PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 11, Pages 3307–3311
S 0002-9939(01)06042-7
Article electronically published on April 2, 2001

# THE STRUCTURE OF QUANTUM SPHERES

ALBERT JEU-LIANG SHEU

(Communicated by David R. Larson)

Abstract. We show that the C*-algebra $C\left(\mathbb{S}_q^{2n+1}\right)$ of a quantum sphere $\mathbb{S}_q^{2n+1}$, $q>1$, consists of continuous fields $\{f_t\}_{t\in\mathbb{T}}$ of operators $f_t$ in a C*-algebra $\mathcal{A}$, which contains the algebra $\mathcal{K}$ of compact operators with $\mathcal{A}/\mathcal{K}\cong C\left(\mathbb{S}_q^{2n-1}\right)$, such that $\rho_*\left(f_t\right)$ is a constant function of $t\in\mathbb{T}$, where $\rho_*:\mathcal{A}\to\mathcal{A}/\mathcal{K}$ is the quotient map and $\mathbb{T}$ is the unit circle.

## Introduction

Some interesting C*-algebras that arise from geometric objects have been successfully studied, using the groupoid C*-algebraic approach [R, CM, MR, SaShU, Sh1, Sh2]. In particular, the C*-algebra $C\left(\mathbb{S}_q^{2n+1}\right)$ of a quantum sphere $\mathbb{S}_q^{2n+1}$ [VSo], $q>1$, was realized as a concrete groupoid C*-algebra $C^*\left(\mathfrak{F}_n\right)$ independent of $q$ [Sh3]. Decomposing the underlying groupoid $\mathfrak{F}_n$, we were able to conclude that $C\left(\mathbb{S}_q^{2n+1}\right)$ is an extension of $C\left(\mathbb{S}_q^{2n-1}\right)$ by $C\left(\mathbb{T}\right)\otimes\mathcal{K}$, which well reflects, at the quantum level, the symplectic leaf space structure [W] of the $SU\left(n+1\right)$-homogeneous Poisson $\mathbb{S}^{2n+1}$ [D] because $\mathbb{S}_q^{2n+1}\backslash\mathbb{S}_q^{2n-1}$ is a disjoint union of a $\mathbb{T}$-family of symplectic leaves $\mathbb{C}^n$, where $\mathbb{T}$ is the unit circle. However since the extensions of C*-algebras are usually not unique, the algebra $C\left(\mathbb{S}_q^{2n+1}\right)$ is not completely determined up to isomorphism. In this paper, we find an explicit recursive description that completely determines the algebra $C\left(\mathbb{S}_q^{2n+1}\right)$ up to isomorphism. This description would be very useful, for example, in the study of the cancellation problem of "vector bundles" over $\mathbb{S}_q^{2n+1}$.

## 1. Quantum sphere and groupoid

In this section, we identify the C*-algebra $C\left(\mathbb{S}_q^{2n+1}\right)$ of a quantum sphere $\mathbb{S}_q^{2n+1}$, $q>1$, with a concrete groupoid C*-algebra $C^*\left(\mathfrak{G}_n\right)$ of a concrete groupoid $\mathfrak{G}_n$, independent of $q$, whose description is simpler and easier to handle than that of $\mathfrak{F}_n$ found in [Sh3]. For the background material of groupoid and group C*-algebras, we refer readers to the books of Renault [R] and Pedersen [P].

Recall that the C*-algebra of the quantum group $SU(n)_q$ is generated by elements $u_{ij}$ satisfying certain commutation relations and the C*-algebra of quantum

Received by the editors March 15, 2000.
2000 *Mathematics Subject Classification.* Primary 46L05; Secondary 17B37, 46L89, 47B35, 58B32, 81R50.
*Key words and phrases.* Quantum spheres, quantum groups, groupoid C*-algebras, Toeplitz algebras.
The author was partially supported by NSF Grant DMS-9623008.

