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SPECTRAL RADIUS OF THE SAMPLING OPERATOR WITH CONTINUOUS SYMBOL

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ABSTRACT. Let $\varphi(\theta) \sim \sum_{-\infty}^{\infty} a_k e^{ik\theta}$ (where a_k is the k-th Fourier coefficient of φ) be a bounded measurable function on the unit circle $\mathbf{T}$. Consider the operator $S_\varphi(m,n)$ on $L^2(\mathbf{T})$ whose matrix with respect to the standard basis $\{e^{ik\theta} : k \in \mathbf{Z}\}$ is given by $(a_{mi-nj})_{i,j\in\mathbf{Z}}$. In this paper, we give upper and lower bound estimation for $r(S_\varphi(m,n))$, the spectral radius of $S_\varphi(m,n)$. Furthermore, we will show that in some cases (for example, if φ is continuous on $\mathbf{T}$ and $\varphi > 0$), the spectral radius of $S_\varphi(m,n)$ can be computed exactly in terms of roots of the norms of some finite Toeplitz matrices.

1. Introduction

Let us consider the Hilbert space $L^2(\mathbf{T}) = \{f : \int_{\mathbf{T}} |f|^2 < \infty\}$ with the usual inner product $\langle f, g\rangle = \int_{\mathbf{T}} f\overline{g}$ and a bounded measurable function $\varphi(\theta) \sim \sum_{-\infty}^{\infty} a_k e^{ik\theta}$ (where a_k is the k-th Fourier coefficient of φ) on the unit circle $\mathbf{T} = \{e^{i\theta} : \theta \in \mathbf{R}\}$. We will denote the operator of multiplication by φ as M_φ. It is well-known that the matrix of M_φ with respect to the standard basis $\{e^{ik\theta} : k \in \mathbf{Z}\}$ is given by $(a_{i-j})_{i,j\in\mathbf{Z}}$. It is also known that the norm and spectral radius of M_φ are the same, given by $\|\varphi\|_\infty = \sup_{0\le\theta<2\pi} |\varphi(e^{i\theta})|$.

Let $m, n \in \mathbf{N}$. The *sampling operator* $S_\varphi(m,n)$ *with symbol* φ *and parameter* m, n is the operator on $L^2(\mathbf{T})$ whose matrix representation with respect to $\{e^{ik\theta} : k \in \mathbf{Z}\}$ is obtained by "keeping" every m-th row and n-th column of the doubly infinite matrix of M_φ, i.e., the matrix $(a_{mi-nj})_{i,j\in\mathbf{Z}}$.

A sampling operator arises from considering some alternative for a given interpolation scheme in approximation theory. Let f be a function on $\mathbf{R}$ whose values (or sometimes called the *outputs* of f) are known at equally spaced points, say for example, the integers, and we are given an interpolation scheme that allows us to estimate the values of f at, say, the set $\{\frac{k}{m} : k \in \mathbf{Z}\}$ while keeping the same values of f at the integers. The alternative scheme here is for one to only approximate the values of f at the subset $\{\frac{nj}{m} : j \in \mathbf{Z}\}$ (for the details on this subject one can check [8]). To approach this problem from the operator theory point of view, one can formulate the problem as follows: Let $\{y_0(k)\}_{-\infty}^{\infty}$ be the values of f at the integers and let $\{y_1(j)\}_{-\infty}^{\infty}$ be the estimates for f on $\{\frac{nj}{m} : j \in \mathbf{Z}\}$, with $y_0(0) = y_1(0) = f(0)$.

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Then

$$y_1(j) = \sum_{k=-\infty}^{\infty} a_{mj-nk} y_0(k),$$

where $\varphi(\theta) \sim \sum_{-\infty}^{\infty} a_k e^{ik\theta}$ is usually a real-valued function on $\mathbf{T}$ associated with the particular scheme chosen.

