

## AREAS OF TWO-DIMENSIONAL MODULI SPACES

TOSHIHIRO NAKANISHI AND MARJATTA NÄÄTÄNEN

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ABSTRACT. Wolpert's formula expresses the Weil-Petersson 2-form in terms of the Fenchel-Nielsen coordinates in case of a closed or punctured surface. The area-form in Fenchel-Nielsen coordinates is invariant under the mapping class group on each 2-dimensional Teichmüller space of a surface with singularities, hence areas with respect to it can be calculated for 2-dimensional moduli spaces in cases when the Teichmüller space admits global Fenchel-Nielsen coordinates: The area of the moduli space for the signature  $(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$  is  $2(\pi^2 - \theta_1^2 - \theta_2^2 - \theta_3^2 - \theta_4^2)$ , the definition of signature is generalized to include punctures, cone points and geodesic boundary curves. In case the surface is represented by a Fuchsian group, the area is the classical Weil-Petersson area.

### 1. INTRODUCTION

By a hyperbolic surface  $R$  we mean a 2-dimensional cone manifold, that is, a surface equipped with a metric of constant curvature  $-1$  which admits conical singular points (or simply *cone-points*) and all of whose boundary components are totally geodesic closed curves. In polar co-ordinates around a cone-point, the metric has the form  $dr^2 + \sinh^2 r d\varphi^2$  where  $r$  is the distance from the cone-point,  $\varphi$  is the angular measure around the cone-point, which is measured modulo  $\theta$  for some  $\theta \in (0, 2\pi)$ . This  $\theta$  is called the *angle* around the cone-point. We assume that  $R$  has a finite area. Then  $R$  has a finite number of *singularities*, which are, in our notation, cone-points, punctures and boundary curves. We assign the number  $\theta$  to a cone-point if the angle around it is  $\theta$ , the number 0 to a puncture and the number  $i\theta$  to a boundary curve if its length is  $\theta$ . If  $R$  has genus  $g$  and the numbers  $\theta_1, \dots, \theta_n$  assigned to its singularities in the way specified above, we call the tuple  $(g; \theta_1, \dots, \theta_n)$  the *signature* of  $R$ . Although this definition of signature is not the usual one, it is convenient since it allows us to treat hyperbolic surfaces in a unified manner.

If the Teichmüller space of a hyperbolic orbifold  $R$  is two-dimensional, the signature is either of type  $(1; \theta)$  or of type  $(0; \theta_1, \theta_2, \theta_3, \theta_4)$  [14, 34.3]. (Remember that the boundary curves have specified lengths.) If the Teichmüller space is parameterized by a Fenchel-Nielsen coordinate system  $(l, s)$ ,  $l > 0$ ,  $-\infty < s < \infty$ , then the 2-form

$$(1.1) \quad \omega_{WP} = dl \wedge ds$$

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is invariant under the mapping class group, and hence it can be considered in the moduli space. The form (1.1) is Wolpert's formula for the Weil-Petersson 2-form [12, Theorem 1.3].

In this paper, we compute the areas of 2-dimensional moduli spaces of hyperbolic cone-surfaces with respect to the form (1.1) when corresponding Teichmüller spaces admit global Fenchel-Nielsen coordinates. The invariance of (1.1) under the mapping class group is proved in Section 3. For the signature  $(1; 0)$ , Wolpert [10] obtained the (Weil-Petersson) area  $\pi^2/6$ . In [8] we found that the area for the signature  $(1; 2\theta)$ ,  $\theta \in [0, \pi) \cup i\mathbf{R}_+$  is  $\frac{1}{6}(\pi^2 - \theta^2)$ . The result is stated only for  $\theta \in \{\pi/p : p = 2, 3, \dots, \infty\} \cup i\mathbf{R}_+$ , but the argument applies for other values of  $\theta \in [0, \pi) \cup i\mathbf{R}_+$  as well. See also Section 5. In order to calculate the area, we used the classical result in [3] that the Teichmüller space for the signature  $(1; 2\theta)$  is represented by the sublocus of the equation

$$(1.2) \quad x^2 + y^2 + z^2 - 2xyz - \sin^2 \frac{\theta}{2} = 0$$

satisfying  $x, y, z > 1$ . In this paper we calculate the area of the moduli space for the signature  $(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$ . We derive in Lemma 2.1 an algebraic equation which represents the Teichmüller space  $\mathbf{T}(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$  for the same signature. After considering the action of the mapping class group in Section 3, in Section 4 we prove:

**Theorem 1.1.** *If  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$  are numbers in  $[0, \pi/2] \cup i\mathbf{R}_+$ , then the area of the moduli space for the signature  $(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$  is*

$$(1.3) \quad 2(\pi^2 - \theta_1^2 - \theta_2^2 - \theta_3^2 - \theta_4^2).$$

Here the moduli space means  $\mathbf{T}(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)/MC$ , where  $MC$  denotes the mapping class group of isotopy classes of homeomorphisms (including orientation-reversing ones) fixing each singularity pointwise. Thus, if two of  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$  are identical,  $MC$  is not the full mapping class group. We remark also that  $(0; \pi, \pi, \pi, \pi)$  is treated here as a degenerate case: the surface of this signature is realized as a parabolic orbifold, rather than a hyperbolic orbifold [4, p. 142]. This case has Weil-Petersson area 0, which will be used in Section 5 to evaluate a constant.

In this paper we treat the case where  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$  are numbers in  $[0, \pi/2]$ . Other cases where some  $\theta_i \in i\mathbf{R}_+$  follow the same way by slight modifications of matrices and figures in Section 2. Since we make use of traces of matrices in  $SL_2(\mathbf{R})$  (see Lemma 2.1), noticing that  $\cos i\theta = \cosh \theta$ , we need no changes in the computation achieved in Section 4.

In [5], [9] and [13], Weil-Petersson volumes of higher-dimensional moduli spaces of punctured surfaces are obtained.

