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QUASICONFORMAL VARIATION OF SLIT DOMAINS

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Abstract. We use quasiconformal variations to study Riemann mappings onto variable single slit domains when the slit is the tail of an appropriately smooth Jordan arc. In the real analytic case our results answer a question of Dieter Gaier and show that the function κ in Löwner's differential equation is real analytic.

Introduction

Throughout this paper $\mathbb{C}$ will be the complex plane, Ω will be a simply connected region in $\mathbb{C}$ that contains the origin, and $f\colon [0,T) \to \Omega$, with $0 < T \leq \infty$, will be a parametrization of a Jordan arc Γ in Ω.

We impose the following standing conditions on f and Γ. First, we assume that f is regular; that is, f' exists and is nonzero at each point of $[0,T)$. Second, we assume that Γ is a closed subset of Ω. This implies that $f(t)$ approaches the boundary of Ω as t approaches T and that for all t in $[0,T)$ the arc $\Gamma_t = f([t,T))$ is a closed subset of Ω and the region $\Omega_t = \Omega \setminus \Gamma_t$ is simply connected. Finally, we assume that the point 0 does not lie on Γ, so that 0 belongs to each region Ω_t, $0 \leq t < T$.

We shall use quasiconformal variations to study the dependence on t of Riemann mappings from the open unit disk Δ to Ω_t. Our method is most easily applied when f is at least of class C^2 on $[0,T)$; it is most successful when f is real analytic on $[0,T)$. Our first theorem studies the conformal radius of Ω_t at the origin as a function of t when f is real analytic.

Definition. Let $\mathcal{D}$ be a proper simply connected subregion of $\mathbb{C}$, and let c be a point in $\mathcal{D}$. The *conformal radius* of $\mathcal{D}$ at c is the positive number $|g'(0)|$, where g is any Riemann mapping of Δ onto $\mathcal{D}$ such that $g(0) = c$.

Theorem 1. *Let Ω and $f\colon [0,T) \to \Omega$ satisfy our standing conditions, and let Ω_t, $t \in [0,T)$, be as above. If f is real analytic on $[0,T)$, then the conformal radius $R(t)$ of Ω_t at the origin is a real analytic function of t on the interval $[0,T)$.*

The special case where Ω is $\mathbb{C}$ and f is the arc length parametrization of Γ solves the last of four problems posed by Dieter Gaier during the March 1996 function

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theory meeting at Oberwolfach. The first three problems inspired the paper [7], and the fourth inspired this paper.

Our proof of Theorem 1, given in §2, uses an important theorem of Rodin [12] about the variation of the Riemann mapping function when the boundary of a simply connected plane region undergoes a holomorphic motion. Rodin's theorem, in turn, is proved by combining the theory of holomorphic motions with a method of quasiconformal variation first introduced by Ahlfors in [1].

Ahlfors's method can be applied directly to both the real analytic and C^n cases. We use it to prove our second theorem, which studies the dependence on t of certain Riemann mappings of Δ onto Ω_t.

Theorem 2. *Let Ω and $f\colon [0,T) \to \Omega$ satisfy our standing conditions. For t in $[0,T)$ let Ω_t be as above, and let $z \mapsto h(z,t)$ be the Riemann mapping of Δ onto Ω_t such that $h(0,t) = 0$ and $h(1,t) = f(t)$, the tip of the slit Γ_t. If f is C^n on $[0,T)$, with $n \geq 2$, then $(z,t) \mapsto h(z,t)$ is a C^{n-1} function on $\Delta \times [0,T)$. In addition there are real valued C^{n-2} functions α and β on $[0,T)$ such that $\alpha(t) > 0$ and*

$$(1)\qquad \frac{\partial h}{\partial t}(z,t) = z\frac{\partial h}{\partial z}(z,t)\left[\alpha(t)\frac{1+z}{1-z} + i\beta(t)\right] \qquad \text{for all } z \in \Delta \text{ and } t \in [0,T).$$

If f is real analytic on $[0,T)$, then the map $(z,t) \mapsto h(z,t)$ and the functions α and β are also real analytic on their respective domains.

We treat the real analytic case of Theorem 2 in §3 and the C^n case in §4. In §5 we put equation (1) into a classical Löwner form by changing the regular parametrization f and the normalization of the Riemann mappings. If f is C^n, $n \geq 2$, we find (see §5) that the "standard parametrization" of Γ is regular and that the standard parametrization and Löwner's κ-function are both C^{n-1}. Both are real analytic if f is real analytic. Finally, in §6 we discuss extending Theorem 2 to situations where f is slightly less than C^2.

The literature contains a number of theorems about the smoothness of κ for certain analytic Jordan arcs Γ. For example Komatu's 1941 paper [9] proves that κ is continuously differentiable when Ω is Δ and Γ terminates at a boundary point of Δ. Komatu's result is reproved in Royden's Stanford University Master's thesis [13]. (We thank Brad Osgood for sending us a copy of [13].) Schiffer [14] proved that κ is analytic if Ω equals $\mathbb{C}$ and Γ has a parametrization that satisfies his famous differential equation (see Chapter 10 of [6] or Chapter 8 of [2]). More generally, Brickman, Leung, and Wilken [4] used a form of Schiffer's interior variation to show that κ is analytic if Ω equals $\mathbb{C}$ and the analytic Jordan arc Γ is also analytic at infinity. The real analytic case of Theorem 3 allows such additional possibilities as $\Gamma = \{x + i\sin x : x \geq 1\}$ and $\Omega = \mathbb{C}$ or $\Gamma = \{1 - t^{-1} + it\sin t : t \geq 1\}$ and $\Omega = \{w \in \mathbb{C} : \mathrm{Re}(w) < 1\}$.

