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A CHARACTERIZATION OF BILATERAL OPERATOR WEIGHTED SHIFTS BEING COWEN-DOUGLAS OPERATORS

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ABSTRACT. It is characterized when a bilateral operator weighted shift is a Cowen-Douglas operator.

1. INTRODUCTION

Let $\mathbb{C}$ denote the complex plane, $\mathbb{C}^n = \bigoplus_{i=1}^n \mathbb{C}$ $(n < \infty)$ and $\{W_k\}_{k=-\infty}^{+\infty}$ a sequence of uniformly bounded invertible linear operators on $\mathbb{C}^n$. A bounded linear operator S on $\mathcal{K} := \bigoplus_{k=-\infty}^{+\infty} \mathbb{C}^n$ is called a (forward) bilateral operator weighted shift with the weight sequence $\{W_k\}_{k=-\infty}^{+\infty}$, denoted by $S \sim \{W_k\}_{k=-\infty}^{+\infty}$, if

$$S(\cdots, x_{-1}, \hat{x}_0, x_1, \cdots) = (\cdots, W_{-1}x_{-2}, \widehat{W_0 x_{-1}}, W_1 x_0, \cdots), \ \forall x = (x_k) \in \mathcal{K}.$$

It can be easily shown that $\mathcal{K}_+ := \bigoplus_{k=0}^{+\infty} \mathbb{C}^n$ is an invariant subspace of S. Also, $S_+ := S|_{\mathcal{K}_+}$ is called a (forward) unilateral operator weighted shift with the weight sequence $\{W_k\}_{k=1}^{+\infty}$, denoted by $S \sim \{W_k\}_{k=1}^{+\infty}$. In general, S and S_+ are called by a joint name operator weighted shift. Their adjoint operators S^* and S_+^* are referred to as backward operator weighted shifts, and n is said to be their multiplicity. Operator weighted shifts were first defined by A. Lambert [4]. When $n = 1$, they are exactly scalar weighted shifts which have been widely studied (see [6]).

First, we recall some notations and terminologies (see, for example, [2]).

Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the set of all bounded linear operators acting on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, let $\sigma(T)$ and $\sigma_p(T)$ denote the spectrum and the point spectrum of T, respectively. Set $\operatorname{nul} T = \dim\ker T$. We write $r(T)$ for the spectral radius of T and let $r_1(T) = \lim_{k\to\infty}(m(T^k))^{1/k}$, where $m(T) := \inf\{\|Tx\| : \ \|x\| = 1\}$. Recall that T is called a Fredholm operator if $\operatorname{ran} T$, the range of T, is closed and both $\operatorname{nul} T$ and $\operatorname{nul} T^*$ are finite. In this case, the intex of T is defined by $\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^*$. Moreover,

$$\rho_F(T) := \{\lambda \in \mathbb{C} : \ T - \lambda \text{ is Fredholm}\}$$

and $\sigma_e(T) := \mathbb{C}\backslash\rho_F(T)$ will denote the Fredholm domain and the essential spectrum of T, respectively.

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Let Ω be a connected open subset of $\mathbb{C}$ and m a natural number. $\mathcal{B}_m(\Omega)$ denotes the set of operators T in $\mathcal{L}(\mathcal{H})$ satisfying:

(a) $\Omega \subset \sigma(T)$,
(b) $\operatorname{ran}(T-\lambda) = \mathcal{H}$, $\forall \lambda \in \Omega$,
(c) $\bigvee\{\ker(T-\lambda) : \lambda \in \Omega\} = \mathcal{H}$ and
(d) $\operatorname{nul}(T-\lambda) = m$, $\forall \lambda \in \Omega$.

Call an operator in $\mathcal{B}_m(\Omega)$ a Cowen-Douglas operator.

Clearly, if $T \in \mathcal{B}_m(\Omega)$, then $\Omega \subset \rho_F(T)$ and $\operatorname{ind}(T-\lambda) = m$ for every $\lambda \in \Omega$.

Originally, Cowen-Douglas operators were introduced as using the method of complex geometry to developing operator theory (see [1]). However, it has been presented recently that they are closely related to the structure of bounded linear operators (see [3]).