## 2. A PARAMETER SPACE FOR TEICHMÜLLER SPACE

Let  $L_0$  and  $L_1$  denote (resp.) the hyperbolic lines formed by the circle  $|z| = 1$  and the circle  $|z| = e^{l/2}$ ,  $l \geq 0$ . Choose a point  $P = -u + iv \in L_0$ ,  $0 \leq u \leq 1$ ,  $v = \sqrt{1 - u^2}$ . Let  $L_2$  denote a hyperbolic line which meets  $L_0$  at  $P$  and intersects  $L_1$ , also. We assume that  $L_2$  does not meet  $L_1$  in the right half plane: If the angle from  $L_0$  to  $L_2$  is  $\theta_1$  and the angle from  $L_2$  to  $L_1$  is  $\theta_2$  (see Figure 1a), this assumption

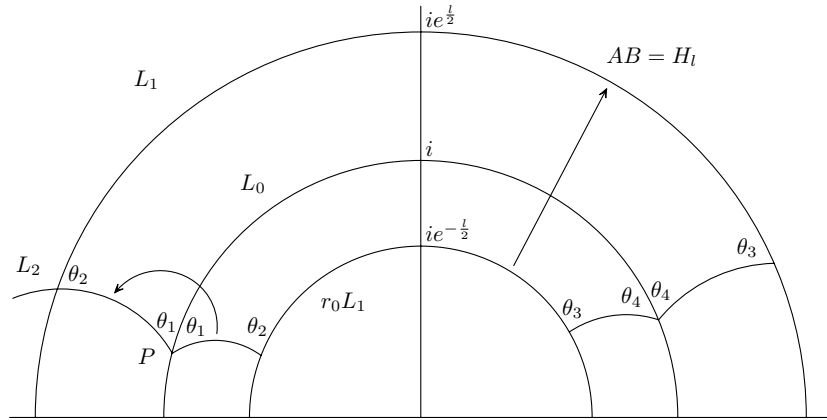


FIGURE 1a.

is equivalent to

$$(2.1) \quad \cos \theta_1 + \cosh \frac{l}{2} \cos \theta_2 \geq 0.$$

Let  $A = r_2 r_0$  and  $H_l = r_1 r_0$ , where  $r_0, r_1$  and  $r_2$  denote (resp.) reflections in  $L_0, L_1$  and  $L_2$ . They have the matrix-representations

$$(2.2) \quad \begin{aligned} A &= \begin{pmatrix} \cos \theta_1 - uv^{-1} \sin \theta_1 & v^{-1} \sin \theta_1 \\ -v^{-1} \sin \theta_1 & \cos \theta_1 + uv^{-1} \sin \theta_1 \end{pmatrix}, \\ H_l &= \begin{pmatrix} -e^{l/2} & 0 \\ 0 & -e^{-l/2} \end{pmatrix}. \end{aligned}$$

The matrix for  $B = A^{-1} H_l$  is

$$(2.3) \quad B = \begin{pmatrix} -e^{l/2}(\cos \theta_1 + uv^{-1} \sin \theta_1) & e^{-l/2} v^{-1} \sin \theta_1 \\ -e^{l/2} v^{-1} \sin \theta_1 & -e^{-l/2}(\cos \theta_1 - uv^{-1} \sin \theta_1) \end{pmatrix}.$$

Since the (1,2)-entry is non-negative and  $B = r_0 r_2 r_0 \cdot r_0 r_1 r_0$  is the positive rotation of angle  $2\theta_2$  about the intersection point of  $r_0 L_1$  and  $r_0 L_2$ , we know  $tr B \geq 0$ . Thus  $tr B = 2 \cos \theta_2$  and we get

$$(2.4) \quad \begin{aligned} u &= \frac{|\cos \theta_2 + \cosh \frac{l}{2} \cos \theta_1|}{(\cosh^2 \frac{l}{2} + 2 \cosh \frac{l}{2} \cos \theta_1 \cos \theta_2 + \cos^2 \theta_1 + \cos^2 \theta_2 - 1)^{1/2}}, \\ v &= \frac{\sinh \frac{l}{2} \sin \theta_1}{(\cosh^2 \frac{l}{2} + 2 \cosh \frac{l}{2} \cos \theta_1 \cos \theta_2 + \cos^2 \theta_1 + \cos^2 \theta_2 - 1)^{1/2}}. \end{aligned}$$

**The Fenchel-Nielsen deformation.** Let  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$  be numbers in  $[0, \pi/2]$ . Then any two of them satisfy the inequality (2.1) for all  $l > 0$ . Thus the numerator of the first expression in (2.4) does not need absolute-values. We define  $A$  and  $B$  by (2.2), (2.3) and (2.4) and  $\tilde{A}, \tilde{B}$  similarly but replacing  $(\theta_1, \theta_2)$  by  $(\theta_4, \theta_3)$ . We take conjugations of  $\tilde{A}^{-1}$  and  $\tilde{B}^{-1}$  by  $H_s r_0 E$ , where  $E(z) = -1/z, H_s(z) = e^s z$ , to

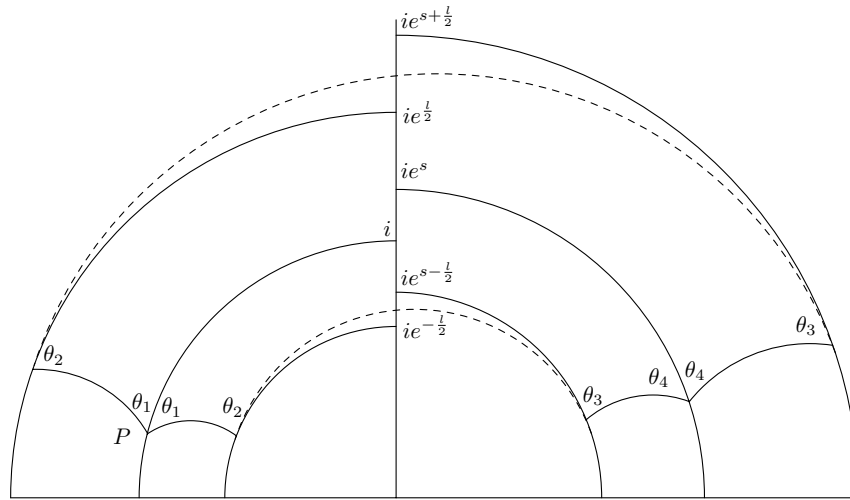


FIGURE 1b.