For a treatment of less regular arcs we call attention to the paper [10] of Marshall and Rohde, where (among other things) it is proved that κ is Hölder continuous with exponent $1/2$ if $\Omega = \mathbb{C}$ and Γ is a quasiarc.

1. A lemma

Our quasiconformal variations will be supported in a small neighborhood of an arbitrary point on Γ. Their construction depends on the following elementary fact.

Lemma 1. *For any complex number t_0 and positive number r let $D(t_0, r)$ be the open disk with center t_0 and radius r. Let η be a compactly supported real valued*

C^∞ function on $D(t_0, r)$ such that $\eta(t_0) = 1$, and let M be the maximum value of $2|\partial\eta/\partial t|$. For any complex number λ with $M|\lambda| < 1$ the function

$$\psi_\lambda(t) = t + \lambda\eta(t), \qquad t \in D(t_0, r), \tag{2}$$

is a bi-Lipschitz (hence quasiconformal) C^∞ diffeomorphism of $D(t_0, r)$ onto itself, and its Beltrami coefficient μ_λ satisfies

$$\mu_\lambda = \frac{\partial\psi_\lambda/\partial\bar{t}}{\partial\psi_\lambda/\partial t} = \frac{\lambda\partial\eta/\partial\bar{t}}{1 + \lambda\partial\eta/\partial t} = \lambda\frac{\partial\eta}{\partial\bar{t}}\sum_{n=0}^{\infty}\left(-\lambda\frac{\partial\eta}{\partial t}\right)^n. \tag{3}$$

In addition ψ_λ is the identity in a neighborhood of the boundary of $D(t_0, r)$, and if t_0 and λ are real, then ψ_λ maps $D(t_0, r) \cap \mathbb{R}$ onto itself.

Proof. The function ψ_λ is a bi-Lipschitz C^∞ diffeomorphism because it differs from the identity by a C^∞ function whose Lipschitz constant $M|\lambda|$ is less than one by hypothesis. The remaining assertions are obviously true. $\square$

2. Proof of Theorem 1

Let t_0 in $[0, T)$ be given. Choose $r > 0$ so small that $t_0 + r < T$ and there is a one-to-one conformal map $\widetilde{f}$ from $D = D(t_0, r)$ into Ω such that $0 \notin \widetilde{f}(D)$, the functions $\widetilde{f}$ and f are equal in their common domain $D \cap [0, T)$, and

$$\Gamma \cap \widetilde{f}(D) = f(D \cap [0, T)).$$

Let $\partial\Omega_{t_0}$ be the boundary of Ω_{t_0}. Define $\phi\colon D(0, 1/M) \times \partial\Omega_{t_0} \to \mathbb{C}$ by

$$\phi(\lambda, w) = \begin{cases} w, & \lambda \in D(0, 1/M), \quad w \in \partial\Omega_{t_0} \setminus \widetilde{f}(D), \\ \widetilde{f}(\psi_\lambda(\widetilde{f}^{-1}(w))), & \lambda \in D(0, 1/M), \quad w \in \partial\Omega_{t_0} \cap \widetilde{f}(D), \end{cases}$$

where ψ_λ is the quasiconformal mapping defined by (2). It is easily seen that

(a) $\phi(0, w) = w$ for all w in $\partial\Omega_{t_0}$,
(b) the map $\phi(\lambda, \cdot)\colon \partial\Omega_{t_0} \to \mathbb{C}$ is injective for each λ in $D(0, 1/M)$, and
(c) the map $\phi(\cdot, w)\colon D(0, 1/M) \to \mathbb{C}$ is holomorphic for each w in $\partial\Omega_{t_0}$.

Therefore, by definition, ϕ is a holomorphic motion of $\partial\Omega_{t_0}$ over $D(0, 1/M)$.

For each λ in $D(0, 1/M)$ the set $\phi(\{\lambda\} \times \partial\Omega_{t_0})$ bounds a simply connected subregion of Ω. Following Rodin [12], we denote by Φ_λ the Riemann mapping of Δ onto that region, normalized so that $\Phi_\lambda(0) = 0$ and $\Phi'_\lambda(0) > 0$.