The spectral radius of the sampling operator $S_\varphi(m,n)$ is known to have close ties to the so-called "smoothness index" of the interpolation scheme that φ is associated with. For example, in the case of the dyadic wavelets in $L^2(\mathbf{R})$ (which corresponds to the interpolation scheme for half integers $\left\{\frac{k}{2} : k \in \mathbf{Z}\right\}$), usually one first obtains a smooth solution f in $L^2(\mathbf{T})$ of the equation

$$f(t) = \sum_k c_k f(2t-k),$$

where c_k is the k-th Fourier coefficient of a trigonometric polynomial φ on $\mathbf{T}$, and then constructs the so-called *mother function* from f to generate the wavelet basis with desired properties. In order to determine the regularity of the generated wavelet basis, and consequently the smoothness of f, one calculates the quantity

$$s_0 = M - \log_2 r(S_{\tilde{\varphi}}(2,1)) - \frac{1}{2},$$

where s_0 is the so-called *Sobolev exponent* of f and M is the multiplicity of φ at $\theta = \pi$. The term $r(S_{\tilde{\varphi}}(2,1))$ is precisely the spectral radius for $S_{\tilde{\varphi}}(2,1)$ on $L^2(\mathbf{T})$ and $\tilde{\varphi}$ is obtained basically by factoring out the zeros of φ at $\theta = \pi$ up to its multiplicity. For information on the subjects concerning the estimation for the regularity of wavelet basis see, for example, [1], [2], [4], [5] and [9].

The purpose of this article is to give an explicit formula to compute $r(S_\varphi(m,n))$ in terms of the asymptotic behavior of the norms of some finite Toeplitz matrices if φ is a positive continuous function on $\mathbf{T}$ (Theorem 3.3).

2. Norms for the sampling operators

Throughout this article we will use the notation $\varphi(\theta) = \varphi(e^{i\theta})$, $\theta \in \mathbf{R}$. Let τ_m be the continuous mapping defined on $\mathbf{T}$ by $\tau_m : e^{i\theta} \to e^{im\theta}$. Consider the *composition operator* $C_m = C_{\tau_m}$ on $L^2(\mathbf{T})$ (i.e., $C_m f = f \circ \tau_m$) and the *average operator* R_m defined by

$$(R_m f)(\theta) := \frac{1}{m} \sum_{l=0}^{m-1} f\left(\frac{\theta + 2l\pi}{m}\right) \quad \text{a.e. } \theta \in \mathbf{R}$$

for all $f \in L^2(\mathbf{T})$. It is easy to see that C_m is an isometry and R_m is a partial isometry. Moreover, $R_m = C_m^*$ for every m. It also follows by the definitions that

$$R_m e^{ik\theta} = \begin{cases} e^{is\theta} & \text{if } k = ms, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

It is also easy to check, with this notation, that we have

$$S_\varphi(m,n) = R_m M_\varphi C_n.$$

From this, it is evident that $S_\varphi(m,n)$ is a bounded operator on $L^2(\mathbf{T})$ since $\|S_\varphi(m,n)\| = \|R_m M_\varphi C_n\| \le \|M_\varphi\| \le \|\varphi\|_\infty$.

The following are some useful facts about R_m and C_n $(m, n \in \mathbf{N})$:

(2.1) $R_m C_m = I$ and $C_m R_m = P_m$, where P_m is the projection from $L^2(\mathbf{T})$ onto the closed span of $\{e^{imk\theta} : k \in \mathbf{Z}\}$. Also, it is easy to see that $R_m R_n = R_{mn}$ and $C_m C_n = C_{mn}$.

(2.2) $P_m = \frac{1}{m}\sum_{l=0}^{m-1} \mathcal{L}_{\zeta_m^l}$, where $\mathcal{L}_\zeta$ is the *rotation operator* $(\mathcal{L}_\zeta f)(\theta) = f(\zeta e^{i\theta})$ $(\zeta \in \mathbf{C})$ and $\zeta_m = e^{i\frac{2\pi}{m}}$.

(2.3) For any φ in $L^\infty(\mathbf{T})$, $R_m M_\varphi C_m = M_\psi$, where $\psi = R_m \varphi$.

(2.4) From the computation

$$\begin{aligned}\|M_{\overline{\varphi}} C_m f\|_2^2 &= \langle M_{\overline{\varphi}} C_m f,\ M_{\overline{\varphi}} C_m f\rangle \\ &= \left\langle R_m M_{|\varphi|^2} C_m f,\ f\right\rangle \\ &= \langle M_\psi f,\ f\rangle,\end{aligned}$$

where $\psi = R_m(|\varphi|^2)$, one has $\|R_m M_\varphi\|^2 = \|M_{\overline{\varphi}} C_m\|^2 = \left\|R_m(|\varphi|^2)\right\|_\infty$.

(2.5) $M_\varphi R_m = R_m M_{\varphi\circ\tau_m}$ and $C_m M_\varphi = M_{\varphi\circ\tau_m} C_m$ for any $m \in \mathbf{N}$ and any $\varphi \in L^\infty(\mathbf{T})$ (in fact, we have $C_\tau M_\varphi = M_{\varphi\circ\tau} C_\tau$ in general, for any τ on a compact metric space X that induces a bounded C_τ on $L^2(X, d\mu)$ with respect to the measure μ).