obtain the transformations  $D$  and  $C$  with matrix-representations

$$C = \begin{pmatrix} -e^{-l/2}(\cos \theta_4 - \xi \eta^{-1} \sin \theta_4) & e^s e^{-l/2} \eta^{-1} \sin \theta_4 \\ -e^{-s} e^{l/2} \eta^{-1} \sin \theta_4 & -e^{l/2}(\cos \theta_4 + \xi \eta^{-1} \sin \theta_4) \end{pmatrix},$$

$$D = \begin{pmatrix} \cos \theta_4 + \xi \eta^{-1} \sin \theta_4 & e^s \eta^{-1} \sin \theta_4 \\ -e^{-s} \eta^{-1} \sin \theta_4 & \cos \theta_4 - \xi \eta^{-1} \sin \theta_4 \end{pmatrix},$$

where

$$\xi = \frac{\cos \theta_3 + \cosh \frac{l}{2} \cos \theta_4}{(\cosh^2 \frac{l}{2} + 2 \cosh \frac{l}{2} \cos \theta_3 \cos \theta_4 + \cos^2 \theta_3 + \cos^2 \theta_4 - 1)^{1/2}},$$

$$\eta = \frac{\sinh \frac{l}{2} \sin \theta_4}{(\cosh^2 \frac{l}{2} + 2 \cosh \frac{l}{2} \cos \theta_3 \cos \theta_4 + \cos^2 \theta_3 + \cos^2 \theta_4 - 1)^{1/2}}.$$

The performed Fenchel-Nielsen deformation is represented in Figure 1a with coordinates  $(l, 0)$  and in Figure 1b with coordinates  $(l, s)$ . Since  $AB = H_l$  and  $CD = H_l^{-1}$ ,  $ABCD = 1$ . We think of  $A, B, C$  and  $D$  as matrix-valued functions defined in the Fenchel-Nielsen coordinate-space  $(l, s) \in \mathbf{R}_+ \times \mathbf{R}$  for the Teichmüller space of signature  $(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$ . The transformations  $A, B$  and  $C$  are (resp.) positive rotations of angles  $2\theta_1, 2\theta_2$  and  $2\theta_3$  about their fixed points. Then, since  $\text{tr}AB = -2 \cosh \frac{l}{2} \leq -2$ , symmetry (by replacing  $(A, B, C, D)$  with  $(B, C, A, A^{-1}DA)$  and  $(C, A, B, B^{-1}DB)$ ) shows that  $\text{tr}BC < -2$  and  $\text{tr}CA < -2$ .

The next lemma gives an equation which corresponds, for the case treated above, to (1.2) for the case of torus.

**Lemma 2.1.** *Let  $x = -\frac{1}{2}\text{tr}BC, y = -\frac{1}{2}\text{tr}CA$  and  $z = -\frac{1}{2}\text{tr}AB$ . Define the constants  $a = \cos \theta_1 \cos \theta_4 + \cos \theta_2 \cos \theta_3, b = \cos \theta_2 \cos \theta_4 + \cos \theta_1 \cos \theta_3, c = \cos \theta_3 \cos \theta_4 + \cos \theta_1 \cos \theta_2$  and*

$$d = 4 \cos \theta_1 \cos \theta_2 \cos \theta_3 \cos \theta_4 + \cos^2 \theta_1 + \cos^2 \theta_2 + \cos^2 \theta_3 + \cos^2 \theta_4 - 1.$$

Then the Teichmüller space for the signature  $(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$  is represented by the sublocus  $T$  satisfying  $x > 1, y > 1$  and  $z > 1$  of the equation

$$(2.5) \quad x^2 + y^2 + z^2 - 2xyz + 2ax + 2by + 2cz + d = 0.$$

The equation (2.5) is obtained by a lengthy calculation starting from the equation  $tr[(ABC)^2] = trD^2$  and expressing the left-hand side as a function of  $x, y$  and  $z$ .

In order to see that  $T$  represents the Teichmüller space, consider the mapping  $\varphi(l, s) = (x, y, z)$  which sends the Fenchel-Nielsen coordinates  $(l, s)$  to  $T$ . From the formulas  $z = \cosh l/2, x = \alpha \cosh s + \beta, y = \gamma \cosh(s - l/2) + \delta$  with  $\alpha, \beta, \gamma$  and  $\delta$  independent of  $s$ , it follows that  $\varphi$  is injective and a real-analytic diffeomorphism. □

We calculate the area form (1.1), where  $l$  and  $s$  are the Fenchel-Nielsen coordinates, in terms of  $x, y$  and  $z$ . Since  $z = -\frac{1}{2}trAB = \cosh \frac{l}{2}$  and  $x = -\frac{1}{2}trBC$ , we get  $l = 2 \log(z + \sqrt{z^2 - 1}), s = \log(L + \sqrt{L^2 - 1})$  with

$$L = \frac{x + \cos \theta_1 \cos \theta_4 - u\xi v^{-1} \eta^{-1} \sin \theta_1 \sin \theta_4}{v^{-1} \eta^{-1} \sin \theta_1 \sin \theta_4}.$$

Since  $\partial(l, s)/\partial(z, x) = (\partial l/\partial z) \cdot (\partial s/\partial x)$ ,

$$(2.6) \quad \omega_{WP} = \frac{4dz \wedge dx}{\sqrt{x^2 z^2 - x^2 - z^2 - 2bxz - 2ax - 2cz + b^2 - d}}.$$

In order to make our description simple, we will often call a boundary curve of length  $\theta$  a cone point of angle  $i\theta$ .

By (2.5) the denominator of (2.6) equals  $xz - y - b$  and again by (2.5),  $dz = z_x dx + z_y dy, dx = x_y dy + x_z dz$ , where

$$z_y = \frac{xz - y - b}{z - xy + c}, \quad x_y = \frac{xz - y - b}{x - yz + a}.$$

Therefore

$$(2.7) \quad \omega_{WP} = \frac{4dz \wedge dx}{xz - y - b} = \frac{4dx \wedge dy}{xy - z - c} = \frac{4dy \wedge dz}{yz - x - a}.$$

Originally (2.6) is obtained from the Fenchel-Nielsen coordinates associated to the simple closed curve which separates the cone points of angles  $\theta_1$  and  $\theta_2$  from those of angles  $\theta_3$  and  $\theta_4$ ; see Figures 1a and 1b. But the symmetry in  $(x, a), (y, b)$  and  $(z, c)$  manifested in the expression (2.7) implies that (2.6) can be obtained also from the simple closed curve separating the cone points of angles  $\theta_1$  and  $\theta_3$  from those of angles  $\theta_2$  and  $\theta_4$  and the one separating the cone points of angles  $\theta_1$  and  $\theta_4$  from those of angles  $\theta_2$  and  $\theta_3$ . Since  $\mathbf{T}(0; \pi, \pi, \pi, \theta) = \mathbf{T}(1; 2\theta)$  (see Section 5), we see also that the 2-form  $dl \wedge ds$  on  $\mathbf{T}(1; 2\theta)$  does not depend on a particular choice of Fenchel-Nielsen coordinates  $(l, s)$ . This fact can be used to show that a 2-form on any Teichmüller space defined analogously to Wolpert’s formula of Weil-Petersson form [12] is invariant under the action of the mapping class group as long as the Teichmüller space admits global Fenchel-Nielsen coordinates.