If $\lambda \in D(0, 1/M) \cap \mathbb{R}$ and $t_0 + \lambda \geq 0$, then the region bounded by $\phi(\{\lambda\} \times \partial\Omega_{t_0})$ is $\Omega_{t_0+\lambda}$, so the definition of the conformal radius $R(t)$ implies that

$$R(t_0 + \lambda) = \Phi'_\lambda(0) \qquad \text{if } \lambda \in D(0, 1/M) \cap \mathbb{R} \text{ and } t_0 + \lambda \geq 0. \tag{4}$$

Rodin's theorem in [12] gives a positive $\varepsilon < 1/M$ such that $(\lambda, z) \mapsto \Phi_\lambda(z)$ is a real analytic function on $D(0, \varepsilon) \times \Delta$. Therefore $\lambda \mapsto \Phi'_\lambda(0)$ is real analytic in $D(0, \varepsilon)$, and equation (4) implies that the function $R(t)$ is real analytic at t_0. $\square$

3. Proof of Theorem 2: The real analytic case

Fix (z_0, t_0) in $\Delta \times [0, T)$ and set $h_0 = h(\cdot, t_0)$. Choose $r > 0$ and a one-to-one conformal map $\widetilde{f}$ as in the proof of Theorem 1, taking r so small that $h_0(z_0)$ is not in $\widetilde{f}(D)$. (Here, as in §2, $D = D(t_0, r)$.)

Set $\varepsilon = 1/M$. For any real number λ in the interval $(-\varepsilon, \varepsilon)$ we define quasiconformal maps φ_λ and W_λ of the plane onto itself as follows.

The map φ_λ is defined by the formula

$$\varphi_\lambda(w) = \begin{cases} w & \text{if } w \in \mathbb{C} \setminus \widetilde{f}(D), \\ \widetilde{f}(\psi_\lambda(\widetilde{f}^{-1}(w))) & \text{if } w \in \widetilde{f}(D), \end{cases} \tag{5}$$

where again ψ_λ is defined by (2). Observe that $\varphi_\lambda(\Omega_{t_0}) = \Omega_{t_0+\lambda}$ if $t_0 + \lambda \geq 0$.

Let E be the (compact) support of the function η in Lemma 1, and let V be the complement of $\widetilde{f}(E)$ in $\mathbb{C}$. The open sets V and $\widetilde{f}(D)$ cover $\mathbb{C}$, and φ_λ is the identity on V.

The Beltrami coefficient ν_λ of φ_λ satisfies

$$\nu_\lambda(w) = \begin{cases} 0 & \text{if } w \in V, \\ \mu_\lambda(t)\widetilde{f}'(t)\Big/\overline{\widetilde{f}'(t)} & \text{if } w \in \widetilde{f}(D) \text{ and } t = \widetilde{f}^{-1}(w), \end{cases} \tag{6}$$

where μ_λ is given by (3), so $\lambda \mapsto \nu_\lambda$ is a real analytic map from $(-\varepsilon, \varepsilon)$ to $L^\infty(\mathbb{C})$.

Following Ahlfors [1] and Rodin [12] we define the quasiconformal map W_λ so that it fixes the points 0 and 1, maps Δ onto itself, has the symmetry property $W_\lambda(1/\bar{z}) = 1/\overline{W_\lambda(z)}$, and its Beltrami coefficient σ_λ satisfies

$$\sigma_\lambda(z) = \nu_\lambda(h_0(z))\overline{h_0'(z)}/h_0'(z), \qquad z \in \Delta. \tag{7}$$

By equation (7), the quasiconformal maps W_λ and $\varphi_\lambda \circ h_0$ of Δ into $\mathbb{C}$ have the same Beltrami coefficient. Therefore $\varphi_\lambda \circ h_0 \circ {W_\lambda}^{-1}$ is a Riemann mapping of Δ onto $\varphi_\lambda(\Omega_{t_0})$. Since this map fixes 0 and sends 1 to $\varphi_\lambda(f(t_0))$, it equals $h(\cdot, t_0 + \lambda)$ when $t_0 + \lambda \geq 0$. Therefore

$$h_0(z) = h(W_\lambda(z), t_0 + \lambda) \quad \text{if } t_0 + \lambda \geq 0,\ \lambda \in (-\varepsilon, \varepsilon), \text{ and } z \in h_0^{-1}(V). \tag{8}$$

Closely following Rodin's reasoning on pages 193 and 194 of [12], we shall use (8) to prove that the function $h(z, t)$ is real analytic at (z_0, t_0). First we observe that $\lambda \mapsto \sigma_\lambda$ is a real analytic map from $(-\varepsilon, \varepsilon)$ to $L^\infty(\Delta)$, by (7). Therefore, by the corollary to Theorem 11 of Ahlfors-Bers [3], $\lambda \mapsto W_\lambda$ is a real analytic map from $(-\varepsilon, \varepsilon)$ to the Banach space $C(\bar{\Delta})$ of continuous functions on the closed unit disk.

Equations (6) and (7) also imply that W_λ is holomorphic in the set $U = h_0^{-1}(V)$, which properly contains the set $\Delta \setminus h_0^{-1}(\widetilde{f}(D))$. By our choice of D both 0 and z_0 belong to U.