spheres $S_q^{2n+1} = SU(n)_q \backslash SU(n+1)_q$ defined as homogeneous quantum spaces [N] can be identified with

$$C(S_q^{2n+1}) = C^*(\{u_{n+1,m} | \, 1 \leq m \leq n+1\}).$$

Let $\mathbb{Z}_{\geq} = \mathbb{N} \cup \{0\}$, and regard $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{+\infty\}$ and $\overline{\mathbb{Z}}_{\geq} := \mathbb{Z}_{\geq} \cup \{+\infty\}$ as topological spaces with their canonical topologies. We use $\mathcal{H}^n := \mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n|_{\overline{\mathbb{Z}}^n_{\geq}}$ to denote the transformation group groupoid $\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n$ restricted to the positive "cone" $\overline{\mathbb{Z}}^n_{\geq}$ of its unit space $\overline{\mathbb{Z}}^n$, and use $\mathcal{F}^n = \mathbb{Z} \times \left(\mathbb{Z}^n \ltimes \overline{\mathbb{Z}}^n|_{\overline{\mathbb{Z}}^n_{\geq}}\right)$ to denote the direct product of the group $\mathbb{Z}$ and the groupoid $\mathcal{H}^n$ [R, MR, CM].

Let $\approx$ be the equivalence relation on $\overline{\mathbb{Z}}^n_{\geq} := \left(\overline{\mathbb{Z}}_{\geq}\right)^n$ that is generated by $w \approx w'$ for $w, w' \in \overline{\mathbb{Z}}^n_{\geq}$ such that for some $1 \leq i \leq n$, $w_j = w'_j$ for all $j \leq i$ and $w'_j = \infty$ for all $j \geq i$. This equivalence relation can be canonically extended to equivalence relations $\sim$ on spaces like $\mathcal{H}^n$ or $\mathcal{F}^n$ by defining $(x,w) \sim (x',w')$ if and only if $x = x'$ and $w \approx w'$ for $(x,w), (x',w') \in \mathcal{H}^n$, and $(z,x,w) \sim (z',x',w')$ if and only if $(z,x) = (z',x')$, and $w \approx w'$ for $(z,x,w), (z',x',w') \in \mathcal{F}^n$.

It is proved in [Sh3] that $C(S_q^{2n+1}) \simeq C^*(\mathfrak{F}_n)$ with $\mathfrak{F}_n := \widetilde{\mathfrak{F}_n}/\sim$ a subquotient groupoid of $\mathcal{F}^n$ where

$$\widetilde{\mathfrak{F}_n} := \{(z,x,w) \in \mathcal{F}^n | \text{ for any } 1 \leq i \leq n, \text{ if } w_i = \infty, \text{ then}$$

$$x_i = -z - x_1 - x_2 - ... - x_{i-1} \text{ and } x_{i+1} = ... = x_n = 0\}$$

is a subgroupoid of $\mathcal{F}^n$.

We first note that by a "change of variables" $k := z + x_1 + x_2 + ... + x_n$, the conditions

$$x_i = -z - x_1 - x_2 - ... - x_{i-1} \text{ and } x_{i+1} = ... = x_n = 0$$

in defining $\widetilde{\mathfrak{F}_n}$, can be replaced by

$$k = 0 \text{ and } x_{i+1} = ... = x_n = 0.$$

More precisely, the bijection

$$(z,x,w) \mapsto (z + x_1 + x_2 + ... + x_n, x, w)$$

defines a homeomorphic groupoid isomorphism from $\widetilde{\mathfrak{F}_n}$ to the subgroupoid

$$\widetilde{\mathfrak{G}_n} := \{(k,x,w) \in \mathcal{F}^n | \text{ for any } 1 \leq i \leq n, \text{ if } w_i = \infty,$$

$$\text{then } k = 0 = x_{i+1} = ... = x_n\}$$

of $\mathcal{F}^n$. Defining $\mathfrak{G}_n := \widetilde{\mathfrak{G}_n}/\sim$, we get a groupoid $\mathfrak{G}_n$ isomorphic to $\mathfrak{F}_n$ since the above groupoid isomorphism preserves the equivalence relation $\sim$.

**Proposition 1.** *For $q > 1$,*

$$C(S_q^{2n+1}) \simeq C^*(\mathfrak{G}_n).$$

## 2. Structure theorem

In this section, we recursively characterize $C(S_q^{2n+1})$ as an algebra of fields of operators and hence determine $C(S_q^{2n+1})$ up to isomorphism.