The backward unilateral shift, i.e., all its weights W_k's are the identity operator on $\mathbb{C}$, is the simplest Cowen-Douglas operator. Thus, it is a proper problem when an operator weighted shift is a Cowen-Douglas operator. It is not hard to show $\sigma_p(S_+) = \emptyset$. Hence, S_+ cannot be a Cowen-Douglas operator. When S_+^* is a Cowen-Douglas operator has been characterized in [5]. In this note, we will prove the following theorem.

Theorem. *Let $S \sim \{W_k\}_{k=-\infty}^{+\infty}$ be a bilateral operator weighted shift with multiplicity n. Then S is a Cowen-Douglas operator if and only if there exists a $\lambda_0 \in \rho_F(S)$ such that ind$(S-\lambda_0) = n$.*

Remark 1. For $S \sim \{W_k\}_{k=-\infty}^{+\infty}$, it can be shown that S^* is unitarily equivalent to the bilateral operator weighted shift with the weight sequence $\{W_{1-k}^*\}_{k=-\infty}^{+\infty}$. Thus, the theorem is also applicable for backward bilateral operator weighted shifts.

2. Some lemmas

Note that a bilateral operator weighted shift $S \sim \{W_k\}_{k=-\infty}^{+\infty}$ can be represented as the following operator matrix:

(2.1)
$$S = \left(\begin{array}{cccc|cccc} \ddots & & & & & & & \\ & \ddots & 0 & & & 0 & & \\ & & W_{-1} & 0 & & & & \\ \hline & & & W_0 & 0 & & & \\ & & & & W_1 & 0 & & \\ & & & & & \ddots & \ddots & \end{array}\right) := \begin{pmatrix} S_- & 0 \\ F & S_+ \end{pmatrix},$$

where S_+ (S_-, resp.) is a forward (backward, resp.) unilateral operator weighted shift with the weight sequence $\{W_k\}_{k=1}^{+\infty}$ ($\{W_{-k}\}_{k=1}^{+\infty}$, resp.), and F is an n rank operator. Therefore, we can immediately obtain the following lemma.

Lemma 1. *Let S be given as in* (2.1)*. Then $\sigma_e(S) = \sigma_e(S_-) \bigcup \sigma_e(S_+)$ and*

$$\operatorname{ind}(S-\lambda) = \operatorname{ind}(S_- - \lambda) + \operatorname{ind}(S_+ - \lambda), \ \forall \lambda \in \rho_F(S). \tag{2.2}$$

In [5], $\sigma_e(S_+)$ and the indices associated with all holes in $\sigma_e(S_+)$ have been described. For convenience, we draw the following conclusions.

Lemma 2 ([5])**.** *Each S_+ is unitarily equivalent to an upper triangular operator matrix $(S_{ij})_{1\le i\le j\le n}$ on $(l^2_+)^{(n)} := \bigoplus_{i=1}^n l^2_+$, where $l^2_+ = \bigoplus_{k=0}^{+\infty}\mathbb{C}$, each S_{ij} is a unilateral scalar weighted shift, and each S_{ii} is injective. Moreover, $r(S_{ii}) \ge r(S_{(i+1)(i+1)})$, $\sigma_e(S_+) = \bigcup_{i=1}^n \sigma_e(S_{ii})$, and $\sigma(S_+) = \bigcup_{i=1}^n \sigma(S_{ii})$.*

By Theorem 4 and Theorem 6 in [6], we have that

$$\sigma(S_{ii}) = \{\lambda \in \mathbb{C} :\ |\lambda| \le r(S_{ii})\}$$

and

$$\sigma_e(S_{ii}) = \{\lambda \in \mathbb{C} :\ r_1(S_{ii}) \le |\lambda| \le r(S_{ii})\}.$$

Let Ω be a hole of $\sigma_e(S_+)$. It follows from Lemma 2 and the above formulas that one and only one of the following two situations must occur:

(a) $\Omega = \{\lambda :\ |\lambda| < \delta\}$, where $\delta = \min\{r_1(S_{ii}) :\ 1 \le i \le n\} > 0$;

(b) there exist i_0 and j_0, $1 \le i_0 < j_0 \le n$, such that $\Omega = \{\lambda :\ r(S_{j_0 j_0}) < |\lambda| < r_1(S_{i_0 i_0})\}$.