Let  $S$  be an oriented closed surface of genus  $g$  and  $P = \{x_1, \dots, x_n\}$  a fixed set of  $n$  points on  $S$ . Let  $(g; \theta_1, \dots, \theta_n)$  be the signature of a hyperbolic surface. A point of the Teichmüller space  $\mathbf{T}(g; \theta_1, \dots, \theta_n)$  is a class of marked hyperbolic surfaces  $(R, [f])$  where  $[f]$  is the isotopy class of an orientation-preserving homeomorphism  $f : S \setminus P \rightarrow R \setminus \{\text{cone points}\}$  such that  $f$  sends the neighborhood of  $x_i$  to the singularity of type  $\theta_i$ .

We assume that  $(g; \theta_1, \dots, \theta_n)$  is a signature such that any hyperbolic surface  $R$  with this signature satisfies the following property: Any simple closed curve  $c$  on  $R \setminus \{\text{cone points}\}$  can be deformed to a unique geodesic curve under a homotopy which meets no cone points. There is an exceptional case where  $c$  bounds a disk with two cone points of angle  $\pi$ . In this case  $c$  is deformed to the degeneration of a simple closed curve and goes between the cone points along a geodesic arc back and forth. Then, given a curve-system  $(c_1, \dots, c_d)$ ,  $d = 3g - 3 + n$ , which decomposes  $S$  into pairs of pants, we can define the Fenchel-Nielsen coordinates  $(l_1, \dots, l_d, s_1, \dots, s_d)$  on the Teichmüller space  $\mathbf{T}(g; \theta_1, \dots, \theta_n)$  and also a 2-form by

$$(2.8) \quad \omega_{WP} = \sum_{i=1}^d dl_i \wedge ds_i.$$

Our aim is to show that the form (2.8) is independent of the choice of the curve-system. Let  $\mathcal{S} = (c_1, \dots, c_d)$  and  $\mathcal{S}' = (c'_1, \dots, c'_d)$  be two curve-systems as above such that  $c_j = c'_j$  for all indices  $j$  except for one  $i$ . Namely,  $\mathcal{S}'$  is a result of a *Thurston-Hatcher type simple move* on  $\mathcal{S}$  ([4]). They define Fenchel-Nielsen coordinates  $(l_1, \dots, l_d, s_1, \dots, s_d)$  and  $(l'_1, \dots, l'_d, s'_1, \dots, s'_d)$  respectively, such that  $l_j = l'_j$  and  $s_j = s'_j$  for all  $j \neq i$ . Fix  $\{(l_j^0, s_j^0)\}_{j=1, j \neq i}^d \in (\mathbf{R}_+ \times \mathbf{R})^{d-1}$ . Then the subspace of  $\mathbf{T}(g; \theta_1, \dots, \theta_d)$  determined by  $l_j = l_j^0$ ,  $s_j = s_j^0$ ,  $j = 1, \dots, i-1, i+1, \dots, d$ , can be identified with a two-dimensional Teichmüller space. For example, if  $c_{i_1} = c'_{i_1}, \dots, c_{i_4} = c'_{i_4}$  separate  $c_i$  (resp.  $c'_i$ ) from the rest in  $\mathcal{S}$  (resp.  $\mathcal{S}'$ ), then the latter Teichmüller space is  $\mathbf{T}(0; \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4)$  with  $\hat{\theta}_k = i2^{-1}l_{i_k}$ . Both  $(l_i, s_i)$  and  $(l'_i, s'_i)$  give Fenchel-Nielsen coordinates for this two-dimensional Teichmüller space and so  $dl_i \wedge ds_i = dl'_i \wedge ds'_i$ , as we have seen above. Since  $\{(l_j^0, s_j^0)\}_{j=1, j \neq i}^d$  are arbitrary, we obtain

$$\sum_{i=1}^d dl_i \wedge ds_i = \sum_{i=1}^d dl'_i \wedge ds'_i.$$

For any pair of curve-systems  $\mathcal{S}$  and  $\mathcal{S}'$ , there exists a finite chain of curve-systems  $\mathcal{S} = \mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_q = \mathcal{S}'$  such that  $\mathcal{S}_{j+1}$  is obtained from  $\mathcal{S}_j$  by a simple move for  $j = 1, \dots, q-1$ ; see [4], [6]. We conclude that the form (2.8) is independent of the choice of curve-system which decomposes  $S$  into pairs of pants.

**Theorem 2.2.** *The 2-form  $\omega_{WP}$  is invariant under the action of the mapping class group.*

This theorem is known for the case of hyperbolic surfaces without cone points and almost trivial because (2.8) is Wolpert's formula for the Weil-Petersson Kähler form [12]. Our statement of the theorem includes the case of conical hyperbolic surfaces with boundary and the following proof is elementary.

*Proof of Theorem 2.2.* Since the action of each mapping class is a composite of full (or Dehn) twists and half-twists about a finite number of simple closed curves, it suffices to show that  $\omega_{WP}$  is invariant under the twist about a simple closed curve. Here the half-twist is performed only about the boundary curve of a disk which contains two cone points of the same angle and avoids all other cone points.