Now we compute the norm for $S_\varphi(m,n)^k$, $k \in \mathbf{N}$:

Lemma 2.1. *Given m, n in $\mathbf{N}$ such that $(m,n) = 1$ (where $(m,n) = g.c.d.(m,n)$). Then C_m and R_n commute, i.e., $C_m R_n = R_n C_m$.*

Proof. Simply check the equality $C_m R_n e^{ik\theta} = R_n C_m e^{ik\theta}$ for all $k \in \mathbf{Z}$, using the fact that $(m,n) = 1$. □

Lemma 2.2. *If $(m,n) = 1$, then for $f \in L^2(\mathbf{T})$ we have*

$$\|S_\varphi(m,n)f\|^2 = \sum_{k=0}^{n-1} \|R_m M_{\varphi_k} f\|_2^2$$

where

$$\varphi_k(\theta) = e^{-ika\theta} R_n(e^{ik\theta}\varphi) = e^{-ika\theta} \sum_{j=0}^{n-1} e^{ik(\frac{\theta+2j\pi}{n})} \varphi(\frac{\theta + 2j\pi}{n})$$

for some a such that $na \equiv 1 \mod m$. Here $\|f\|_2 = (\int_{\mathbf{T}} |f|^2)^{\frac{1}{2}}$.

Proof. First let us adopt a decomposition of M_φ due to Zizler in Theorem 8, [10]. For each $0 \le k \le n-1$, let $H_k = \bigvee\{e^{i(k+na)\theta} : a \in \mathbf{Z}\}$ and consider $h_k = e^{-ik\theta} C_n R_n(e^{ik\theta}\varphi)$. Then it follows from (1) that $\varphi = h_0 + h_1 \cdots + h_{n-1}$ and $M_{h_k} C_n f \in H_k$ for all k. Therefore, $M_\varphi C_n = M_{h_0} C_n + M_{h_1} C_n + \cdots + M_{h_{n-1}} C_n$ and $\langle M_{h_s} C_n f, M_{h_t} C_n f\rangle = 0$ for all $f \in L^2(\mathbf{T})$ if $s \ne t$ $(0 \le s, t \le n-1)$.

On the other hand, it also follows from (1) that $R_m(e^{i(s+na)\theta}) \ne 0$ only if $s + na = m(t + nb)$ for some $0 \le t \le n-1$ and $b \in \mathbf{Z}$ and consequently, only if $s \equiv mt \mod n$ for some $0 \le t \le n-1$. Suppose that $0 \le s_1 \ne s_2 \le n-1$ and $s_1 \equiv mt_1 \mod n$, $s_2 \equiv mt_2 \mod n$ for some $0 \le t_1, t_2 \le n-1$. Then $t_1 \ne t_2$. This means that $\langle R_m M_{h_s} C_n f, R_m M_{h_t} C_n f\rangle = 0$ for all $f \in L^2(\mathbf{T})$ if $s \ne t$, and from this we get

$$\|S_\varphi(m,n)f\|_2^2 = \sum_{k=0}^{n-1} \|R_m M_{h_k} C_n f\|_2^2,\ \ f \in L^2(\mathbf{T}).$$

Now let $\alpha_k \in \mathbf{Z}$ such that $n\alpha_k + k \equiv 0 \mod m$ for each k. Then

$$\|R_m(e^{i(k+n\alpha_k)\theta} g)\|_2 = \|R_m g\|_2$$

for any $g \in H_k$ and for each k. Therefore, for any $f \in L^2(\mathbf{T})$,

$$\begin{aligned}\|R_m(e^{i(k+n\alpha_k)\theta} M_{h_k} C_n f)\|_2 &= \|R_m C_n(e^{\alpha_k\theta} R_n(e^{ik\theta}\varphi)f)\|_2 \\ &= \|C_n R_m(e^{\alpha_k\theta} R_n(e^{ik\theta}\varphi)f)\|_2 \\ &= \|R_m(e^{\alpha_k\theta} R_n(e^{ik\theta}\varphi)f)\|_2\end{aligned}$$

for each k. Notice that the above equations follow from the definitions, Lemma 2.1 and the fact that C_n is an isometry.