Lemma 3 ([5])**.** (1) *If the situation* (a) *occurs, then* $\operatorname{ind}(S_+ - \lambda) = -n$, $\forall \lambda \in \Omega$;

(2) *If the situation* (b) *occurs, then* $\operatorname{ind}(S_+ - \lambda) = -i_0$, $\forall\lambda \in \Omega$.

3. Proof of the Theorem and its corollaries

Now, we are ready to prove our theorem.

Proof of the Theorem. Suppose that there exists a $\lambda_0 \in \rho_F(S)$ such that $\operatorname{ind}(S - \lambda_0) = n$. Taking a basis $\{e_j\}_{j=1}^n$ of $\mathbb{C}^n$, there exists a basis $\{d_j\}_{j=1}^n$ of $\mathbb{C}^n$ such that $W_0 = \sum_{j=1}^n e_j \otimes d_j$, identifying d_j and e_j with $(\cdots, 0, d_j)$ in $\mathcal{K}_-$ and $(e_j, 0, \cdots)$ in $\mathcal{K}_+$, respectively. Then F, as in (2.1), is identical with $\sum_{j=1}^n e_j \otimes d_j$.

Since $0 \le \operatorname{nul}(S_- - \lambda_0) \le n$ and $\operatorname{nul}(S_- - \lambda_0)^* = \operatorname{nul}(S_+ - \lambda_0) = 0$ it follows from (2.1) that

$$\operatorname{nul}(S_+ - \lambda_0)^* = \operatorname{ind}(S_+ - \lambda_0) = 0$$

and

$$\operatorname{ind}(S_- - \lambda_0) = \operatorname{nul}(S_- - \lambda_0) = \operatorname{ind}(S - \lambda_0) = n.$$

Thus, we have that $\lambda_0 \notin \sigma(S_+) = \{\lambda :\ |\lambda| \le r(S_+)\}$ (see [4]). Let Ω_0 be the component of $\rho_F(S_-)$ containing λ_0. Then it follows from Lemma 3 that Ω_0 is an open disk with the center at 0. It implies that $\sigma(S_+) \subset \Omega_0$. By Proposition 2.3 in [5], moreover, we have that $S_- \in \mathcal{B}_n(\Omega_0)$. Set $\Omega = \Omega_0 \backslash \sigma(S_+)$. Then Ω is a connected open subset of $\mathbb{C}$. We will prove that $S \in \mathcal{B}_n(\Omega)$. For this purpose, it suffices to show the following statements:

(i) $\Omega \subset \rho_F(S)$;

(ii) $\operatorname{ind}(S - \lambda) = n$ and $\ker(S - \lambda)^* = \{0\}$, $\forall \lambda \in \Omega$;

(iii) $\mathcal{K} = \mathcal{K}_- \bigoplus \mathcal{K}_+ = \bigvee\{\ker(S - \lambda) :\ \lambda \in \Omega\}$.

Obviously, $\Omega \subset \rho_F(S_-) \bigcap \rho_F(S_+)$. From Lemma 1, (i) is immediate. Also, by the continuity of index, (ii) holds. Finally, it follows from Proposition 1.51 in [3] that there exist holomorphic $\mathcal{K}_-$-valued functions $\{f_i(\lambda)\}_{i=1}^n$ on Ω such that $\{f_i(\lambda)\}_{i=1}^n$ forms a basis of $\ker(S_- - \lambda)$ for each $\lambda \in \Omega$. Now, set

$$g_i(\lambda) = \sum_{j=1}^n (f_i(\lambda), d_j)(\lambda - S_+)^{-1} e_j,\ \forall \lambda \in \Omega,$$

and

$$h_i(\lambda) = f_i(\lambda) \oplus g_i(\lambda),\ \forall \lambda \in \Omega.$$