Let  $c$  be a simple closed curve on  $S \setminus P$ . Then there is a curve-system  $\mathcal{S} = (c_1, \dots, c_d)$  with  $c = c_1$  which decomposes  $S$  into pairs of pants. Let  $(l_1, \dots, l_d,$

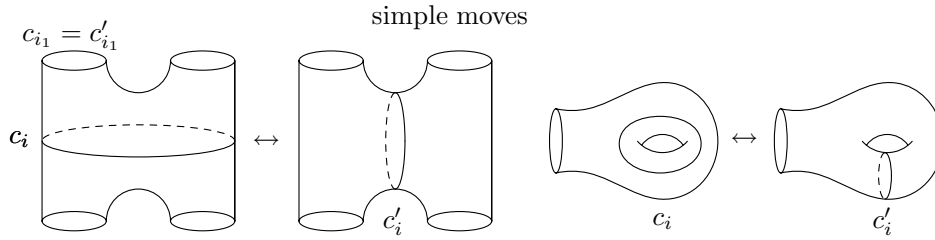


FIGURE 2.

$s_1, \dots, l_d$ ) be the Fenchel-Nielsen coordinates defined by  $\mathcal{S}$ . Express  $\omega_{WP}$  as in (2.8). Then invariance of  $\omega_{WP}$  is obvious, because the full twist about  $c$  causes the translation  $s_1 \mapsto s_1 \pm l_1$  and the half-twist causes  $s_1 \mapsto s_1 \pm \frac{1}{2}l_1$ .

### 3. ACTION OF THE MAPPING CLASS GROUP

If, for instance,  $y$  and  $z$  are fixed, then (2.5) regarded as a quadratic equation in  $x$  has the pair of roots  $x$  and  $2yz - x - 2a$ . Thus the transformations in  $\mathbf{R}_+^3$  defined by

$$\begin{aligned}
 \alpha &: (x, y, z) \mapsto (2yz - x - 2a, y, z), \\
 \beta &: (x, y, z) \mapsto (x, 2zx - y - 2b, z), \\
 \gamma &: (x, y, z) \mapsto (x, y, 2xy - z - 2c)
 \end{aligned}
 \tag{3.1}$$

exchange the pairs of roots and preserve the surface  $T$ . The geometric meaning of these transformations is as follows. Let  $p_A, p_B, p_C$  and  $p_D$  denote the cone points of angles  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$ , respectively. Then  $\alpha$  corresponds to the topological reflection of the sphere in a Jordan curve going through  $p_A, p_B, p_D, p_C$  in this order; see Figure 3. Likewise  $\beta$  corresponds to the topological reflection in a Jordan curve going through  $p_B, p_C, p_D, p_A$  in this order and  $\gamma$  to the topological reflection in a Jordan curve going through  $p_C, p_A, p_D, p_B$  in this order. Then  $\beta \cdot \alpha$  is a Dehn twist. In the generic case, the group generated by  $\alpha, \beta$  and  $\gamma$  has the same action as the mapping class group  $\mathcal{MC}$ . If two of  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$  are equal,  $\mathcal{MC}$  contains classes of mappings which exchange cone points of the same angle. Thus the group has the same action as a subgroup of  $\mathcal{MC}$  of finite index. For the rest of this paper, we consider the generic case. We shall remark on the special case at the end of Section 4.

Let  $V : T \rightarrow \mathbf{R}$  be  $V(x, y, z) = x + y + z$  and let  $\Delta' \subset T$  be the set where conditions  $V(x, y, z) < V(\alpha(x, y, z))$ ,  $V(x, y, z) < V(\beta(x, y, z))$  and  $V(x, y, z) < V(\gamma(x, y, z))$  are satisfied. These conditions are equivalent to  $x < yz - a$ ,  $y < zx - b$  and  $z < xy - c$ . We show that  $\Delta'$  is a fundamental domain for  $\mathcal{MC}$ . Since the action of  $\mathcal{MC}$  is discontinuous and since  $V(x, y, z)$  tends to  $+\infty$  as  $(x, y, z) \in T$  tends to the boundary of  $T$ , the function  $V$  achieves minima in each orbit  $\mathcal{MC}(p)$ ,  $p \in T$  and if  $q \in \mathcal{MC}(p)$  represents a minimum, then  $q$  must be in the closure of  $\Delta'$ . Hence each  $\mathcal{MC}$ -orbit meets the closure of  $\Delta'$ . Next we show that two distinct points of  $\Delta'$  are not  $\mathcal{MC}$ -equivalent. First observe that the three subsets  $\Delta'_x = \{(x, y, z) \in T : x \geq yz - a\}$ ,  $\Delta'_y = \{(x, y, z) \in T : y \geq zx - b\}$  and  $\Delta'_z = \{(x, y, z) \in T : z \geq xy - c\}$  are mutually disjoint. Now assume that  $p \in \Delta'$  and that  $\delta(p) \in \Delta'$  for some non-trivial  $\delta \in \mathcal{MC}$ . Express  $\delta$  by a shortest word in

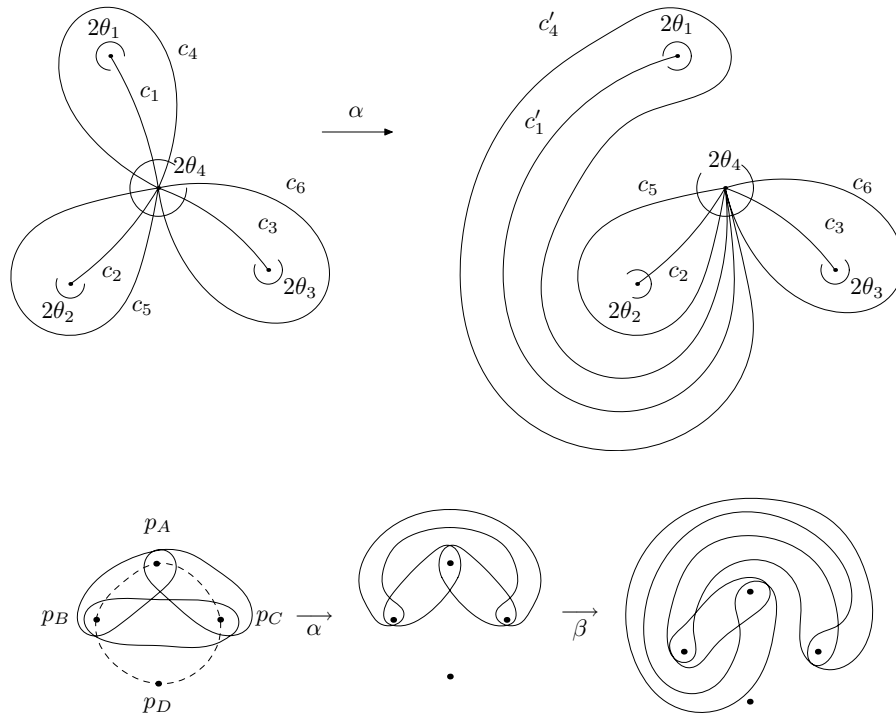


FIGURE 3.