To finish the proof, notice that since $(m,n) = 1$, there are $a, b \in \mathbf{Z}$ so that $na + mb = 1$. This means that $na \equiv 1 \mod m$, and therefore $n(-ka) + k \equiv 0 \mod m$ for each k, which implies that we may choose $\alpha_k = -ka$. □

Remark. Here we would like to point out that $R_n(|\varphi|^2) = \sum_{k=0}^{n-1} R_n(|h_k|^2) = \sum_{k=0}^{n-1} |\varphi_k|^2$. The following is a simple proof: Write

$$\varphi(\theta) = \sum_{k=0}^{n-1} h_k(\theta) = \sum_{k=0}^{n-1} e^{-ik\theta}\varphi_k(\theta);$$

then

$$\begin{aligned}R_n(|\varphi|^2)(\theta) &= \sum_{k=0}^{n-1} |\varphi(\frac{\theta+2k\pi}{n})|^2 \\ &= \frac{1}{n}\sum_{s,s'=0}^{n-1} \varphi_s(\theta)\overline{\varphi}_{s'}(\theta) e^{\frac{-i(s-s')\theta}{n}} \sum_{k=0}^{n-1} e^{\frac{-i2(s-s')k\pi}{n}} \\ &= \sum_{k=0}^{n-1} |\varphi(\theta)|^2 \quad = \quad \sum_{k=0}^{n-1} |h_k(\theta)|^2.\end{aligned}$$

Theorem 2.3. *Suppose that $(m,n) = r$. Then $\|S_\varphi(m,n)\|^2$ equals the essential supreme of*

$$\|S(\theta)\|$$

over $[0, 2\pi)$, where $S(\theta)$ is a $q \times q$ positive definite matrix on $\mathbf{C}^p$ whose st-th entry is the multiplication by $R_p(\psi_s\overline{\psi_t})(\theta)$ a.e. $\theta \in [0, 2\pi)$ $(0 \le s, t \le q-1)$, with $m = pr$, $n = qr$, $\psi = R_r\varphi$ and ψ_s is defined by

$$\psi_s(\theta) = e^{-isa\theta} R_q(e^{is\theta}\psi) = e^{-isa\theta} \sum_{j=0}^{q-1} e^{is(\frac{\theta+2j\pi}{q})} \psi(\frac{\theta+2j\pi}{q})$$

for some a so that $qa \equiv 1 \mod p$.

Proof. First observe $S_\varphi(m,n) = R_m M_\varphi C_n = R_p(R_r M_\varphi C_r)C_q = R_p M_\psi C_q$ (see (2.1)–(2.5)). Therefore, by Lemma 2.2, we have

$$\|S_\varphi(m,n)\|^2 = \sup_{\|f\|_2=1} \sum_{k=0}^{q-1} \|R_p M_{\psi_k} f\|_2^2 .$$

Now consider the operator T on $K = \bigoplus_1^q L^2(\mathbf{T})$ defined by

$$T := \begin{pmatrix} R_p M_{\psi_0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ R_p M_{\psi_{q-1}} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \end{pmatrix}.$$

It is easy to see that $\|T\|^2 = \|S_\varphi(m,n)\|^2$. Therefore, the proof follows from (2.3), $\|T\|^2 = \|T^*\|^2 = \|TT^*\|$ and the fact that TT^* is a block multiplication operator on K. □

Remark. There are two immediate consequences of Theorem 2.3. First, we are now able to obtain upper bound for $\|S_\varphi(m,n)\|$. Specifically, since for a.e. θ, $S(\theta)$ is positive definite, we have (see the remark after Lemma 2.2)

$$\|S(\theta)\| \le \mathrm{Tr}(S(\theta)) = R_{pq}(|\psi|^2)(\theta) \quad \text{a.e. } \theta$$

(where $\mathrm{Tr}(A)$ is the trace of A). Therefore, $\|S_\varphi(m,n)\| \le \|R_{pq}(|\psi|^2)\|_\infty^{\frac{1}{2}}$. Second, since the largest eigenvalue $\lambda(\theta)$ of $S(\theta)$ satisfies

$$\lambda(\theta) \ge q^{-1}\mathrm{Tr}(S(\theta)) = q^{-1} R_{pq}(|\psi|^2)(\theta) \quad \text{a.e. } \theta,$$

$\|S_\varphi(m,n)\|$ is bounded below by $q^{-\frac{1}{2}}\|R_{pq}(|\psi|^2)\|_\infty^{\frac{1}{2}}$. The readers can find detailed discussion on the subject of positive definite matrices in, for example, Chapter 7 in [6]. Here we point out that the upper bound above for $\|S_\varphi(m,n)\|$ was first obtained by Zizler in [10].

