A CHARACTERIZATION OF BILATERAL OPERATOR WEIGHTED SHIFTS BEING COWEN-DOUGLAS OPERATORS

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Abstract. It is characterized when a bilateral operator weighted shift is a Cowen-Douglas operator.

1. Introduction

Let $\mathbb{C}$ denote the complex plane, $\mathbb{C}^n = \bigoplus_{k=1}^{n} \mathbb{C}$ ($n < \infty$) and $\{W_k\}_{k=-\infty}^{+\infty}$ a sequence of uniformly bounded invertible linear operators on $\mathbb{C}^n$. A bounded linear operator $S$ on $K := \bigoplus_{k=-\infty}^{+\infty} \mathbb{C}^n$ is called a (forward) bilateral operator weighted shift with the weight sequence $\{W_k\}_{k=-\infty}^{+\infty}$, denoted by $S \sim \{W_k\}_{k=-\infty}^{+\infty}$, if

$$S(\cdots, x_{-1}, x_0, x_1, \cdots) = (\cdots, W_{-1}x_{-2}, W_0x_{-1}, W_1x_0, \cdots), \quad \forall x = (x_k) \in K.$$ 

It can be easily shown that $K_+ := \bigoplus_{k=0}^{+\infty} \mathbb{C}^n$ is an invariant subspace of $S$. Also, $S_+ := S|_{K_+}$ is called a (forward) unilateral operator weighted shift with the weight sequence $\{W_k\}_{k=1}^{+\infty}$, denoted by $S \sim \{W_k\}_{k=1}^{+\infty}$. In general, $S$ and $S_+$ are called by a joint name operator weighted shift. Their adjoint operators $S^*$ and $S_+^*$ are referred to as backward operator weighted shifts, and $n$ is said to be their multiplicity. Operator weighted shifts were first defined by A. Lambert [4]. When $n = 1$, they are exactly scalar weighted shifts which have been widely studied (see [6]).

First, we recall some notations and terminologies (see, for example, [2]).

Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the set of all bounded linear operators acting on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, let $\sigma(T)$ and $\sigma_p(T)$ denote the spectrum and the point spectrum of $T$, respectively. Set $\text{null } T = \dim \ker T$. We write $r(T)$ for the spectral radius of $T$ and let $r_1(T) = \lim_{k \to \infty} (m(T^k))^{1/k}$, where $m(T) := \inf\{\|Tx\| : \|x\| = 1\}$. Recall that $T$ is called a Fredholm operator if $\text{ran } T$, the range of $T$, is closed and both $\text{null } T$ and $\text{null } T^*$ are finite. In this case, the index of $T$ is defined by $\text{ind } T = \text{null } T - \text{null } T^*$. Moreover,

$$\rho_F(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is Fredholm}\}$$

and $\sigma_c(T) := \mathbb{C} \setminus \rho_F(T)$ will denote the Fredholm domain and the essential spectrum of $T$, respectively.

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Let \( \Omega \) be a connected open subset of \( \mathbb{C} \) and \( m \) a natural number. \( B_m(\Omega) \) denotes the set of operators \( T \) in \( \mathcal{L}(\mathcal{H}) \) satisfying:

(a) \( \Omega \subset \sigma(T) \),
(b) \( \text{ran } (T - \lambda) = \mathcal{H}, \forall \lambda \in \Omega \),
(c) \( \bigvee \{ \ker (T - \lambda) : \lambda \in \Omega \} = \mathcal{H} \) and
(d) \( \text{mul } (T - \lambda) = m, \forall \lambda \in \Omega \).

Call an operator in \( B_m(\Omega) \) a Cowen-Douglas operator.

Clearly, if \( T \in B_m(\Omega) \), then \( \rho_F(T) \) and \( \text{ind } (T - \lambda) = m \) for every \( \lambda \in \Omega \).

Originally, Cowen-Douglas operators were introduced as using the method of complex geometry to developing operator theory (see [1]). However, it has been presented recently that they are closely related to the structure of bounded linear operators (see [3]).