$\{\alpha, \beta, \gamma\}$ :  $\delta = \delta_n \delta_{n-1} \cdots \delta_1, \delta_i \in \{\alpha, \beta, \gamma\}$ . Set  $p_0 = p$  and inductively  $p_i = \delta_i(p_{i-1})$ , ( $i = 1, \dots, n$ ). Since  $V(p) < V(p_1)$  and  $V(p_{n-1}) > V(\delta(p))$ , there exists  $i$  such that  $V(p_{i-1}) \leq V(p_i)$  and  $V(p_i) > V(p_{i+1})$ . Suppose that  $\delta_i = \alpha$ . Then  $\delta_{i+1} \neq \alpha$  due to the shortest word-representation of  $\delta$  because  $\alpha^2 = 1$ . Let, for example,  $\delta_{i+1} = \beta$ . Then  $V(p_{i-1}) \leq V(p_i)$  means  $p_i \in \Delta'_x$  and  $V(p_i) > V(p_{i+1})$  means  $p_i \in \Delta'_y$ , which is a contradiction. Other cases can be treated similarly. In conclusion, each  $MC$ -orbit meets  $\Delta'$  at most once and thus  $\Delta'$  is a fundamental domain for  $MC$ .

4. THE AREA OF THE MODULI SPACE FOR THE SIGNATURE  $(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$

It follows from (2.6) and symmetry in  $x, y$  and  $z$  shown in (2.7) and also from the discussion and results of the preceding sections, that the area of the moduli space  $\mathbf{T}(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)/MC$  is

$$V = V(\theta_1, \theta_2, \theta_3, \theta_4) = \iint_{\Delta} \frac{4dx dy}{\sqrt{x^2 y^2 - x^2 - y^2 - 2cxy - 2ax - 2by + c^2 - d}},$$

where  $\Delta$  denotes the image of  $\Delta'$  under the projection  $(x, y, z) \mapsto (x, y)$ , which is one-to-one in  $\Delta'$  because (2.5) and the inequality  $z < xy - c$  yield

$$(4.1) \quad z = xy - c - \sqrt{x^2 y^2 - x^2 - y^2 - 2cxy - 2ax - 2by + c^2 - d}.$$

The domain  $\Delta$  is bounded by three curves  $C_z : z = xy - c$ ,  $C_y : y = zx - b$  and  $C_x : x = yz - a$  with  $z$  satisfying (4.1); see Figure 4.



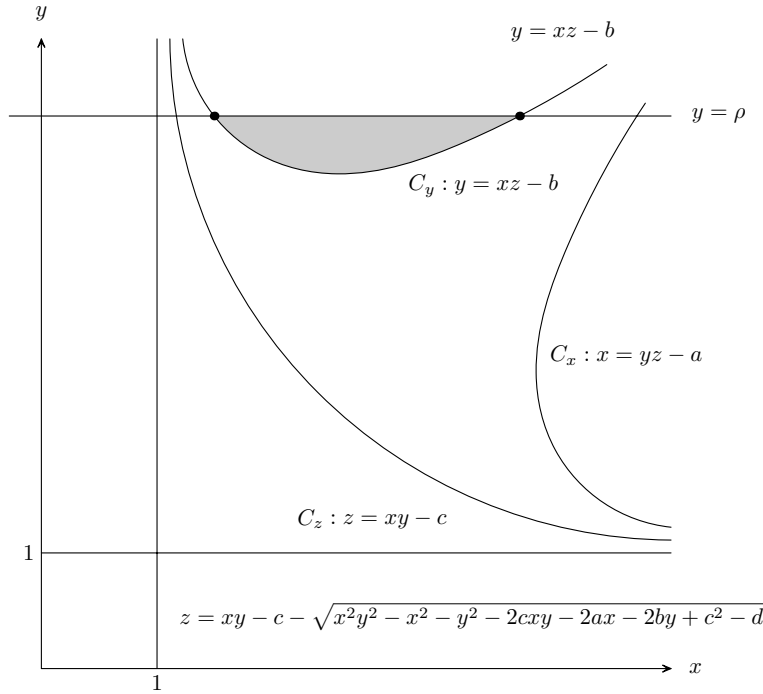


FIGURE 4.

By making a change of variables  $F_1 : (x, y) \mapsto (X, Y)$  with

$$x = \frac{cy + a}{y^2 - 1} + X \frac{\sqrt{(cy + a)^2 + (y^2 - 1)(y^2 + 2by - c^2 + d)}}{y^2 - 1}, \quad y = Y,$$

we obtain

$$V = \iiint_{F_1\Delta} \frac{4dXdY}{\sqrt{(X^2 - 1)(Y^2 - 1)}}.$$

The domain  $F_1\Delta$  is bounded by  $F_1C_z : X = 1$ ,  $F_1C_x : X = Y$  and  $F_1C_y$ . Unfortunately, it is not clear how to express  $F_1C_y$  by a graph of an explicit function of  $X$ , hence we take a roundabout course to our goal. Observe, that if  $(x, y) \in C_y$ , then  $y$  diverges to  $+\infty$  as  $x$  tends to 1 or to  $+\infty$  and, if  $\rho$  is chosen to be sufficiently large, then the line  $y = \rho$  meets  $C_y$  in only two points. Let  $D(\rho)$  denote the domain between  $y = \rho$  and  $C_y$ . Then

$$V = \lim_{\rho \rightarrow +\infty} \left\{ \int_1^\rho \left[ \int_X^\rho \frac{4dY}{\sqrt{(X^2 - 1)(Y^2 - 1)}} \right] dX - \iint_{F_1D(\rho)} \frac{4dXdY}{\sqrt{(X^2 - 1)(Y^2 - 1)}} \right\}.$$