The map $(\Phi, z) \mapsto \Phi(z)$ from $H^\infty(U) \times U$ to $\mathbb{C}$ is locally bounded and is holomorphic in the variables Φ and z separately, so it is holomorphic (see Chapter 14 of [5]). Since W_λ is holomorphic and one-to-one in U, the map

$$G(z, \lambda) = (W_\lambda(z), \lambda)$$

of $U \times (-\varepsilon, \varepsilon)$ into $\mathbb{C} \times (-\varepsilon, \varepsilon)$ is real analytic and one-to-one. The Jacobian of G is never zero, so the inverse map $(\zeta, \lambda) \mapsto ({W_\lambda}^{-1}(\zeta), \lambda)$ is real analytic in the open set $G(U \times (-\varepsilon, \varepsilon))$. Therefore the map $(\zeta, \lambda) \mapsto h_0({W_\lambda}^{-1}(\zeta))$ is real analytic in the same set. By (8), $h(\zeta, t_0 + \lambda) = h_0({W_\lambda}^{-1}(\zeta))$ if $(\zeta, \lambda) \in G(U \times (-\varepsilon, \varepsilon))$ and $t_0 + \lambda \geq 0$, so h is real analytic at (z_0, t_0), as required.

To prove equation (1) we start with the identity $\varphi_\lambda(h_0(z)) = h(W_\lambda(z), t_0 + \lambda)$, valid for z in Δ and $0 \leq \lambda < \varepsilon$. Differentiating with respect to λ and setting $\lambda = 0$ we obtain

$$\frac{\partial h}{\partial t}(z, t_0) = \dot{\varphi}(h_0(z)) - h_0'(z)\dot{W}(z), \qquad z \in \Delta,$$

where

$$(9) \qquad \dot{W}(z) = \lim_{\lambda \to 0} \frac{W_\lambda(z) - z}{\lambda}, \ z \in \mathbb{C}, \quad \text{and} \quad \dot{\varphi}(w) = \lim_{\lambda \to 0} \frac{\varphi_\lambda(w) - w}{\lambda}, \ w \in \mathbb{C}.$$

Since

$$(10) \qquad \frac{\partial^2 h}{\partial \bar{z} \partial t}(z,t) = \frac{\partial^2 h}{\partial t \partial \bar{z}}(z,t) = 0, \quad (z,t) \in \Delta \times [0,T),$$

the function $z \mapsto \frac{\partial h}{\partial t}(z,t_0)$ is holomorphic in Δ, and $\dot{W}$ is holomorphic in the open set U, where $\dot{\varphi} \circ h_0$ vanishes identically. Both $\dot{W}$ and $\frac{\partial h}{\partial t}(\cdot, t_0)$ vanish at $z = 0$, so the function

$$(11) \qquad H(z) = \frac{\partial h}{\partial t}(z,t_0)/zh_0'(z) = \frac{\dot{\varphi}(h_0(z))}{zh_0'(z)} - \frac{\dot{W}(z)}{z}, \qquad z \in \Delta,$$

is holomorphic in Δ. We examine its boundary behavior.

Since $|W_\lambda(z)| = 1$ when $|z| = 1$, equation (9) implies that the real part of $\dot{W}(z)/z$ vanishes identically on the boundary of Δ. Therefore the real part of $H(z)$ vanishes at all boundary points of Δ that do not belong to the closure of $h_0^{-1}(\widetilde{f}(D))$. Since the radius of D can be arbitrarily small and H does not depend on the choice of D, the real part of H vanishes at all boundary points of Δ except $z = 1$. We can therefore extend H by Schwarz reflection to a function $\widetilde{H}$ that is holomorphic at all points of the Riemann sphere $\widehat{\mathbb{C}}$ except for an isolated singularity at $z = 1$. By (11), that singularity is a simple pole. Indeed $\dot{W}(1) = 0$, h_0' has a simple zero at $z = 1$, and $\dot{\varphi}(h_0(1)) = \dot{\varphi}(f(t_0)) = f'(t_0) \neq 0$.

We conclude that $\widetilde{H}$ is a Möbius transformation that maps 1 to ∞ and the unit circle to the extended imaginary axis. Every such transformation is the composition of the map $z \mapsto \frac{1+z}{1-z}$ with a map $w \mapsto \alpha w + i\beta$, where α and β are real and $\alpha \neq 0$. Therefore, by (11),

$$\frac{\partial h}{\partial t}(z,t_0) = zh_0'(z)\left[\alpha \frac{1+z}{1-z} + i\beta\right], \qquad z \in \Delta,$$

for some real constants $\alpha \neq 0$ and β. Since t_0 in $[0,T)$ was chosen arbitrarily, equation (1) holds in $\Delta \times [0,T)$, with $\alpha(t)$ and $\beta(t)$ real and $\alpha(t) \neq 0$ for all t in $[0,T)$.

It remains to prove that α and β are real analytic functions of t and that α is positive. Differentiating (1) with respect to z and setting $z = 0$, we obtain

$$(12) \qquad \frac{\partial^2 h}{\partial t \partial z}(0,t) = \frac{\partial^2 h}{\partial z \partial t}(0,t) = \frac{\partial h}{\partial z}(0,t)[\alpha(t) + i\beta(t)], \quad t \in [0,T).$$

Therefore α and β are real analytic functions, as they are the real and imaginary parts of the real analytic function

$$t \mapsto \frac{\partial^2 h}{\partial t \partial z}(0,t) \Big/ \frac{\partial h}{\partial z}(0,t)\ , \quad t \in [0,T).$$

The positivity of α also follows since, by (12), α is the derivative of the increasing function $t \mapsto \log\left|\frac{\partial h}{\partial z}(0,t)\right|$. Thus α can never be negative, and we already know that it is never zero. ☐

Remark. Since $\left|\frac{\partial h}{\partial z}(0,t)\right|$ is the conformal radius $R(t)$ of Ω_t at the origin, $\alpha(t)$ is the derivative of $\log R(t)$. Theorem 2 in the analytic case is a by-product of our efforts to understand the derivative of the function $R(t)$.