Then $h_i(\lambda) \in \ker(S-\lambda)$. To prove (iii), it suffices to show that $\mathcal{M} := \bigvee\{h_i(\lambda) : \lambda \in \Omega, 1 \le i \le n\} = \mathcal{K}$. Suppose that there exists some $x \in \mathcal{K} \ominus \mathcal{M}$. Write $x = y \oplus z$ with y in $\mathcal{K}_-$ and z in $\mathcal{K}_+$. It follows that

$$0 = (h_i(\lambda), x) = (f_i(\lambda), y) + (g_i(\lambda), z),\ \forall \lambda \in \Omega.$$

Namely,

$$(f_i(\lambda), y) = -(g_i(\lambda), z) = \sum_{j=1}^{n} (f_i(\lambda), d_j)((S_+ - \lambda)^{-1} e_j, z),\ \forall \lambda \in \Omega.$$

Thus, we have that

$$\begin{pmatrix} (f_1(\lambda), y) \\ \vdots \\ (f_n(\lambda), y) \end{pmatrix} = G(\lambda) \begin{pmatrix} ((S_+ - \lambda)^{-1} e_1, z) \\ \vdots \\ ((S_+ - \lambda)^{-1} e_n, z) \end{pmatrix},\ \forall \lambda \in \Omega,$$

where $G(\lambda) = ((f_i(\lambda), d_j))_{1 \le i,j \le n}$ is a Gram matrix. So, it is invertible because both $\{f_i(\lambda)\}_{i=1}^n$ and $\{d_j\}_{j=1}^n$ are linear independent. It implies that

$$(G(\lambda))^{-1} \begin{pmatrix} (f_1(\lambda), y) \\ \vdots \\ (f_n(\lambda), y) \end{pmatrix} = \begin{pmatrix} ((S_+ - \lambda)^{-1} e_1, z) \\ \vdots \\ ((S_+ - \lambda)^{-1} e_n, z) \end{pmatrix},\ \forall \lambda \in \Omega.$$

The left side of the equality is holomorphic on Ω_0. Meanwhile, the right side is holomorphic on $\{\lambda : \ |\lambda| > r(S_+)\}$ and $((S_+ - \lambda)^{-1} e_i, z) \longrightarrow 0$ $(\lambda \longrightarrow \infty)$ for $i = 1, 2, \cdots, n$. Hence, it follows from the Liouville Theorem that

$$(f_i(\lambda), y) = 0,\ \forall \lambda \in \Omega_0,\ i = 1, 2, \cdots, n,$$

and

$$((\lambda - S_+)^{-1} e_i, z) = 0,\ \forall \lambda \in \mathbb{C} \setminus \sigma(S_+),\ i = 1, 2, \cdots, n.$$

However, $\bigvee\{f_i(\lambda) :\ \lambda \in \Omega_0,\ 1 \le i \le n\} = \bigvee\{\ker(S_- - \lambda) :\ \lambda \in \Omega_0\} = \mathcal{K}_-$. So, $y = 0$. Also, note that

$$0 = ((\lambda - S_+)^{-1} e_i, z) = \sum_{k=0}^{+\infty} (S_+^k e_i, z) \lambda^{-(k+1)},\ \forall \lambda > \|S_+\|.$$

It follows that $(S_+^k e_i, z) = 0$ for all $k \ge 0$ and $1 \le i \le n$. Moreover, it can be verified that $\bigvee\{S_+^k e_i :\ k \ge 0,\ 1 \le i \le n\} = \mathcal{K}_+$. Thus, $z = 0$. This proves that $\mathcal{M} = \mathcal{K}$.