We first note that $\widetilde{\mathfrak{G}_n} \subset \mathbb{Z} \times \widetilde{\mathfrak{H}_n} \subset \mathcal{F}^n$ and

$$\mathfrak{G}_n \subset \mathbb{Z} \times \mathfrak{H}_n$$

where $\widetilde{\mathfrak{H}_n}$ is the subgroupoid

$$\widetilde{\mathfrak{H}_n} := \{(x,w) \in \mathcal{H}^n \mid \text{for any } 1 \le i \le n, \text{ if } w_i = \infty,$$

$$\text{then } x_{i+1} = ... = x_n = 0\}$$

of $\mathcal{H}^n$ and $\mathfrak{H}_n := \widetilde{\mathfrak{H}_n}/\sim$. The unit space of $\widetilde{\mathfrak{H}_n}$ (or $\mathbb{Z} \times \widetilde{\mathfrak{H}_n}$, or $\widetilde{\mathfrak{G}_n}$) is $\widetilde{W} := \overline{\mathbb{Z}}_{\ge}^n$ while the unit space of $\mathfrak{H}_n$ (or $\mathbb{Z} \times \mathfrak{H}_n$, or $\mathfrak{G}_n$) is the quotient space $W := \widetilde{W}/\approx$.

The closed subset $\widetilde{W}_n := \overline{\mathbb{Z}}_{\ge}^n \backslash \mathbb{Z}_{\ge}^n$ of $\widetilde{W}$ and its complement $\widetilde{W} \backslash \widetilde{W}_n = \mathbb{Z}_{\ge}^n$ are closed under the equivalence relation $\approx$ and are invariant (under the $\widetilde{\mathfrak{H}_n}$-action) subsets of $\widetilde{W}$. Correspondingly, we have the closed subset $W_n := \widetilde{W}_n/\approx$ of $W$ and its complement $W \backslash W_n$ as invariant subsets of the unit space $W$ of $\mathfrak{H}_n$. By the general theory of groupoid C*-algebras [R], we have the short exact sequence

$$0 \to C^*\left(\mathfrak{H}_n|_{W\backslash W_n}\right) \xrightarrow{\iota_*} C^*\left(\mathfrak{H}_n\right) \xrightarrow{\rho_*} C^*\left(\mathfrak{H}_n|_{W_n}\right) \to 0$$

where $\rho_*$ is induced by the restriction map $\rho$ on $C_c\left(\mathfrak{H}_n\right)$ and $\iota_*$ is induced by the inclusion map $\iota$ on $C_c\left(\mathfrak{H}_n|_{W\backslash W_n}\right)$, and similarly the short exact sequence

$$0 \to C^*\left(\left(\mathbb{Z} \times \mathfrak{H}_n\right)|_{W\backslash W_n}\right) \to C^*\left(\mathbb{Z} \times \mathfrak{H}_n\right) \to C^*\left(\left(\mathbb{Z} \times \mathfrak{H}_n\right)|_{W_n}\right) \to 0.$$

Since clearly $\left(\mathbb{Z} \times \mathfrak{H}_n\right)|_{W\backslash W_n} \cong \mathbb{Z} \times \left(\mathfrak{H}_n|_{W\backslash W_n}\right)$ and $\left(\mathbb{Z} \times \mathfrak{H}_n\right)|_{W_n} \cong \mathbb{Z} \times \left(\mathfrak{H}_n|_{W_n}\right)$, we get the commuting diagram

$$\begin{array}{ccccccccc}
0 & \to & C^*\left(\mathbb{Z} \times \left(\mathfrak{H}_n|_{W\backslash W_n}\right)\right) & \to & C^*\left(\mathbb{Z} \times \mathfrak{H}_n\right) & \to & C^*\left(\mathbb{Z} \times \left(\mathfrak{H}_n|_{W_n}\right)\right) & \to & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \to & C^*(\mathbb{Z}) \otimes C^*\left(\mathfrak{H}_n|_{W\backslash W_n}\right) & \overset{\mathrm{id}\otimes\iota_*}{\to} & C^*(\mathbb{Z}) \otimes C^*\left(\mathfrak{H}_n\right) & \overset{\mathrm{id}\otimes\rho_*}{\to} & C^*(\mathbb{Z}) \otimes C^*\left(\mathfrak{H}_n|_{W_n}\right) & \to & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \to & C(\mathbb{T}) \otimes C^*\left(\mathfrak{H}_n|_{W\backslash W_n}\right) & \overset{\mathrm{id}\otimes\iota_*}{\to} & C(\mathbb{T}) \otimes C^*\left(\mathfrak{H}_n\right) & \overset{\mathrm{id}\otimes\rho_*}{\to} & C(\mathbb{T}) \otimes C^*\left(\mathfrak{H}_n|_{W_n}\right) & \to & 0
\end{array}$$

of exact rows.