We fix  $\theta_1, \theta_2$  and  $\theta_3$  and think of  $V$  as a function of  $\theta_4$ . Let  $V_\rho(\theta_4)$  denote the term inside the brackets of the expression above. Then only the last integral depends on  $\theta_4$  and changing the variables back gives

$$\frac{dV_\rho(\theta_4)}{d\theta_4} = -\frac{d}{d\theta_4} \iint_{D(\rho)} \frac{4dxdy}{\sqrt{x^2y^2 - x^2 - y^2 - 2cxy - 2ax - 2by + c^2 - d}}.$$

The change of variables  $F_2 : (x, y) \mapsto (X, Y)$  with

$$x = X, \quad y = \frac{cx + b}{x^2 - 1} + Y \frac{\sqrt{(cx + b)^2 + (x^2 - 1)(x^2 + 2ax - c^2 + d)}}{x^2 - 1},$$

deforms the curve  $C_y$  to  $F_2C_y : X = Y$  and  $y = \rho$  to  $Y = Y_\rho(X)$ , where

$$Y_\rho(X) = Y_\rho(X, \theta_4) = \frac{(X^2 - 1)\rho - (cX + b)}{\sqrt{(cX + b)^2 + (X^2 - 1)(X^2 + 2aX - c^2 + d)}},$$

and the last integral becomes

$$\iint_{F_2D(\rho)} \frac{4dXdY}{\sqrt{(X^2 - 1)(Y^2 - 1)}} = \int_{\alpha(\rho)}^{\beta(\rho)} \frac{4}{\sqrt{X^2 - 1}} \log \frac{Y_\rho(X) + \sqrt{Y_\rho(X)^2 - 1}}{X + \sqrt{X^2 - 1}} dX,$$

where  $\alpha(\rho) = \alpha(\rho, \theta_4)$ ,  $\beta(\rho) = \alpha(\rho, \theta_4)$  denote the  $X$ -coordinates of the intersection points of  $Y = X$  and  $Y = Y_\rho(X)$ . Since the integrand of the last integral vanishes at  $X = \alpha(\rho)$  and  $X = \beta(\rho)$ , by denoting the variable  $X$  by  $x$  we get

$$\begin{aligned} - \frac{d}{d\theta_4} \int_{\alpha(\rho)}^{\beta(\rho)} \frac{4}{\sqrt{x^2 - 1}} \log \frac{Y_\rho(x) + \sqrt{Y_\rho(x)^2 - 1}}{x + \sqrt{x^2 - 1}} dx \\ = - \int_{\alpha(\rho)}^{\beta(\rho)} \left( \frac{4 \frac{\partial Y_\rho}{\partial \theta_4}(x, \theta_4)}{\sqrt{Y_\rho(x)^2 - 1}} \right) \frac{dx}{\sqrt{x^2 - 1}}. \end{aligned}$$

When  $\rho \rightarrow +\infty$ , the last integral converges locally uniformly in  $\theta_4$  to

$$\int_1^{+\infty} \frac{4}{\sqrt{x^2 - 1}} \left[ \frac{\partial}{\partial \theta_4} \log ((cx + b)^2 + (x^2 - 1)(x^2 + 2ax - c^2 + d)) \right] dx.$$

From (2.5) we obtain the factorization

$$\begin{aligned} (cx + b)^2 + (x^2 - 1)(x^2 + 2ax - c^2 + d) \\ = (x + \cos(\theta_1 + \theta_4))(x + \cos(\theta_1 - \theta_4))(x + \cos(\theta_2 + \theta_3))(x + \cos(\theta_2 - \theta_3)). \end{aligned}$$

Finally, we obtain

$$\lim_{\rho \rightarrow \infty} \frac{dV_\rho}{d\theta_4}(\theta_4) = -8 \int_1^{+\infty} \frac{1}{\sqrt{x^2 - 1}} \left( \frac{(\cos \theta_1 x + \cos \theta_4) \sin \theta_4}{(x + \cos(\theta_1 + \theta_4))(x + \cos(\theta_1 - \theta_4))} \right) dx.$$

By [7, §55 (ii) 1°] the integral has value  $-4\theta_4$  and the locally uniform convergence in  $\theta_4$  yields also  $\partial V / \partial \theta_4 = -4\theta_4$ . By symmetry,  $V = C - 2(\theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2)$ , and the constant  $C$  can be determined to be  $2\pi^2$  because the area tends to 0 as the signature  $(0; \pi/2, \pi/2, \pi/2, \theta)$ ,  $\theta < \pi/2$ , tends to the degenerate case  $(0; \pi/2, \pi/2, \pi/2, \pi/2)$ . This gives the result claimed in Theorem 1.1.

*Remark.* If two of  $\theta_1, \theta_2, \theta_3$  and  $\theta_4$  in a signature  $(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$  are equal,  $\mathcal{MC}$  has a finite extension  $\widetilde{\mathcal{MC}}$ , which contains classes of mappings which permute cone points of the same angle. We call  $\widetilde{\mathcal{MC}}$  the *full* mapping class group and denote by  $\widetilde{\mathcal{MC}}^+$  the subgroup of index 2 consisting of all classes of orientation preserving mappings. The areas of the spaces  $\mathbf{T}(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4) / \widetilde{\mathcal{MC}}$  and  $\mathbf{T}(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4) / \widetilde{\mathcal{MC}}^+$  are calculated by dividing the value in (1.3) by the index  $[\widetilde{\mathcal{MC}} : \mathcal{MC}]$  or by  $[\widetilde{\mathcal{MC}}^+ : \mathcal{MC}] = \frac{1}{2}[\widetilde{\mathcal{MC}} : \mathcal{MC}]$ , respectively.

Case 1. If  $\theta_1 = \theta_2$ , then  $[\widetilde{\mathcal{MC}} : \mathcal{MC}] = 2$ . In this case  $\widetilde{\mathcal{MC}}$  is obtained by adding to  $\mathcal{MC}$  the class  $\delta : (x, y, z) \mapsto (y, x, z)$ .

Case 2. If  $\theta_1 = \theta_2 = \theta_3$ ,  $\widetilde{\mathcal{MC}}$  is obtained by adding to  $\mathcal{MC}$  the classes  $\delta_1 : (x, y, z) \mapsto (y, x, z)$  and  $\delta_2 : (x, y, z) \mapsto (y, z, x)$ . Thus  $\langle \delta_1, \delta_2 \rangle$  is isomorphic to the symmetry group  $S_2$  and  $[\widetilde{\mathcal{MC}} : \mathcal{MC}] = 6$ .