4. Proof of Theorem 2: The C^n case

We repeat the proof in §3, with minor technical changes. Fix (z_0, t_0) in $\Delta \times [0, T)$ and set $h_0 = h(\cdot, t_0)$. Since $f'(t_0) \neq 0$, there is a C^n diffeomorphism $\widetilde{f}$ from an open disk $D = D(t_0, r)$ into Ω such that $\widetilde{f} = f$ in $D \cap [0, T)$ and $\Gamma \cap \widetilde{f}(D) = f(D \cap [0, T))$. We take r so small that $t_0 + r < T$, neither 0 nor $h_0(z_0)$ belongs to $\widetilde{f}(D)$, and $\widetilde{f}$ is quasiconformal in D. We denote the Beltrami coefficient of $\widetilde{f}$ in D by $\widetilde{\mu}$.

Choose $\varepsilon > 0$ so small that the map ψ_λ defined by (2) is a quasiconformal map of D onto itself for any real number λ in the interval $(-\varepsilon, \varepsilon)$. For such λ we define the quasiconformal map φ_λ of the plane onto itself by formula (5). As before, φ_λ is the identity on $V = \mathbb{C} \setminus \widetilde{f}(E)$, where E is the support of η. Because $\widetilde{f}$ is generally not conformal, the Beltrami coefficient ν_λ of φ_λ will not in general satisfy equation (6). Instead, differentiation of the identity $\widetilde{f} \circ \psi_\lambda = \varphi_\lambda \circ \widetilde{f}$ in D leads to the formula

$$\nu_\lambda(w) = \begin{cases} 0 & \text{if } w \in V, \\ \dfrac{\vartheta_\lambda(t)\psi_\lambda^*(\widetilde{\mu})(t) + \rho_\lambda(t)}{\overline{\rho_\lambda}(t)\psi_\lambda^*(\widetilde{\mu})(t) + \overline{\vartheta_\lambda}(t)} & \text{if } w \in \widetilde{f}(D) \text{ and } t = \widetilde{f}^{-1}(w), \end{cases} \tag{13}$$

where $\vartheta_\lambda = \frac{\partial \widetilde{f}}{\partial t} - \overline{\mu_\lambda}\frac{\partial \widetilde{f}}{\partial \bar{t}}$, $\rho_\lambda = \mu_\lambda \frac{\partial \widetilde{f}}{\partial t} - \frac{\partial \widetilde{f}}{\partial \bar{t}}$, and $\psi_\lambda^*(\widetilde{\mu}) = (\widetilde{\mu} \circ \psi_\lambda)\, \left(\overline{\partial \psi_\lambda / \partial t}\right) \Big/ (\partial \psi_\lambda / \partial t)$.

Formula (13) implies that the map $\lambda \mapsto \nu_\lambda$ from $(-\varepsilon, \varepsilon)$ to $L^\infty(\mathbb{C})$ is of class C^{n-1}. Indeed, ϑ_λ, ρ_λ, and $\partial \psi_\lambda / \partial t$ depend analytically on λ, by (2) and (3), and it is easy to see that the map $\lambda \mapsto \widetilde{\mu} \circ \psi_\lambda$ from $(-\varepsilon, \varepsilon)$ to $L^\infty(D)$ is of class C^{n-1}. (Simply prove by induction on k that the map $\lambda \mapsto F \circ \psi_\lambda$ from $(-\varepsilon, \varepsilon)$ to $L^\infty(D)$ is of class C^k if $F \colon D \to \mathbb{C}$ is a bounded function of class C^k, $k \geq 1$.)

As in §3, define the quasiconformal map W_λ of $\mathbb{C}$ onto itself so that it fixes 0 and 1, maps Δ onto itself, has the symmetry property $W_\lambda(1/\bar{z}) = 1/\overline{W_\lambda(z)}$, and its Beltrami coefficient σ_λ satisfies equation (7) (with ν_λ given by (13), not by (6)).

Equation (7) again implies that $\varphi_\lambda \circ h_0 \circ {W_\lambda}^{-1}$ equals $h(\cdot, t_0 + \lambda)$ when $t_0 + \lambda \geq 0$, so equation (8) also holds. Since $\lambda \mapsto \sigma_\lambda$ is a C^{n-1} map from $(-\varepsilon, \varepsilon)$ to $L^\infty(\Delta)$, by (7), we can use the argument following equation (8) (changing the words "real analytic" to "C^{n-1}" whenever they occur) to show that $(z, t) \mapsto h(z, t)$ is a C^{n-1} map.

The derivation of equation (1) requires a few more adjustments. We need the following simple lemma, which can be proved by representing $F(z, t)$ as a Cauchy integral and differentiating under the integral sign (see page 96 of [2]).