Clearly the equivalence relation $\approx$ on $\widetilde{W}\backslash\widetilde{W}_n = \mathbb{Z}_{\ge}^n$ is trivial, and hence $\mathfrak{G}_n|_{W\backslash W_n} \cong \widetilde{\mathfrak{G}_n}|_{\widetilde{W}\backslash\widetilde{W}_n}$ and $\mathfrak{H}_n|_{W\backslash W_n} \cong \widetilde{\mathfrak{H}_n}|_{\widetilde{W}\backslash\widetilde{W}_n}$. Furthermore

$$\widetilde{\mathfrak{G}_n}|_{\widetilde{W}\backslash\widetilde{W}_n} = \left\{(k,x,w) \in \mathcal{F}^n \middle|\ w \in \mathbb{Z}_{\ge}^n\right\} = \mathbb{Z} \times \mathcal{H}^n|_{\mathbb{Z}_{\ge}^n}.$$

and similarly $\widetilde{\mathfrak{H}_n}|_{\widetilde{W}\backslash\widetilde{W}_n} = \mathcal{H}^n|_{\mathbb{Z}_{\ge}^n}$. So we get $\mathfrak{G}_n|_{W\backslash W_n} = \mathbb{Z} \times \left(\mathfrak{H}_n|_{W\backslash W_n}\right)$, and the commuting diagram

$$\begin{array}{ccccccccc}
0 & \to & C^*\left(\mathfrak{H}_n|_{W\backslash W_n}\right) & \xrightarrow{\iota_*} & C^*\left(\mathfrak{H}_n\right) & \xrightarrow{\rho_*} & C^*\left(\mathfrak{H}_n|_{W_n}\right) & \to & 0 \\
 & & \downarrow\cong & & \cap & & & & \\
0 & \to & \mathcal{K}\left(\ell^2\left(\mathbb{Z}_{\ge}^n\right)\right) & \to & \mathcal{B}\left(\ell^2\left(\mathbb{Z}_{\ge}^n\right)\right) & & & &
\end{array}$$

via the faithful regular representation [R, MR] of $C^*\left(\mathfrak{H}_n\right)$ on $\ell^2\left(\mathbb{Z}_{\ge}^n\right)$.

On the other hand, $\widetilde{\mathfrak{G}_n}|_{\widetilde{W}_n}$ consists of $(k,x,w) \in \widetilde{\mathfrak{G}_n}$ with $w_i = \infty$ for some $i \le n$ and hence $k = 0$. So $\widetilde{\mathfrak{G}_n}|_{\widetilde{W}_n} = \{0\} \times \widetilde{\mathfrak{H}_n}|_{\widetilde{W}_n}$ and

$$\mathfrak{G}_n|_{W_n} = \{0\} \times \mathfrak{H}_n|_{W_n} \subset \mathbb{Z} \times \mathfrak{H}_n|_{W_n}.$$

Now it is clear that

$$\begin{aligned}\mathfrak{G}_n &= \left(\mathfrak{G}_n|_{W\setminus W_n}\right) \cup \left(\mathfrak{G}_n|_{W_n}\right) = \left(\mathbb{Z} \times \left(\mathfrak{H}_n|_{W\setminus W_n}\right)\right) \cup \left(\{0\} \times \mathfrak{H}_n|_{W_n}\right) \\ &= \left(\mathbb{Z} \times \left(\mathfrak{H}_n|_{W\setminus W_n}\right)\right) \cup \left(\{0\} \times \mathfrak{H}_n\right)\end{aligned}$$

is an open subgroupoid of $\mathbb{Z} \times \mathfrak{H}_n$, and we have the commuting diagram