Conversely, if S is a Cowen-Douglas operator, i.e., $S \in \mathcal{B}_m(\Omega)$, then it is clear that $m \le n$. Imitating the case $n = 1$, we can show that $\sigma(S)$, $\sigma_e(S)$ and $\sigma_p(S)$ are circular symmetric about the origin. Thus, without loss of generality, we can assume that Ω is an annular domain that contains 1. For each natural number l, set $\lambda_l = e^{-i\frac{\pi}{l}}$, where i denotes the imaginary unit. Then $\lambda_l \in \Omega$ for all l and $\lim_{l \longrightarrow \infty} \lambda_l = 1$. Now, it follows from Proposition 1.41 in [3] that

$$\bigvee\{\ker(S - \lambda_l) :\ l \ge 1\} = \bigvee\{\ker(S - \lambda) :\ \lambda \in \Omega\} = \mathcal{K}.$$

Taking a basis $\{f_j^{(0)}\}_{j=1}^m$ of $\ker(S-1)$, let $f_j^{(0)} = (x_k^{(j)})_{-\infty<k<+\infty}$, where $x_k^{(j)} \in \mathbb{C}^n$ for all k. Then $Sf_j^{(0)} = f_j^{(0)}$ implies that $W_{k+1}x_k^{(j)} = x_{k+1}^{(j)}$ for all k and $1 \le j \le m$. Since $\sum_{k=-\infty}^{+\infty} \|e^{i\frac{k\pi}{l}} x_k^{(j)}\|^2 = \sum_{k=-\infty}^{+\infty} \|x_k^{(j)}\|^2 < \infty$, $f_j^{(l)} := (e^{i\frac{k\pi}{l}} x_k^{(j)})_{-\infty<k<+\infty}$ is in $\mathcal{K}$ and

$$W_{k+1}e^{i\frac{k\pi}{l}}x_k^{(j)} = e^{i\frac{k\pi}{l}}W_{k+1}x_k^{(j)} = e^{i\frac{k\pi}{l}}x_{k+1}^{(j)} = \lambda_l e^{i\frac{(k+1)\pi}{l}}x_{k+1}^{(j)}.$$

Thus, $Sf_j^{(l)} = \lambda_l f_j^{(l)}$. Clearly, $\{f_1^{(l)}, \cdots, f_m^{(l)}\}$ is linear independent. So, it is a basis of $\ker(S-\lambda_l)$. Hence,

$$\bigvee\{f_j^{(l)} :\ l \ge 1,\ 1 \le j \le m\} = \mathcal{K}.$$

Assume that $m < n$. Then there exists a non-zero vector $x \in \mathbb{C}^n$ such that $x \perp x_0^{(j)}$ for $j = 1, 2, \cdots, m$. So, $x \perp \bigvee\{f_j^{(l)} :\ l \ge 1,\ 1 \le j \le m\}$. This contradiction completes the proof. □

By the Theorem and its proof, we immediately get the following corollaries.

Corollary 1. *Let S be a bilateral scalar weighted shift. Then either S or S^* is a Cowen-Douglas operator if and only if $\sigma_e(S) \ne \sigma(S)$.*

Corollary 2. *Let S be a bilateral operator weighted shift. If either S or S^* is a Cowen-Douglas operator, then $\sigma_e(S)$ is not connected.*

Remark 2. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be strongly irreducible if it does not commute with any nontrivial idempotents. A lot of work has been done studying strongly irreducible operators (cf. [3]). By the Riesz decomposition theorem, if $\sigma(T)$ is not connected, then T is not strongly irreducible. For a unilateral operator weighted shift S_+, it follows from Theorem 3.1 in [5] that if $\sigma_e(S_+)$ is not connected, then S_+ is not strongly irreducible. For a bilateral operator weighted shift S, however, the following example shows that a similar claim is false even though S is a Cowen-Douglas operator.

Example. Let $w_k = 2$ for $k < 0$, $w_k = 1$ for $k \ge 0$ and let $S \sim \{w_k\}_{k=-\infty}^{+\infty}$ be a bilateral scalar weighted shift. By Lemma 1, $\sigma_e(S) = \{\lambda :\ |\lambda| = 1 \text{ or } 2\}$, which is not connected. Also, by Theorem 9 in [6], $\operatorname{ind}(S-\lambda) = 1$ for all $\lambda \in \Omega := \{\lambda :\ 1 < |\lambda| < 2\}$. Thus, it follows from our theorem that $S \in \mathcal{B}_1(\Omega)$. Hence, S is strongly irreducible (see [2]).

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