Lemma 2. *Let I be an interval in $\mathbb{R}$, and let $F \colon \Delta \times I \to \mathbb{C}$ be a C^k map, $k \geq 1$. If the function $z \mapsto F(z, t)$ is holomorphic in Δ for all t in I, then*

(a) *the map $z \mapsto \dfrac{\partial F}{\partial t}(z, t)$ is holomorphic in Δ for all t in I and*

(b) *the map $\dfrac{\partial F}{\partial z} \colon \Delta \times I \to \mathbb{C}$ is C^k and $\dfrac{\partial^2 F}{\partial t \partial z} = \dfrac{\partial^2 F}{\partial z \partial t}$ in $\Delta \times I$.*

Our first use of Lemma 2 is to guarantee that the function $z \mapsto \frac{\partial h}{\partial t}(z, t_0)$ is holomorphic in Δ even when $n = 2$ and the calculation in formula (10) is questionable.

Now consider the holomorphic function H defined by (11). Since h_0 has no analytic continuation to a neighborhood of $z = 1$ in the non-analytic case, we need a new argument to show that the analytic continuation $\widetilde{H}$ of H has a simple pole at $z = 1$. Fortunately, since f is a regular C^n parametrization with $n \geq 2$, the slit Γ_{t_0} has a Dini-smooth corner of opening 2π at $f(t_0)$. A theorem of Warschawski (see Theorem 3.9 in [11]) therefore guarantees that $h_0'(1)/(z-1)$ is continuous and bounded away from zero and ∞ in some neighborhood of 1 in $\bar{\Delta}$, so we can conclude from (11) as before that $\widetilde{H}$ has a simple pole at $z=1$.

The derivation of (1) from (11) now proceeds exactly as in §3, but we need Lemma 2 to study the smoothness of the functions α and β. Since $(z,t)\mapsto h(z,t)$ is a C^{n-1} function, Lemma 2 implies that the function

$$t \mapsto \frac{\partial^2 h}{\partial t \partial z}(0,t) \Big/ \frac{\partial h}{\partial z}(0,t)\ , \quad t\in[0,T),$$

is of class C^{n-2}. Since α and β are the real and imaginary parts of that function, by (12), they are both of class C^{n-2}.

The proof that α is positive is unchanged. $\square$

5. Löwner's equation

A change in the parametrization of Γ and the normalization of the Riemann mappings reduces equation (1) to a traditional Löwner form (see for example Exercise 8 in Chapter 3 of [6]).

Theorem 3. *Let Ω and $f\colon[0,T)\to\Omega$ satisfy our standing conditions, and let Ω_t, $t\in[0,T)$, be as above. If f is C^n on $[0,T)$, with $n\geq 2$, let $a(t)$ be the C^{n-1} function on $[0,T)$ such that $a(0)=0$ and $a'(t)$ is the function $\alpha(t)$ in equation (1). Let $[0,\widehat{T})$ be the image of $[0,T)$ under a. For each τ in $[0,\widehat{T})$ let $z\mapsto g(z,\tau)$ be the Riemann mapping of Δ onto $\Omega_{a^{-1}(\tau)}$ such that $g(0,\tau)=0$ and $\frac{\partial g}{\partial z}(0,\tau)>0$. Then $(z,\tau)\mapsto g(z,\tau)$ is a C^{n-1} function on $\Delta\times[0,\widehat{T})$. In addition there is a C^{n-1} function κ on $[0,\widehat{T})$ such that $|\kappa(\tau)|=1$, $g(1/\kappa(\tau),\tau)=f(a^{-1}(\tau))$, and*

$$\frac{\partial g}{\partial \tau}(z,\tau)=z\frac{\partial g}{\partial z}(z,\tau)\frac{1+z\kappa(\tau)}{1-z\kappa(\tau)} \qquad \text{for all } z\in\Delta \text{ and } \tau\in[0,\widehat{T}). \tag{14}$$

If f is real analytic on $[0,T)$, then the map $(z,\tau)\mapsto g(z,\tau)$ and the functions a and κ are also real analytic on their respective domains.

Proof. By Theorem 2, the hypothesis that f is C^n implies that a is C^{n-1}, as its derivative α is C^{n-2}. The function $a^{-1}\colon[0,\widehat{T})\to[0,T)$ is also C^{n-1} because α is positive on $[0,T)$. Set $\widehat{h}(z,\tau)=h(z,a^{-1}(\tau))$, $(z,\tau)\in\Delta\times[0,\widehat{T})$. Equation (1) and a chain rule calculation yield the equation

$$\frac{\partial \widehat{h}}{\partial \tau}(z,\tau)=z\frac{\partial \widehat{h}}{\partial z}(z,\tau)\left[\frac{1+z}{1-z}+i\widehat{\beta}(\tau)\right], \qquad (z,\tau)\in\Delta\times[0,\widehat{T}), \tag{15}$$

where $\widehat{\beta}$ is the function on $[0,\widehat{T})$ such that $\widehat{\beta}(a(t))a'(t)=\beta(t)$ for all t in $[0,T)$.

Now $g(\cdot,0)$ and $\widehat{h}(\cdot,0)$ fix the origin and are Riemann mappings of Δ onto the same region, so there is a real number b_0 such that $g(z,0)=\widehat{h}(e^{-ib_0}z,0)$ for all z in Δ. Let $b(\tau)$ be the C^{n-1} function on $[0,\widehat{T})$ such that $b(0)=b_0$ and $b'=\widehat{\beta}$.