$$\begin{array}{ccccccccc} & & & & & & C^*\left(\mathfrak{G}_n|_{W_n}\right) & & \\ & & & & & & \downarrow\cong & & \\ 0 & \to & C^*\left(\mathfrak{G}_n|_{W\setminus W_n}\right) & \to & C^*\left(\mathfrak{G}_n\right) & \to & C^*\left(\{0\} \times \left(\mathfrak{H}_n|_{W_n}\right)\right) & \to & 0 \\ & & \downarrow\cong & & \cap & & \cap & & \\ 0 & \to & C^*\left(\mathbb{Z} \times \left(\mathfrak{H}_n|_{W\setminus W_n}\right)\right) & \to & C^*\left(\mathbb{Z} \times \mathfrak{H}_n\right) & \to & C^*\left(\mathbb{Z} \times \left(\mathfrak{H}_n|_{W_n}\right)\right) & \to & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \to & C\left(\mathbb{T}\right) \otimes \mathcal{K}\left(\ell^2\left(\mathbb{Z}^n_{\geq}\right)\right) & \overset{\mathrm{id}\otimes\iota_*}{\to} & C\left(\mathbb{T}\right) \otimes C^*\left(\mathfrak{H}_n\right) & \overset{\mathrm{id}\otimes\rho_*}{\to} & C\left(\mathbb{T}\right) \otimes C^*\left(\mathfrak{H}_n|_{W_n}\right) & \to & 0 \end{array}$$

of exact rows, in which $C^*\left(\mathfrak{G}_n\right)$ is embedded in $C\left(\mathbb{T}\right) \otimes C^*\left(\mathfrak{H}_n\right) \cong C\left(\mathbb{T}, C^*\left(\mathfrak{H}_n\right)\right)$ as an algebra containing $C\left(\mathbb{T}\right) \otimes \mathcal{K}\left(\ell^2\left(\mathbb{Z}^n_{\geq}\right)\right)$ and $C^*\left(\mathfrak{G}_n|_{W_n}\right)$ is embedded in $C\left(\mathbb{T}\right) \otimes C^*\left(\mathfrak{H}_n|_{W_n}\right)$ as

$$C^*\left(\{0\} \times \left(\mathfrak{H}_n|_{W_n}\right)\right) \cong C^*\left(\{0\}\right) \otimes C^*\left(\left(\mathfrak{H}_n|_{W_n}\right)\right) \cong \mathbb{C} \otimes C^*\left(\left(\mathfrak{H}_n|_{W_n}\right)\right).$$

So

$$C^*\left(\mathfrak{G}_n\right) \cong \left(\mathrm{id} \otimes \rho_*\right)^{-1}\left(\mathbb{C} \otimes C^*\left(\left(\mathfrak{H}_n|_{W_n}\right)\right)\right).$$

We claim that $\mathfrak{G}_n|_{W_n}$ is isomorphic to the groupoid $\mathfrak{G}_{n-1}$. In fact, $\widetilde{\mathfrak{G}_n}|_{\widetilde{W}_n}$ consists of $(k, x, w) \in \widetilde{\mathfrak{G}_n}$ with $w_i = \infty$ for some $i \leq n$ and hence $k = 0$. So by considering the smallest $i$ with $w_i = \infty$, we get

$$\begin{aligned}\widetilde{\mathfrak{G}_n}|_{\widetilde{W}_n} = \{(0, x, w) \in \mathcal{F}^n | \text{ for some } i \leq n,\ w_i = \infty,\ x_{i+1} = ... = x_n = 0 \\ \text{but } w_j < \infty \text{ for all } j < i\}.\end{aligned}$$

Note that the map $\tilde{\phi}$ sending $(0, x, w) \in \widetilde{\mathfrak{G}_n}|_{\widetilde{W}_n}$ to $(k', x', w') \in \mathcal{F}^{n-1}$, where $k' = x_n$, and $x'_i = x_i$ and $w'_i = w_i$ for all $i \leq n - 1$, takes values in $\widetilde{\mathfrak{G}_{n-1}}$, because if $w'_i = \infty$ for some $i \leq n - 1$, then $w_i = \infty$ and hence $k' = x_n = 0$ and $x'_j = x_j = 0$ for all $i < j \leq n - 1$. It is not hard to verify that $\tilde{\phi}$ is a surjective groupoid morphism from $\widetilde{\mathfrak{G}_n}|_{\widetilde{W}_n}$ to $\widetilde{\mathfrak{G}_{n-1}}$. Furthermore $\tilde{\phi}$ preserves the equivalence relation $\sim$ and hence induces a homeomorphic groupoid isomorphism $\phi$ from the quotient groupoid $\mathfrak{G}_n|_{W_n} = \widetilde{\mathfrak{G}_n}|_{\widetilde{W}_n}/\sim$ to the quotient groupoid $\mathfrak{G}_{n-1} = \widetilde{\mathfrak{G}_{n-1}}/\sim$. So we have

$$C^*\left(\mathfrak{H}_n|_{W_n}\right) \cong C^*\left(\mathfrak{G}_n|_{W_n}\right) \cong C^*\left(\mathfrak{G}_{n-1}\right) \cong C\left(\mathbb{S}_q^{2n-1}\right).$$

We conclude the above discussion in the following theorem.

