - P_c z - s_c Q_c z}{1 - \langle z, c \rangle}, \qquad z \in B,$$

where P_c is the projection of $\mathbb{C}^n$ onto the one-dimensional space $\{\lambda c : \lambda \in \mathbb{C}\}$, $Q_c = I - P_c$ (with I the identity), and $s_c = \sqrt{1 - \|c\|^2}$.

Since $\varphi((a+b)/2) = 0$, $\varphi(E)$ is the intersection of a one-dimensional subspace of $\mathbb{C}^n$ with B. Therefore we may choose a unitary mapping U such that the mapping $\hat{\Psi} = U \circ \varphi$ maps E onto E_1. Let η and ζ satisfy $\hat{\Psi}(a) = \eta e_1$ and $\hat{\Psi}(b) = \zeta e_1$. Here, $|\eta| = |\zeta| = 1$. Let T be a holomorphic automorphism of the unit disk satisfying

$$T(\lambda) = \gamma \frac{\lambda + \beta}{1 + \overline{\beta}\lambda}, \qquad T(\eta) = 1,\ T(\zeta) = -1.$$

Finally, let Ψ be given by (2.2), and define $\Upsilon = \Psi \circ \hat{\Psi}$. Then $\Upsilon(E) = E_1$, $\Upsilon(a) = e_1$, and $\Upsilon(b) = -e_1$.

Lemma 2.11. *Define, for $z \in B$,*

$$\begin{aligned} G(z) &= \Lambda_\Upsilon(F)(z), \\ \hat{u} &= D\Upsilon(0)^{-1} DF(\Upsilon(0))^{-1} u, \\ \Phi_t(z) &= G^{-1}(G(z) + t\hat{u}). \end{aligned} \tag{2.3}$$

Then

(a) $G(B) + t\hat{u} = G(B)$ *for all* $t \in \mathbb{R}$, *and*
(b) $\Phi_t = \Upsilon^{-1} \circ \psi_t \circ \Upsilon$,

where ψ_t is given by (2.1).

Proof. The proof is straightforward, using the definition of G and G^{-1}. □

Lemma 2.12. *The automorphisms Φ_t have the form*

$$\Phi_t(z) = \left(\frac{z_1 + x}{1 + x z_1}, \frac{\sqrt{1 - x^2}\, U z'}{1 + x z_1} \right), \qquad z \in B, \tag{2.4}$$

where x is a real differentiable function of t such that $x \to \pm 1$ as $t \to \pm\infty$ and U is unitary on $\mathbb{C}^{n-1}$. Also, up to a positive constant multiple, $\hat{u} = e_1$.

Proof. From Lemma 2.11 and the properties of ψ_t, E_1 is invariant under the mapping Φ_t, and hence is given by (2.2). Since $\pm e_1$ are fixed points of Φ_t, Φ_t has the form asserted. By combining (2.3) and (2.4), we have that $x = \langle G^{-1}(t\hat{u}), e_1 \rangle$, hence is differentiable, and by Corollary 2.7, the limit assertions hold. By (2.3), we may write

$$G(\Phi_t(z)) - t\hat{u} = G(z), \qquad z \in B, \tag{2.5}$$

with $\Phi_0 = I$ (so that $x(0) = 0$). Set $z = 0$, differentiate with respect to t, and evaluate at 0. The result is $x'(0)e_1 - \hat{u} = 0$. This completes the proof. □

We now return to the equality (2.5). After possibly replacing t by σt for some positive number σ, we may take $\hat{u} = e_1$. It then follows that $G(-xe_1) = -te_1$. Therefore the analytic function $\lambda \mapsto \langle G(\lambda e_1), e_j \rangle$ is zero for real λ and $2 \le j \le n$. Hence it is zero for all λ in the unit disk. Thus we may write $G(\lambda e_1) = g(\lambda) e_1$, where g is analytic in the unit disk with $g(0) = 0$ and $g'(0) = 1$. In fact, g maps the unit disk onto a strip domain containing the real axis. In view of the fact that g has infinite singularities at ± 1, we conclude $g(\lambda) = \frac{1}{2} \log \frac{1+\lambda}{1-\lambda}$. We may now prove the following theorem.

Theorem 2.13. *The unitary map U in (2.4) is the identity and G has the form*

$$G(z)=\left(\frac{1}{2}\log\frac{1+z_1}{1-z_1}\right)e_1+H\left(\frac{z'}{\sqrt{1-z_1^2}}\right),$$

where $H : B^{n-1}\to\mathbb{C}^n$.

Proof. Using the fact that $G(\lambda e_1) = (\frac{1}{2}\log\frac{1+\lambda}{1-\lambda})e_1$, we conclude that G can be written as $G(z) = (\frac{1}{2}\log\frac{1+z_1}{1-z_1})e_1 + H_0(z_1, \frac{z'}{\sqrt{1-z_1^2}})$, where $H_0(z_1, 0) = 0$. From (2.5) with $\hat{u} = e_1$, by setting $z_1 = 0$, we see that $t = \frac{1}{2}\log\frac{1+x}{1-x}$. Writing (2.5) in terms of H_0, it follows that $H_0(z_1, \frac{z'}{\sqrt{1-z_1^2}}) = H_0(\frac{z_1+x}{1+xz_1}, \frac{Uz'}{\sqrt{1-z_1^2}})$. Setting $z_1 = 0$, we find that H_0 is independent of the first argument. In addition, since $H_0(0, z')$ is univalent in a neighborhood of $z' = 0 \in \mathbb{C}^{n-1}$, U is the identity. $\square$

3. Conclusions

We believe that the only convex maps of the ball in $\mathbb{C}^n$ that have two infinite singularities on the boundary are the mappings given by Theorem 2.13. In fact, together with Professor John Pfaltzgraff of UNC, Chapel Hill, we make the following conjecture.

Conjecture 3.1. *If $F : B \to \mathbb{C}^n$ is a normalized univalent holomorphic mapping of the ball onto a convex domain, then*:

(a) *$F(B)$ is bounded and F extends continuously to ∂B, or*
(b) *F extends continuously to ∂B except for one point that is an infinite discontinuity, or*
(c) *F is given by Theorem 2.13, and H extends continuously to ∂B^{n-1}.*

In (c) above, the mapping F extends continuously to ∂B except at the two infinite discontinuities. This is exactly the situation in the plane.

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