The mapping class group  $\mathcal{M}(0, 4)$  (by the notation in [1]) of the 4 punctured sphere  $S^2 \setminus \{p_1, p_2, p_3, p_4\}$ , which is treated in detail in [1, Section 5.3] is one of the mapping class groups which has a special action on Teichmüller spaces; see [14, E. 3.17, p. 203]. For instance, choose a simple closed curve  $c$  which separates the pair  $p_1, p_2$  from  $p_3$  and  $p_4$ . Then a half-twist about  $c$  interchanging  $p_1$  and  $p_2$  can cause the same action as a half-twist about  $c$  interchanging  $p_3$  and  $p_4$ . So Case 1 includes the case where  $\theta_1 = \theta_2$  and  $\theta_3 = \theta_4$  and Case 2 includes the case where  $\theta_1 = \theta_2 = \theta_3 = \theta_4$ .

### 5. THE AREA OF THE MODULI SPACE OF THE TORUS WITH ONE SINGULARITY

In [8], the area of the moduli space for the signature  $(1; 2\theta)$  was calculated to be

$$(5.1) \quad \frac{1}{6}(\pi^2 - \theta^2).$$

Here we consider the mapping class group consisting of only classes of orientation-preserving mappings so that (5.1) is comparable with Wolpert's result [10].

The value in the expression (5.1) can be deduced also from the fact that  $\mathbf{T}(1; 2\theta) = \mathbf{T}(0; \pi, \pi, \pi, \theta)$ . This fact follows from that every hyperbolic cone surface with signature  $(1, 2\theta)$  admits the hyperelliptic involution: The same matrices  $A, B, C$  and  $D$  as in Section 2 satisfy  $A^2 = B^2 = C^2 = ABCD = 1$  (= the identity in  $PSL_2(\mathbf{R})$ ) and the commutator  $[AC, CB]$  equals  $D^{-2}$ . The map  $(A, B, C, D) \mapsto (AC, CB)$  induces a homeomorphism between the two Teichmüller spaces. Since  $a = b = c = 0$  in (2.5) for  $(0; \pi, \pi, \pi, \theta)$ , (2.5) is symmetric in  $x, y, z$  and therefore in (3.1), the order can be reversed to obtain orientation preserving mappings

$$\begin{aligned} \alpha : (x, y, z) &\mapsto (z, y, 2yz - x), \\ \beta : (x, y, z) &\mapsto (x, 2xy - z, y), \\ \gamma : (x, y, z) &\mapsto (2xz - y, x, z) \end{aligned}$$

instead of those in (3.1). Each of these transformations relates to the half-twist around a pair of points of order 2 on the sphere, which lifts to the Dehn twist along a simple loop on the torus, the two-sheeted covering surface. The transformations  $\alpha$  and  $\beta$  generate the mapping class group. (For more details on the action of the mapping class group, see [8]). Since  $\beta\alpha(x, y, z) = (z, x, y)$  generates a cyclic subgroup of order 3, the stabilizing subgroup of  $\Delta'$  (see [8]), the area is one-third of the area (1.3) for the signature  $(0; \pi, \pi, \pi, \theta)$ , the same value as the area of  $\mathbf{T}(0; \pi, \pi, \pi, \theta)/\widetilde{\mathcal{MC}}^+$ .

### REFERENCES

1. Birman, J. S., *Braids, Links, and Mapping Class Groups*, Ann. of Math. Studies 82, Princeton Univ. Press, Princeton, 1974. MR 51:11477; erratum MR 54:13894
2. Ford, L. R., *Automorphic Functions*, Second edition, Chelsea, New York, 1951.
3. Fricke R. and F. Klein, *Vorlesungen über die Theorie der Automorphen Funktionen*, Teubner, 1965. MR 32:1348
4. Hatcher, A. and W. Thurston, *A presentation for the mapping class group of a closed orientable surface*, Topology 19 (1980), 221–237. MR 81k:57008
5. Kaufmann, R, Yu. Manin and D. Zagier, *Higher Weil-Petersson volumes of moduli spaces of stable  $n$ -pointed curves*, Commun. Math. Phys. 181 (1996), 763–787. MR 98i:14029

6. Luo, Feng, *On Heegaard diagrams*, Math. Res. Lett. **4** (1997), no. 2–3, 365–373. MR **95h**:57030
7. Moriguchi, S, K. Udagawa and S. Hitotsumatsu, *Mathematical formulae*, I, Iwanami Shoten, Tokyo, 1989 (in Japanese).
8. Näätänen, M. and T. Nakanishi, *Weil-Petersson areas of the moduli space of tori*, Results in Mathematics **33** (1998), 120–133. MR **2000a**:32028
9. Penner, R. C., *Weil-Petersson volumes*, J. Differential Geometry **35** (1992), 559–608. MR **93d**:32029
10. Wolpert, S., *On the Kähler form of the moduli space of once punctured tori*, Comment. Math. Helv. **58** (1983), 246–256. MR **84j**:32028
11. Wolpert, S., *On the symplectic geometry of deformations of a hyperbolic surface*, Ann. of Math. **117** (1983), 207–234. MR **85e**:32028
12. Wolpert, S., *On the Weil-Petersson geometry of the moduli space of curves*, Amer. J. Math. **107** (1985), 969–997. MR **87b**:32040
13. Zograf, P., *The Weil-Petersson volume of the moduli space of punctured spheres*, Mapping Class Groups and Moduli Spaces of Riemann Surfaces (C.F. Bödigheimer and R. M. Hain, eds.), Contemporary Mathematics, vol. 150, American Mathematical Society, 1993, pp. 367–372. MR **94g**:32030
14. Zieschang, H., *Finite Groups of Mapping Classes of Surfaces*, Lecture Notes in Math. **875**, Springer, 1981. MR **86g**:57001

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FURO-CHO, CHIKUSA-KU, NAGOYA 464-01, JAPAN

*E-mail address*: `tosihiro@math.nagoya-u.ac.jp`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HELSINKI, P.O. BOX 4 (YLIOPISTONKATU 5), 00014 HELSINKI, FINLAND

*E-mail address*: `marjatta.naatanen@helsinki.fi`