Define the C^{n-1} function $\widehat{g}\colon\Delta\times[0,\widehat{T})\to\mathbb{C}$ by

$$\widehat{g}(z,\tau)=\widehat{h}(e^{-ib(\tau)}z,\tau), \qquad (z,\tau)\in\Delta\times[0,\widehat{T}).$$

Equation (15) and another chain rule calculation yield

$$\frac{\partial \widehat{g}}{\partial \tau}(z,\tau)=z\frac{\partial \widehat{g}}{\partial z}(z,\tau)\frac{1+e^{-ib(\tau)}z}{1-e^{-ib(\tau)}z} \qquad \text{for all } (z,\tau)\in\Delta\times[0,\widehat{T}). \tag{16}$$

Differentiating (16) with respect to z, setting $z = 0$, and using Lemma 2 if necessary, we obtain

$$\frac{\partial^2 \widehat{g}}{\partial\tau\partial z}(0,\tau) = \frac{\partial^2 \widehat{g}}{\partial z\partial\tau}(0,\tau) = \frac{\partial\widehat{g}}{\partial z}(0,\tau), \quad \tau \in [0,\widehat{T}),$$

so

$$\frac{\partial\widehat{g}}{\partial z}(0,\tau) = e^{\tau}\frac{\partial\widehat{g}}{\partial z}(0,0), \qquad \tau \in [0,\widehat{T}). \tag{17}$$

By definition $\widehat{g}(z,0) = \widehat{h}(e^{-ib_0}z,0) = g(z,0)$ for all z in Δ, so (17) implies that $\frac{\partial\widehat{g}}{\partial z}(0,\tau)$ is a positive real number for all τ in $[0,\widehat{T})$. Therefore $\widehat{g}(\cdot,\tau)$ is the normalized Riemann mapping $g(\cdot,\tau)$ for all τ in $[0,\widehat{T})$, and $g\colon \Delta\times[0,\widehat{T})\to\mathbb{C}$ is the function $\widehat{g}$.

To obtain equation (14) from (16) set $\kappa(\tau) = e^{-ib(\tau)}$, $0 \le \tau < \widehat{T}$. Clearly κ is C^{n-1} on $[0,\widehat{T})$ and its values lie on the unit circle. Further, if $\tau \in [0,\widehat{T})$, then $g(1/\kappa(\tau),\tau) = \widehat{g}(e^{ib(\tau)},\tau) = \widehat{h}(1,\tau) = h(1,a^{-1}(\tau)) = f(a^{-1}(\tau))$.

Finally, if f is real analytic, Theorem 2 implies that the functions a, a^{-1}, $\widehat{\beta}$, b, $\widehat{g}$, g, and κ are real analytic on their respective domains. □

Remarks. By setting $\tau = a(t)$ in (17) we see that the conformal radius $R(t)$ of Ω_t at the origin satisfies $R(t) = e^{a(t)}R(0)$ for t in $[0,T)$, so

$$\widehat{T} = \lim_{t\to T} a(t) = \lim_{t\to T} \log(R(t)/R(0)).$$

Therefore $\widehat{T} = \infty$ if $\Omega = \mathbb{C}$. If $\Omega \neq \mathbb{C}$, then $\widehat{T} = \log(R/R(0))$, where R is the conformal radius of Ω at the origin.

Equation (15) is the form taken by equation (1) when we use the regular parametrization $\widehat{f} = f\circ a^{-1}$ of Γ. Following Duren's terminology in §3.3 of [6], we call $\widehat{f}$ the standard parametrization of Γ. The proof of Theorem 3 shows that $\widehat{f}$ is C^{n-1} if f is C^n and real analytic if f is real analytic.

6. The non-C^2 case

Until now, in order to use the corollary to Theorem 11 in Ahlfors-Bers [3], we have required the regular parametrization f to be at least C^2. Now we require a bit less. We assume that f is C^1 on $[0,T)$ and has the following property: there exist a locally bounded function F on $[0,T)$ and a set E of full measure in $[0,T)$ such that at each point of E the function F is continuous, f'' exists, and $f'' = F$.

Under that assumption Theorem 10 of [3] allows us to conclude that the map $(z,t)\mapsto h(z,t)$ defined in Theorem 2 is C^1 and that equation (1) holds with continuous real valued functions α and β with $\alpha > 0$. Hence the standard parametrization of Γ is regular and C^1, and the function κ in Löwner's equation (14) is C^1.