Combining $S_\varphi(m,n)^k = R_{p^k} M_{\psi_k} C_{q^k}$ (by (2.4)), (2.5), Lemma 2.1 and Theorem 2.3, we immediately obtain

Proposition 2.4. *Suppose that $(m,n) = r$. Then $\|S_\varphi(m,n)^k\|^2$ equals the essential supreme of*

$$\|S_k(\theta)\|$$

over $[0,2\pi)$, where $S_k(\theta)$ is a $q^k \times q^k$ positive definite matrix on $\mathbf{C}^{p^k}$ whose entries are given by $R_{p^k}(\psi_{k,s}\overline{\psi}_{k,t})(\theta)$ a.e. $\theta \in [0,2\pi)$ $(0 \le s,t \le p^k - 1)$, with $m = pr$, $n = qr$, $\psi = R_r\varphi$, $\psi_{k,s} = e^{-isa_k\theta} R_{p^k}(e^{is\theta}\psi_k)$ for all s, where

$$\psi_k(\theta) = \prod_{\gamma=0}^{k-1} \psi(p^\gamma q^{k-1-\gamma}\theta)$$

and $q^k a_k \equiv 1 \mod p^k$.

Corollary 2.5. *$r(S_\varphi(m,n)) > 0$ if $\log|\psi| \in L^1(\mathbf{T})$.*

Proof. By Proposition 2.4 and the remark after Theorem 2.3, we have

$$\begin{aligned}\|S_\varphi(m,n)^k\|^2 &\geq \frac{1}{q^k}\|R_{p^kq^k}(|\psi_k|^2)\|_\infty \geq \frac{1}{q^k}\int_0^{2\pi} R_{p^kq^k}(|\psi_k|^2)\frac{d\theta}{2\pi}\\ &= \frac{1}{q^k}\int_0^{2\pi}|\psi_k|^2\frac{d\theta}{2\pi} \geq \frac{1}{q^k}\exp(\int_0^{2\pi}\log|\psi_k|^2\frac{d\theta}{2\pi})\\ &= \frac{1}{q^k}\exp(k\int_0^{2\pi}\log|\psi|^2\frac{d\theta}{2\pi}).\end{aligned}$$

This means that

$$r(S_\varphi(m,n)) = \lim_{k\to\infty}\|S_\varphi(m,n)^k\|^{\frac{1}{k}} \geq \frac{1}{\sqrt{q}}\exp\left(\int_0^{2\pi}\log|\psi|\frac{d\theta}{2\pi}\right),$$

and therefore the result follows from the assumption. □

3. Spectral radius computation for $S_\varphi(m,n)$

As a consequence of Proposition 2.4, we see that for any $\varphi \in L^\infty(\mathbf{T})$

$$\frac{1}{\sqrt{q}}\limsup_{k\to\infty}\|R_{p^kq^k}(|\psi_k|^2)\|_\infty^{\frac{1}{2k}} \leq r(S_\varphi(m,n)) \leq \liminf_{k\to\infty}\|R_{p^kq^k}(|\psi_k|^2)\|_\infty^{\frac{1}{2k}}$$

(it can be shown that $\lim_{k\to\infty}\|R_{p^kq^k}(|\psi_k|^2)\|_\infty^{\frac{1}{2k}}$ actually exists). But there is an obvious problem for these upper and lower bounds estimates, that is, they may be far apart when q is large. However, for the case of estimating $r(S_\varphi(m,n))$ when φ is continuous and positive, we will show that we can do much better. Before we begin, let us first introduce some very useful notation:

Definition. Let $\{f_k\}$ and $\{g_k\}$ be two sequences of positive functions on $\mathbf{T}$. We say that f_k *is comparable to* g_k, denoted by $f_k \approx g_k$, if given any $0 < \delta < 1 < \rho$, there exists constant $c > 0$ such that

$$c^{-1}\delta^k f_k \leq g_k \leq c\rho^k f_k \quad \text{uniformly} \tag{2}$$

if $k \geq 1$. Two sequences of positive numbers $\{\alpha_k(s_k(1),\cdots,s_k(n)) : a_k \leq s_k(i) \leq b_k,\ a_k, b_k \in \mathbf{N}\}$ and $\{\beta_k(s_k(1),\cdots,s_k(n)) : a_k \leq s_k(i) \leq b_k,\ a_k, b_k \in \mathbf{N}\}$ with multiple indices are said to be comparable uniformly in k (with the same notation) if they satisfy (2) for some constants that, in addition, do not depend on $1 \leq i \leq n$. Two sequences of positive numbers are said to be comparable if they satisfy (2) for some constants when they are regarded as either sequences of constant functions or a special case for sequences with multiple indices.