The backward unilateral shift, i.e., all its weights \( W_k \)'s are the identity operator on \( \mathbb{C} \), is the simplest Cowen-Douglas operator. Thus, it is a proper problem when an operator weighted shift is a Cowen-Douglas operator. It is not hard to show \( \sigma_p(S) = \emptyset \). Hence, \( S_+ \) cannot be a Cowen-Douglas operator. When \( S_+^* \) is a Cowen-Douglas operator has been characterized in [5]. In this note, we will prove the following theorem.

**Theorem.** Let \( S \sim \{W_k\}_{k=-\infty}^{+\infty} \) be a bilateral operator weighted shift with multiplicity \( n \). Then \( S \) is a Cowen-Douglas operator if and only if there exists a \( \lambda_0 \in \rho_F(S) \) such that \( \text{ind } (S - \lambda_0) = n \).

**Remark 1.** For \( S \sim \{W_k\}_{k=-\infty}^{+\infty} \), it can be shown that \( S^* \) is unitarily equivalent to the bilateral operator weighted shift with the weight sequence \( \{W_{1-k}\}_{k=-\infty}^{+\infty} \). Thus, the theorem is also applicable for backward bilateral operator weighted shifts.

## 2. Some lemmas

Note that a bilateral operator weighted shift \( S \sim \{W_k\}_{k=-\infty}^{+\infty} \) can be represented as the following operator matrix:

\[
S = \begin{pmatrix}
\vdots & \vdots & \vdots \\
\cdots & 0 & 0 \\
\cdots & \text{W}_{-1} & 0 \\
\text{W}_0 & 0 & \text{W}_1 \\
\vdots & \vdots & \text{\ddots}
\end{pmatrix} := \begin{pmatrix}
\text{S}_{-} & 0 \\
0 & \text{S}_{+}
\end{pmatrix},
\]

where \( S_{+} (S_{-}, \text{resp.}) \) is a forward (backward, resp.) unilateral operator weighted shift with the weight sequence \( \{W_k\}_{k=1}^{+\infty} \) (\( \{W_{-k}\}_{k=-\infty}^{+\infty} \), resp.), and \( F \) is an \( n \) rank operator. Therefore, we can immediately obtain the following lemma.

**Lemma 1.** Let \( S \) be given as in (2.1). Then \( \sigma_e(S) = \sigma_e(S_{-}) \cup \sigma_e(S_{+}) \) and

\[
\text{ind } (S - \lambda) = \text{ind } (S_{-} - \lambda) + \text{ind } (S_{+} - \lambda), \quad \forall \lambda \in \rho_F(S).
\]
In [5], \( \sigma_e(S_+) \) and the indices associated with all holes in \( \sigma_e(S_+) \) have been described. For convenience, we draw the following conclusions.

Lemma 2 ([5]). Each \( S_+ \) is unitarily equivalent to an upper triangular operator matrix \((S_{ij})_{1 \leq i \leq j \leq n} \) on \((l_2^+)^{n} \) such that \( l_2^+ = \bigoplus_{k=0}^{\infty} \mathbb{C} \), each \( S_{ij} \) is a unilateral scalar weighted shift, and each \( S_{ii} \) is injective. Moreover, \( r(S_{ii}) \geq r(S_{i(i+1)}) \), \( \sigma_e(S_+) = \bigcup_{j=1}^{n} \sigma_e(S_{ii}) \), and \( \sigma(S_+) = \bigcup_{j=1}^{n} \sigma(S_{ii}) \).

By Theorem 4 and Theorem 6 in [6], we have that
\[
\sigma(S_{ii}) = \{ \lambda \in \mathbb{C} : |\lambda| \leq r(S_{ii}) \}
\]
and
\[
\sigma_e(S_{ii}) = \{ \lambda \in \mathbb{C} : r_1(S_{ii}) \leq |\lambda| \leq r(S_{ii}) \}.
\]

Let \( \Omega \) be a hole of \( \sigma_e(S_+) \). It follows from Lemma 2 and the above formulas that one and only one of the following two situations must occur:

(a) \( \Omega = \{ \lambda : |\lambda| < \delta \} \), where \( \delta = \min\{r_1(S_{ii}) : 1 \leq i \leq n \} > 0 \);

(b) there exist \( i_0 \) and \( j_0 \), \( 1 \leq i_0 < j_0 \leq n \), such that \( \Omega = \{ \lambda : r(S_{j_0j_0}) < |\lambda| < r_1(S_{i_0i_0}) \} \).