The proof, which we shall only sketch, follows the lines of §§3 and 4. Fix (z_0,t_0) in $\Delta\times[0,T)$, set $h_0 = h(\cdot,t_0)$, and define

$$\widetilde{f}(t) = f(\mathrm{Re}(t)) + if'(t_0)\mathrm{Im}(t), \qquad t\in D = D(t_0,r).$$

Take r so small that $t_0 + r < T$, $\widetilde{f}$ is a C^1 quasiconformal diffeomorphism of D into Ω, $\Gamma\cap\widetilde{f}(D) = f(D\cap[0,T))$, and neither 0 nor $h_0(z_0)$ belongs to $\widetilde{f}(D)$. The

Beltrami coefficient $\widetilde{\mu}$ of $\widetilde{f}$ in D satisfies

$$\widetilde{\mu}(t) = \frac{f'(\mathrm{Re}(t)) - f'(t_0)}{f'(\mathrm{Re}(t)) + f'(t_0)}, \qquad t \in D. \tag{18}$$

For real numbers λ sufficiently close to zero, define the quasiconformal maps φ_λ and W_λ of $\mathbb{C}$ onto itself as in §§3 and 4. There we saw that h is a C^1 function on $\Delta \times [0, T)$ if the map $\lambda \mapsto W_\lambda$ from $(-\varepsilon, \varepsilon)$ to $C(\bar{\Delta})$ is of class C^1. Theorem 10 of [3] implies that $\lambda \mapsto W_\lambda$ has the desired property if the Beltrami coefficient σ_λ of W_λ depends differentiably on λ in a rather weak sense. Together with equations (7), (13), and (18), our assumptions on f are precisely what is required to ensure that σ_λ has the correct behavior and $h(z, t)$ is of class C^1.

To derive equation (1) we must again consider the holomorphic function H defined by (11) and the analytic continuation $\widetilde{H}$ of H to $\widehat{\mathbb{C}} \setminus \{1\}$. To see that the singularity at 1 is at worst a simple pole we observe first that $|H(z)| \leq C/(1 - |z|)$ for all z in Δ since, by the Koebe distortion theorem, $|h_0'(z)| \geq \delta(1 - |z|)$ in Δ, for some $\delta > 0$. Therefore the entire function $G(\zeta) = \widetilde{H}((\zeta - 1)/(\zeta + 1))$ satisfies

$$2|\mathrm{Re}(\zeta)| \times |G(\zeta)| \leq C(|\zeta| + 1)^2, \tag{19}$$

first for ζ in the right half plane, then for all ζ by symmetry, since G maps the imaginary axis into itself.

Now suppose that $1 \leq 2|\zeta| \leq R \leq 3|\zeta|$ and that $G(Re^{i\theta}) \neq 0$ for all real θ. Al Baernstein (e-mail communication) pointed out to us that (19) implies estimates of the following type:

$$\begin{aligned} \log|G(\zeta)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \log|G(Re^{i\theta})| \frac{R^2 - |\zeta|^2}{|Re^{i\theta} - \zeta|^2} \, d\theta \\ &\leq \log C + \log(2R) + \frac{R + |\zeta|}{R - |\zeta|} \frac{1}{2\pi} \int_0^{2\pi} (-\log|\cos\theta|) \, d\theta \leq \log 48C|\zeta|. \end{aligned}$$

Therefore $|G(\zeta)| \leq 48C|\zeta|$ if $2|\zeta| \geq 1$, so G is a polynomial of degree at most one, and $\widetilde{H}$ has at worst a simple pole at $z = 1$.

To conclude that (1) holds with $\alpha \neq 0$ we must show that the singularity of $\widetilde{H}$ at 1 is not removable. For that purpose we observe that since $f'(t_0) \neq 0$ there are points z_0 in Δ such that $h_0(1) = f(t_0)$ is the unique boundary point of Ω_{t_0} closest to $h_0(z_0)$. Therefore, by Exercise 2 in §4.3 of [11], the angular derivative $h_0'(1)$ exists and is finite. Since h_0 is not angle preserving at 1, we must have $h_0'(1) = 0$. Therefore, by (11), $H(z) \to \infty$ as $z \to 1$ along the positive real axis, and $\widetilde{H}$ has a (simple) pole at 1. The proof that α and β are continuous and $\alpha > 0$ now proceeds exactly as in §4. The statements about Löwner's equation follow as in §5.

Examples. The stated condition on f holds trivially if f is C^1 and piecewise C^2 on $[0, T)$. It also holds if $f'(t) = t + i\,\omega(t)$, t in $[0, \infty)$, where ω is real valued, continuous, and nonconstant, with $\omega(0) = 1$ and $\omega' = 0$ almost everywhere.

For an instructive example that is C^1 and piecewise analytic, consider the Jordan arc Γ in $\Omega = \mathbb{C}$ formed by the union of the real interval $[1, \infty)$ and the semicircle $\{1 - i - ie^{-i\theta} : 0 \leq \theta \leq \pi\}$ in the w-plane. By our general result, (1) holds with functions α and β that are continuous even at the parameter value corresponding to the point 1, where Γ is not C^2. In this example that continuity can be confirmed directly by computing the limits of α and β as the tip of Γ_t approaches the point 1. This is easy unless Γ_t contains an arc of the semicircle.

If Γ_t contains such an arc, the Möbius transformation $\zeta = 2\pi/(w-1)$ will carry Γ_t to the union of the two horizontal slits $[0,\infty)$ and $\{\xi + i\pi : \xi \leq \xi_t\}$, joined at ∞ in the extended ζ-plane. Riemann mappings of the upper half plane onto the complement of these slits are well known (see for example page 153 of [8]). They can be used, with suitable normalizations, to confirm the predicted continuity.

We studied this example to test the sharpness of Theorem 2. Only after our computations showed α and β to be continuous did we set out to derive the continuity from Theorem 10 of [3], obtaining the results reported in this section.

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