It is easy to see that the relation "$\approx$" is transitive, i.e., if $f_k \approx g_k$ and $g_k \approx h_k$, then $f_k \approx h_k$. Also, if $f_k \approx g_k$, then $\frac{\sqrt[k]{f_k}}{\sqrt[k]{g_k}} \to 1$ uniformly on $\mathbf{T}$ as $k \to \infty$ since ε and ρ are arbitrary. A similar result also holds for sequences.

Lemma 3.1. *Suppose that* $\psi \in C(\mathbf{T})$ *and* $|\psi| > 0$. *If* $p > q$, *then*

$$|\psi_k(\theta+\alpha)| \approx |\psi_k(\theta)|$$

uniformly over $\mathbf{R}$ *if* $|\alpha| \leq \frac{2\pi}{p^k}$, *where* $\psi_k(\theta) = \prod_{\gamma=0}^{k-1}\psi(p^\gamma q^{k-1-\gamma}\theta)$.

Proof. Since $\mathbf{T}$ is compact and $|\psi| > 0$, $|\psi|$ is uniformly continuous in the sense that given any $0 < \delta < 1 < \rho$, there exists $\varepsilon > 0$ so that $\delta|\psi(\theta_1)| < |\psi(\theta_2)| < \rho|\psi(\theta_1)|$ if $|\theta_1 - \theta_2| < \varepsilon$. On the other hand, choose k_0 large enough so that $2\pi q^{k-s} < \varepsilon p^{k-s}$

whenever $k - s \geq k_0 - s_0$ for some s_0 depending only on p and q. This is possible since $p > q$. It follows that for $\theta \in \mathbf{R}$,

$$K^{-s_0}\delta^{k-s_0}|\psi_k(\theta+\alpha)| \leq |\psi_k(\theta)| \leq K^{s_0}\rho^{k-s_0}|\psi_k(\theta+\alpha)|$$

if $k \geq k_0$, where $K = \frac{m}{M}$ and M and m are the maximum and minimum of $|\psi|$ on $\mathbf{T}$, respectively. This completes the proof. □

Lemma 3.2. *Let $p > q$ in $\mathbf{N}$, $(p, q) = 1$ and $\psi > 0$ in $C(\mathbf{T})$. For each k, let Ψ_k be the step function on $\mathbf{T}$ satisfying*

$$\Psi_k(\theta) = \psi_k(\frac{2\pi l}{p^k}), \quad 0 \leq l \leq p^k - 1,$$

if $\theta \equiv \phi \mod 2\pi$ for some $\frac{2\pi b_l}{p^k} \leq \phi < \frac{2\pi(b_l+1)}{p^k}$, where $0 \leq b_l \leq p^k - 1$ is the unique integer so that $b_l \equiv lq^k \mod p^k$, and $\psi_k(\theta) = \prod_{\gamma=0}^{k-1}\psi(p^\gamma q^{k-1-\gamma}\theta)$. Then

$$\sup_{\|f\|_2^2=1}\sum_{t=0}^{q^k-1}\|R_{p^k}M_{\Psi_k}\mathcal{L}_{\zeta_k^t}f\|_2^2 \approx \|U_k\|, \quad \zeta_k = e^{i2\pi/p^k},$$

where U_k is a positive definite q^k by q^k matrix on $\mathbf{C}^{q^k}$ whose ij-entry is

$$\frac{1}{p^k}\sum_{l=0}^{p^k-1}\psi_k(\frac{2\pi l}{p^k}+\frac{2\pi i}{q^k})\psi_k(\frac{2\pi l}{p^k}+\frac{2\pi j}{q^k}), \quad 0 \leq i, j \leq q^k - 1.$$