Lemma 3 ([5]).

(1) If the situation (a) occurs, then \( \text{ind}(S_+ - \lambda) = -n \), \( \forall \lambda \in \Omega \);

(2) If the situation (b) occurs, then \( \text{ind}(S_+ - \lambda) = -i_0 \), \( \forall \lambda \in \Omega \).

3. Proof of the Theorem and its Corollaries

Now, we are ready to prove our theorem.

Proof of the Theorem. Suppose that there exists a \( \lambda_0 \in \rho_F(S) \) such that \( \text{ind}(S - \lambda_0) = n \). Taking a basis \( \{e_j\}_{j=1}^{n} \) of \( \mathbb{C}^n \), there exists a basis \( \{d_j\}_{j=1}^{n} \) of \( \mathbb{C}^n \) such that \( W_0 = \sum_{j=1}^{n} e_j \otimes d_j \), identifying \( d_j \) and \( e_j \) with \( (\cdots, 0, d_j) \) in \( K_- \) and \( (e_j, 0, \cdots) \) in \( K_+ \), respectively. Then \( F \), as in (2.1), is identical with \( \sum_{j=1}^{n} e_j \otimes d_j \).

Since \( 0 \leq \text{nul}(S - \lambda_0)^* \leq n \) and \( \text{nul}(S - \lambda_0)^* = \text{nul}(S + \lambda_0) = 0 \) it follows from (2.1) that
\[
\text{nul}(S + \lambda_0)^* = \text{ind}(S + \lambda_0) = 0
\]
and
\[
\text{ind}(S - \lambda_0) = \text{nul}(S - \lambda_0) = \text{ind}(S - \lambda_0) = n.
\]
Thus, we have that \( \lambda_0 \notin \sigma(S_+) = \{ \lambda : |\lambda| \leq r(S_+) \} \) (see [4]). Let \( \Omega_0 \) be the component of \( \rho_F(S_-) \) containing \( \lambda_0 \). Then it follows from Lemma 3 that \( \Omega_0 \) is an open disk with the center at 0. It implies that \( \sigma(S_+) \subset \Omega_0 \). By Proposition 2.3 in [3], moreover, we have that \( S_- \in B_n(\Omega_0) \). Set \( \Omega = \Omega_0 \setminus \sigma(S_+) \). Then \( \Omega \) is a connected open subset of \( \mathbb{C} \). We will prove that \( S \in B_n(\Omega) \). For this purpose, it suffices to show the following statements:

(i) \( \Omega \subset \rho_F(S) \);

(ii) \( \text{ind}(S - \lambda) = n \) and \( \ker(S - \lambda)^* = \{0\} \), \( \forall \lambda \in \Omega \);

(iii) \( K = K_- \bigoplus K_+ = \bigcup \{ \ker(S - \lambda) : \lambda \in \Omega \} \).

Obviously, \( \Omega \subset \rho_F(S_-) \cap \rho_F(S_+) \). From Lemma 1, (i) is immediate. Also, by the continuity of index, (ii) holds. Finally, it follows from Proposition 1.5.1 in [3] that there exist holomorphic \( K_- \)-valued functions \( \{f_i(\lambda)\}_{i=1}^{n} \) on \( \Omega \) such that \( \{f_i(\lambda)\}_{i=1}^{n} \) forms a basis of \( \ker(S_- - \lambda) \) for each \( \lambda \in \Omega \). Now, set
\[
g_i(\lambda) = \sum_{j=1}^{n} (f_i(\lambda), d_j)(\lambda - S_+)^{-1}e_j, \quad \forall \lambda \in \Omega,
\]
and 
\[ h_i(\lambda) = f_i(\lambda) \oplus g_i(\lambda), \quad \forall \lambda \in \Omega. \]