**Theorem 2.** *There is a $C^*$-subalgebra $\mathcal{A} \supset \mathcal{K}\left(\ell^2\left(\mathbb{Z}^n_{\geq}\right)\right)$ of $\mathcal{B}\left(\ell^2\left(\mathbb{Z}^n_{\geq}\right)\right)$ and a short exact sequence*

$$0 \quad \to \quad \mathcal{K}\left(\ell^2\left(\mathbb{Z}^n_{\geq}\right)\right) \quad \subset \quad \mathcal{A} \quad \overset{\rho_*}{\to} \quad C\left(\mathbb{S}_q^{2n-1}\right) \quad \to \quad 0$$

*such that*

$$\begin{aligned}C\left(\mathbb{S}_q^{2n+1}\right) &\cong \left(\mathrm{id}_{C(\mathbb{T})} \otimes \rho_*\right)^{-1}\left(\mathbb{C} \otimes C\left(\mathbb{S}_q^{2n-1}\right)\right) \\ &\cong \{f \in C\left(\mathbb{T}, \mathcal{A}\right) \mid \rho_* \circ f \text{ is a constant function on } \mathbb{T}\}\end{aligned}$$

*where $\mathrm{id}_{C(\mathbb{T})} \otimes \rho_* : C\left(\mathbb{T}\right) \otimes \mathcal{A} \to C\left(\mathbb{T}\right) \otimes C\left(\mathbb{S}_q^{2n-1}\right)$ and $C\left(\mathbb{T}, \mathcal{A}\right)$ is the algebra of continuous fields of operators in $\mathcal{A}$ over the unit circle $\mathbb{T}$.*

## References


[CM] R. E. Curto and P. S. Muhly, *C*-algebras of multiplication operators on Bergman spaces*, J. Funct. Anal. 64 (1985), 315-329. MR **86m:**47044

[D] V. G. Drinfeld, *On Poisson homogeneous spaces of Poisson Lie groups*, Theo. Math. Phys. 95 (1993), 226-227. MR **94k:**58045

[MR] P. S. Muhly and J. N. Renault, *C*-algebras of multivariable Wiener-Hopf operators*, Trans. Amer. Math. Soc. 274 (1982), 1-44. MR **84h:**46074

[N] G. Nagy, *On the Haar measure of the quantum $SU(N)$ group*, Comm. Math. Phys. 153 (1993), 217-228. MR **94e:**46124

[P] G. K. Pedersen, *"C*-algebras and their Automorphism Groups"*, Academic Press, New York, 1979. MR **81e:**46037

[R] J. Renault, *A Groupoid Approach to C*-algebras*, Lecture Notes in Mathematics, Vol. 793, Springer-Verlag, New York, 1980. MR **82h:**46075

[SaShU] N. Salinas, A. J. L. Sheu, and H. Upmeier, *Toeplitz operators on pseudoconvex domains and foliation C*-algebras*, Annals of Mathematics 130 (1989), 531-565. MR **91e:**47026

[Sh1] A. J. L. Sheu, *Reinhardt domains, boundary geometry and Toeplitz C*-algebras*, J. Func. Anal. 92 (1990), 264-311. MR **92a:**46068

[Sh2] ———, *Compact quantum groups and groupoid C*-algebras*, J. Func. Anal. 144 (1997), 371-393. MR **98e:**46090

[Sh3] ———, *Quantum spheres as groupoid C*-algebras*, Quarterly J. Math. 48 (1997), 503-510. MR **99b:**46112

[VSo] L. L. Vaksman and Ya. S. Soibelman, *The algebra of functions on the quantum group $SU(n+1)$, and odd-dimensional quantum spheres*, Leningrad Math. J. 2 (1991), 1023-1042. MR **92e:**58021

[W] A. Weinstein, *The local structure of Poisson manifolds*, J. Diff. Geom. 18 (1983), 523-557. MR **86i:**58059



Department of Mathematics, University of Kansas, Lawrence, Kansas 66045

*E-mail address*: sheu@falcon.cc.ukans.edu