Proof. By applying the same technique used in the proof of Theorem 2.3, we see that

$$\sup_{\|f\|_2^2=1}\sum_{t=0}^{q^k-1}\|R_{p^k}M_{\Psi_k}\mathcal{L}_{\zeta_k^t}f\|_2^2 = \|\mathcal{W}_k\|^2,$$

where $\mathcal{W}_k$ is the operator on $\bigoplus_1^{q^k} L^2(\mathbf{T})$ defined by

$$\mathcal{W}_k(f_0, \cdots, f_{q^k-1}) := (g_0, \cdots, g_{q^k-1}), \quad g_t = R_{p^k}M_{\Psi_k}\mathcal{L}_{\zeta_k^t}f_0,$$

with adjoint

$$\mathcal{W}_k^*(f_0, \cdots, f_{q^k-1}) := (g, 0, 0, \cdots, 0), \quad g = \sum_{t=0}^{q^k-1}\mathcal{L}_{\overline{\zeta}_k^t}M_{\Psi_k}C_{p^k}f_t,$$

for $(f_0, \cdots, f_{q^k-1}) \in \bigoplus_1^{q^k} L^2(\mathbf{T})$. Since $\|\mathcal{W}_k\|^2 = \|\mathcal{W}_k\mathcal{W}_k^*\|$, it follows from (2.3) that $\|\mathcal{W}_k\|^2 = \|\mathcal{Q}_k\|$, where

$$\mathcal{Q}_k(f_0, \cdots, f_{q^k-1}) := (g_0, \cdots, g_{q^k-1}), \quad g_t = \sum_{s=0}^{q^k-1}(R_{p^k}(\Psi_{k,t}\Psi_{k,s}))f_s,$$

and $\Psi_{k,t}(\theta) = \Psi_k(\theta - \frac{2\pi t}{p^k})$, $\theta \in \mathbf{R}$.

Now suppose that $0 \leq x \leq p^k - 1$ and $0 \leq y \leq p^k - 1$ are integers so that $x \equiv yq^k \mod p^k$, then by the Chinese remainder theorem, the number

$$y = \frac{p^k z + x}{q^k}$$

satisfies $x \equiv yq^k \mod p^k$, where z satisfies $-x \equiv zp^k \mod q^k$. On the other hand, since $(p^k, q^k) = 1$, there exist $\alpha_k, \beta_k \in \mathbf{Z}$ such that $\alpha_k p^k + \beta_k q^k = 1$. Therefore, if we choose $z = -\alpha_k x$, the number

$$y = \frac{(1-\alpha_k p^k)x}{q^k} = \beta_k x$$

satisfies $x \equiv yq^k \mod p^k$, and this means that $\Psi_k(\frac{\theta}{p^k} + \frac{2\pi x}{p^k}) = \psi_k(\frac{2\pi\beta_k x}{p^k})$ for $\theta \in [0, 2\pi)$. So for $0 \le l \le p^k - 1$ and $0 \le t \le q^k - 1$, we have

$$\begin{aligned}
\Psi_{k,t}(\frac{\theta+2\pi l}{p^k}) &= \Psi_k(\frac{\theta+2\pi l}{p^k} - \frac{2\pi t}{p^k}) = \psi_k(\frac{2\pi\beta_k l}{p^k} - \frac{2\pi\beta_k t}{p^k}) \\
&= \psi_k(\frac{2\pi\beta_k l}{p^k} + \frac{2\pi\alpha_k t}{q^k} - \frac{2\pi t}{p^k q^k}) \\
&\approx \psi_k(\frac{2\pi\beta_k l}{p^k} + \frac{2\pi\alpha_k t}{q^k}) \qquad (0 \le \theta < 2\pi)
\end{aligned}$$

uniformly in k by Lemma 3.1 since $0 \le \frac{2\pi t}{p^k q^k} \le \frac{2\pi}{p^k}$ if $0 \le t \le q^k - 1$. Hence

$$R_{p^k}(\Psi_{k,t}\Psi_{k,s}) \approx \frac{1}{p^k}\sum_{l=0}^{p^k-1} \psi_k(\frac{2\pi l}{p^k} + \frac{2\pi\alpha_k t}{q^k})\psi_k(\frac{2\pi l}{p^k} + \frac{2\pi\alpha_k s}{q^k})$$

uniformly in k since the fact that $(\beta_k, p^k) = 1$ implies that $\{\beta_k l : 0 \le l \le p^k - 1\}$ is simply a rearrangement of $\{0, 1, 2, \cdots, p^k - 1\}$. By the same reason, since $(\alpha_k, q^k) = 1$, $\{\alpha_k t : 0 \le t \le q^k - 1\} = \{0, 1, 2, \cdots, q^k - 1\}$, and this completes the proof. □

An $n \times n$ matrix $A = (a_{ij})$, $1 \le i, j \le n$, is called a *Toeplitz matrix* if $a_{ij} = a_{j-i}$ for all i, j. Toeplitz matrices have found a wide variety of applications in areas including analytic functions, stochastic process and statistics (see, for example, [5]). The next result is interesting since it states that the spectral radius of $S_\varphi(m,n)$ can be expressed in terms of the limit of roots of the norms of some finite Toeplitz matrices.