Then \( h_i(\lambda) \in \ker (S - \lambda). \) To prove (iii), it suffices to show that \( \mathcal{M} := \bigvee \{ h_i(\lambda) : \lambda \in \Omega, 1 \leq i \leq n \} = \mathcal{K}. \) Suppose that there exists some \( x \in \mathcal{K} \ominus \mathcal{M}. \) Write \( x = y + z \) with \( y \) in \( \mathcal{K}_- \) and \( z \) in \( \mathcal{K}_+. \) It follows that
\[ 0 = (h_i(\lambda), x) = (f_i(\lambda), y) + (g_i(\lambda), z), \quad \forall \lambda \in \Omega. \]

Namely,
\[ (f_i(\lambda), y) = -(g_i(\lambda), z) = \sum_{j=1}^{n} (f_i(\lambda), d_j)((S_+ - \lambda)^{-1}e_j, z), \quad \forall \lambda \in \Omega. \]

Thus, we have that
\[ \left( \begin{array}{c}
(f_1(\lambda), y) \\
\vdots \\
(f_n(\lambda), y)
\end{array} \right) = G(\lambda) \left( \begin{array}{c}
((S_+ - \lambda)^{-1}e_1, z) \\
\vdots \\
((S_+ - \lambda)^{-1}e_n, z)
\end{array} \right), \quad \forall \lambda \in \Omega,
\]
where \( G(\lambda) = (f_i(\lambda), d_j))_{1 \leq i, j \leq n} \) is a Gram matrix. So, it is invertible because both \( \{f_i(\lambda)\}_{n=1}^{n} \) and \( \{d_j\}_{j=1}^{n} \) are linear independent. It implies that
\[ (G(\lambda))^{-1} \left( \begin{array}{c}
(f_1(\lambda), y) \\
\vdots \\
(f_n(\lambda), y)
\end{array} \right) = \left( \begin{array}{c}
((S_+ - \lambda)^{-1}e_1, z) \\
\vdots \\
((S_+ - \lambda)^{-1}e_n, z)
\end{array} \right), \quad \forall \lambda \in \Omega.
\]

The left side of the equality is holomorphic on \( \Omega_0. \) Meanwhile, the right side is holomorphic on \( \{\lambda : |\lambda| > r(S_+)\} \) and \( ((S_+ - \lambda)^{-1}e_i, z) \rightarrow 0 (\lambda \rightarrow \infty) \) for \( i = 1, 2, \ldots, n. \) Hence, it follows from the Liouville Theorem that
\[ (f_i(\lambda), y) = 0, \quad \forall \lambda \in \Omega_0, \quad i = 1, 2, \ldots, n,
\]
and
\[ ((\lambda - S_+)^{-1}e_i, z) = 0, \quad \forall \lambda \in \mathbb{C} \setminus \sigma(S_+), \quad i = 1, 2, \ldots, n.
\]

However, \( \bigvee \{f_i(\lambda) : \lambda \in \Omega_0, 1 \leq i \leq n\} = \bigvee \{\ker (S_- - \lambda) : \lambda \in \Omega_0\} = \mathcal{K}_-. \) So, \( y = 0. \) Also, note that
\[ 0 = ((\lambda - S_+)^{-1}e_i, z) = \sum_{k=0}^{+\infty} (S_+^k e_i, z)\lambda^{-(k+1)}, \quad \forall \lambda > \|S_+\|.
\]

It follows that \( (S_+^k e_i, z) = 0 \) for all \( k \geq 0 \) and \( 1 \leq i \leq n. \) Moreover, it can be verified that \( \bigvee \{S_+^k e_i : k \geq 0, 1 \leq i \leq n\} = \mathcal{K}_+. \) Thus, \( z = 0. \) This proves that \( \mathcal{M} = \mathcal{K}. \)

