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FOURIER RESTRICTION FOR AFFINE ARCLENGTH MEASURES IN THE PLANE

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Abstract. We obtain an analog, uniform for a large class of curves in the plane, of the Fefferman-Zygmund theorem on restriction of the Fourier transform.

The purpose of this note is to prove an analog, uniform over a certain class of curves in $\mathbb{R}^2$, of the Fefferman-Zygmund restriction theorem for the circle.

Theorem. *If $1 \le p < \frac{4}{3}$ and $\frac{1}{p} + \frac{1}{3q} = 1$, there is a constant $C = C(p)$ such that the estimate*

$$\Big(\int_a^b |\widehat{f}\big(t,\phi(t)\big)|^q \phi''(t)^{\frac{1}{3}} dt\Big)^{\frac{1}{q}} \le C(p)\ \|f\|_{L^p(\mathbb{R}^2)} \tag{1}$$

holds whenever ϕ is a real-valued function on an interval (a,b) satisfying $\phi''(t) > 0$, $\phi^{(3)}(t) \ge 0$ on (a,b).

Writing $d\lambda$ for the measure $\phi''(t)^{1/3}dt$ on the curve $\gamma(t) = \big(t,\phi(t)\big)$, $a < t < b$, we will follow the broad outline of the Fefferman-Zygmund proof and thus establish the dual estimate

$$\|\widehat{fd\lambda}\|_{L^q(\mathbb{R}^2)} \le\ C(p)\|f\|_{L^p(d\lambda)}, \tag{2}$$

where $1 \le p < 4$ and $\frac{1}{p} + \frac{3}{q} = 1$.

Interpolation with the case $(p,q) = (1,\infty)$ shows that it is enough to prove a restricted weak type version of (2). Thus if $\frac{1}{p} = \frac{3}{2}\frac{1}{r} - \frac{1}{2}$, so that $\frac{1}{r} + \frac{2}{q} = 1$, we will show that

$$\|(\chi_{\gamma(E)}d\lambda) * (\chi_{\gamma(E)}d\lambda)^{\sim}\|_{L^{r,\infty}(\mathbb{R}^2)} \le\ C(p)(\int_E \phi''(t)^{1/3}dt)^{\frac{2}{p}} \tag{3}$$

whenever E is a Borel subset of (a,b). Then it will follow that

$$\|\widehat{\chi_{\gamma(E)}d\lambda}\|_{L^{q,\infty}(\mathbb{R}^2)} = \|\ |\widehat{\chi_{\gamma(E)}d\lambda}|^2\|^{\frac{1}{2}}_{L^{\frac{q}{2},\infty}(\mathbb{R}^2)}$$

$$\le C(p)\ \|(\chi_{\gamma(E)}d\lambda) * (\chi_{\gamma(E)}d\lambda)^{\sim}\|^{\frac{1}{2}}_{L^{r,\infty}(\mathbb{R}^2)} \le\ C(p)\Big(\int_E \phi''(t)^{1/3}dt\Big)^{\frac{1}{p}},$$

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by Hunt's generalization of the Hausdorff-Young theorem. Now (3) is true for all (p,r) of interest if it is true for the two extreme cases $(p,r)=(1,1)$ and $(p,r)=(4,2)$. The first of these is easy and so it is enough to establish the inequality

$$\|(\chi_{\gamma(E)}d\lambda)*(\chi_{\gamma(E)}d\lambda)^{\sim}\|_{L^{2,\infty}(\mathbb{R}^2)}\le C\,\Big(\int_E \phi''(t)^{1/3}dt\Big)^{\frac12} \tag{4}$$

for some absolute constant C. Inequality (4) may be regarded as a weak endpoint estimate for (the dual of) Fourier restriction. It is a consequence of

$$\int_a^b\int_t^b \chi_T\big(\gamma(t)-\gamma(s)\big)\ \chi_E(s)\ \chi_E(t)\ \phi''(s)^{1/3}\ \phi''(t)^{1/3}ds\ dt \tag{5}$$

$$\le C\Big(\int_E \phi''(t)^{1/3}dt\Big)^{\frac12}\ |T|^{1/2},$$

where $|T|$ is the two-dimensional Lebesgue measure of an arbitrary Borel subset T of $\mathbb{R}^2$. But the LHS of (5) is bounded by

$$\Big(\int_a^b\Big(\int_t^b \chi_T\big(\gamma(t)-\gamma(s)\big)\ \phi''(s)^{1/3}ds\Big)^2\phi''(t)^{1/3}dt\Big)^{\frac12}\Big(\int_E\phi''(t)^{1/3}dt\Big)^{\frac12}.$$

Thus it is enough to establish the inequality

$$\int_a^b\Big(\int_t^b \chi_T\big(\gamma(t)-\gamma(s)\big)\ \phi''(s)^{1/3}ds\Big)^2\phi''(t)^{1/3}dt\le 4\ |T|. \tag{6}$$

Inequality (6) is (2) in [O]. We repeat the short proof for the reader's convenience.

The convexity of the graph of ϕ shows that the change of variables

$$(s,t)\to\gamma(s)-\gamma(t)=\big(s-t,\phi(s)-\phi(t)\big)$$

is one-to-one. Thus

$$\int_a^b\int_a^b \chi_T\big(\gamma(t)-\gamma(s)\big)\ |\phi'(s)-\phi'(t)|\ ds\ dt\le|T|$$

and (6) will follow from the inequality

$$\phi''(t)^{1/3}\Big(\int_t^b\chi_A(s)\ \phi''(s)^{1/3}ds\Big)^2\le 4\int_t^b\chi_A(s)\big(\phi'(s)-\phi'(t)\big)\ ds \tag{7}$$

if $A\subseteq(t,b)$. To prove (7) we let $|A_u|$ stand for the (one-dimensional) Lebesgue measure of $A\cap(u,b)$ whenever $t\le u\le b$. Then

$$\int_t^b\chi_A(s)\big(\phi'(s)-\phi'(t)\big)ds=\int_t^b\chi_A(s)\int_t^s\phi''(u)\ du\ ds=\int_t^b\phi''(u)|A_u|\ du. \tag{8}$$

Also,

$$\int_t^b\chi_A(s)\ \phi''(s)^{1/3}ds=\int_t^b\chi_A(s)\ \phi''(s)^{1/3}|A_s|^{1/3}|A_s|^{-1/3}ds$$

$$\le\Big(\int_t^b\chi_A(s)\ \phi''(s)\ |A_s|\ ds\Big)^{1/3}\Big(\int_t^b\chi_A(s)|A_s|^{-1/2}ds\Big)^{2/3}.$$

Thus it follows from (8) that

$$\left(\int_t^b \chi_A(s)\phi''(s)^{1/3}ds\right)^3 \leq \left(\int_t^b \chi_A(s)\ \big(\phi'(s)-\phi'(t)\big)ds\right)\left(\int_t^b \chi_A(s)|A_s|^{-1/2}ds\right)^2. \tag{9}$$

If $0 \leq \rho \leq |A|$, then $|\{s \in A : \ |A_s| \leq \rho\}| = \rho$, and so

$$\int_t^b \chi_A(s)|A_s|^{-1/2}ds = \int_0^{|A|} y^{-1/2}dy = 2\ |A|^{1/2}.$$

With this and the fact that ϕ'' is nondecreasing, (9) yields (7) to complete the proof of the theorem.

The measure $d\lambda$ is called the affine arclength measure on the curve γ. Drury [D] was the first to point out its relevance to certain problems in harmonic analysis. In particular, [D] contains a restriction theorem using affine arclength measure on certain curves in $\mathbb{R}^3$.

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