Conversely, if \( S \) is a Cowen-Douglas operator, i.e., \( S \in \mathcal{B}_m(\Omega), \) then it is clear that \( m \leq n. \) Imitating the case \( n = 1, \) we can show that \( \sigma(S), \sigma_e(S) \) and \( \sigma_{e}(S) \) are circular symmetric about the origin. Thus, without loss of generality, we can assume that \( \Omega \) is an annular domain that contains \( 1. \) For each natural number \( l, \) set \( \lambda_l = e^{-i\pi l}, \) where \( i \) denotes the imaginary unit. Then \( \lambda_l \in \Omega \) for all \( l \) and \( \lim_{l \to -\infty} \lambda_l = 1. \) Now, it follows from Proposition 1.41 in [3] that
\[ \bigvee \{\ker (S - \lambda_l) : l \geq 1\} = \bigvee \{\ker (S - \lambda) : \lambda \in \Omega\} = \mathcal{K}.\]
Taking a basis \( \{ f_j^{(0)} \}_{j=1}^m \) of \( \ker (S - 1) \), let \( f_j^{(0)} = (x_k^{(j)})_{-\infty < k < +\infty} \), where \( x_k^{(j)} \in \mathbb{C}^n \) for all \( k \). Then \( S f_j^{(0)} = f_j^{(0)} \) implies that \( W_{k+1} x_k^{(j)} = x_{k+1}^{(j)} \) for all \( k \) and \( 1 \leq j \leq m \). Since \( \sum_{k=-\infty}^{+\infty} \| e^{i\frac{\pi}{2} x_k^{(j)}} \|^2 = \sum_{k=-\infty}^{+\infty} \| x_k^{(j)} \|^2 < \infty \), \( f_j^{(0)} := (e^{i\frac{\pi}{2} x_k^{(j)})}_{-\infty < k < +\infty} \) is in \( \mathcal{K} \) and
\[
W_{k+1} e^{i\frac{\pi}{2} x_k^{(j)}} = e^{i\frac{\pi}{2} W_{k+1} x_k^{(j)}} = e^{i\frac{\pi}{2} x_{k+1}^{(j)}} = \lambda_l e^{i(\frac{\pi}{2} + l) x_k^{(j)}} x_{k+1}^{(j)}.
\]
Thus, \( S f_j^{(l)} = \lambda_l f_j^{(l)} \). Clearly, \( \{ f_1^{(l)}, \cdots, f_m^{(l)} \} \) is linear independent. So, it is a basis of \( \ker (S - \lambda_l) \).

Hence,
\[
\bigvee \{ f_j^{(l)} : l \geq 1, 1 \leq j \leq m \} = \mathcal{K}.
\]
Assume that \( m < n \). Then there exists a non-zero vector \( x \in \mathbb{C}^n \) such that \( x \perp x_0^{(j)} \) for \( j = 1, 2, \cdots, m \). So, \( x \perp \bigvee \{ f_j^{(l)} : l \geq 1, 1 \leq j \leq m \} \). This contradiction completes the proof.

By the Theorem and its proof, we immediately get the following corollaries.

**Corollary 1.** Let \( S \) be a bilateral scalar weighted shift. Then either \( S \) or \( S^* \) is a Cowen-Douglas operator if and only if \( \sigma_e(S) \neq \sigma(S) \).

**Corollary 2.** Let \( S \) be a bilateral operator weighted shift. If either \( S \) or \( S^* \) is a Cowen-Douglas operator, then \( \sigma_e(S) \) is not connected.

**Remark 2.** An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to be strongly irreducible if it does not commute with any nontrivial idempotents. A lot of work has been done studying strongly irreducible operators (cf. [3]). By the Riesz decomposition theorem, if \( \sigma(T) \) is not connected, then \( T \) is not strongly irreducible. For a unilateral operator weighted shift \( S_+ \), it follows from Theorem 3.1 in [5] that if \( \sigma_e(S_+) \) is not connected, then \( S_+ \) is not strongly irreducible. For a bilateral operator weighted shift \( S \), however, the following example shows that a similar claim is false even though \( S \) is a Cowen-Douglas operator.

**Example.** Let \( w_k = 2 \) for \( k < 0 \), \( w_k = 1 \) for \( k \geq 0 \) and let \( S \sim \{ w_k \}_{k=+\infty}^{-\infty} \) be a bilateral scalar weighted shift. By Lemma 1, \( \sigma_e(S) = \{ \lambda : |\lambda| = 1 \text{ or } 2 \} \), which is not connected. Also, by Theorem 9 in [6], \( \text{ind} (S - \lambda) = 1 \) for all \( \lambda \in \Omega := \{ \lambda : 1 < |\lambda| < 2 \} \). Thus, it follows from our theorem that \( S \in B_1(\Omega) \). Hence, \( S \) is strongly irreducible (see [2]).

**References**


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