Theorem 3.3. *Let φ be continuous on $\mathbf{T}$ and $m > n$. Let $m = pr$, $n = qr$, $(p,q) = 1$, $\psi = R_r\varphi$ and $\psi_k(\theta) = \prod_{\gamma=0}^{k-1}\psi(p^\gamma q^{k-1-\gamma}\theta)$. Then*

$$r(S_\varphi(m,n)) = \lim_{k\to\infty} \|S_k\|^{\frac{1}{2k}},$$

if $\psi > 0$, where S_k is a real symmetric $q^k \times q^k$ matrix whose ij-entry is

$$\frac{1}{p^k q^k}\sum_{l=0}^{p^k-1} \psi_k(\frac{2\pi l}{p^k} + \frac{2\pi i}{q^k})\psi_k(\frac{2\pi l}{p^k} + \frac{2\pi j}{q^k}), \quad 0 \le i, j \le q^k - 1.$$

Furthermore, the norm of S_k is comparable to that of a real symmetric Toeplitz matrix T_k on $\mathbf{C}^{q^k}$ for each k. Specifically, the ij-entry for T_k is

$$\frac{1}{p^k q^k}\sum_{l=0}^{p^k-1} \psi_k(\frac{2\pi l}{p^k})\psi_k(\frac{2\pi(l+\beta_{j-i})}{p^k}), \quad 0 \le i, j \le q^k - 1,$$

where $\beta_s = [\frac{p^k s}{q^k}]$ $(s \in \mathbf{Z})$, and $[x]$ is the smallest integer less than or equal to x.

Proof. Since $S_\varphi(m,n)^k = S_{\psi_k}(p^k, q^k)$, we have

$$\begin{aligned}\|S_\varphi(m,n)^k f\|_2^2 &= \frac{1}{p^{2k}}\int_0^{2\pi} |\sum_{l=0}^{p^k-1}\psi_k(\frac{\theta+2\pi l}{p^k})f(\frac{q^k(\theta+2\pi l)}{p^k})|^2\frac{d\theta}{2\pi}\\ &= \frac{1}{p^{2k}q^k}\sum_{t=0}^{q^k-1}\int_0^{2\pi}|\sum_{l=0}^{p^k-1}\psi_k(\frac{\theta+2\pi t}{p^kq^k}+\frac{2\pi l}{p^k})f(\frac{\theta+2\pi(t+lq^k)}{p^k})|^2\frac{d\theta}{2\pi}\\ &\approx \frac{1}{p^{2k}q^k}\sum_{t=0}^{q^k-1}\int_0^{2\pi}|\sum_{l=0}^{p^k-1}\psi_k(\frac{2\pi l}{p^k})f(\frac{\theta+2\pi(t+lq^k)}{p^k})|^2\frac{d\theta}{2\pi}\\ &= \frac{1}{q^k}\sum_{t=0}^{q^k-1}\|R_{p^k}M_{\Psi_k}C_{\zeta_k^t}f\|_2^2,\end{aligned}$$

for any $f \geq 0$ in $L^2(\mathbf{T})$, where Ψ_k and ζ_k are defined in Lemma 3.2. Note also that the last equation follows from the fact that $(p^k, q^k) = 1$. Therefore, by Lemma 3.2, $\|S_\varphi(m,n)^k\|^2 \approx \|S_k\|$ since $q^k S_k = U_k$ (it is enough to assume that $f \geq 0$ since $\psi \geq 0$).

Now let $0 \leq i \leq q^k - 1$. Suppose that $\frac{2\pi\alpha}{p^k} \leq \frac{2\pi i}{q^k} < \frac{2\pi(\alpha+1)}{p^k}$ for some integer $0 \leq \alpha \leq p^k - 1$; then $[\frac{p^k i}{q^k}] - 1 < \alpha \leq [\frac{p^k i}{q^k}]$. It follows that

$$\psi_k(\frac{2\pi l}{p^k}+\frac{2\pi i}{q^k}) \approx \psi_k(\frac{2\pi l}{p^k}+\frac{2\pi\beta_i}{p^k}),$$

uniformly in k, where $\beta_i = [\frac{p^k i}{q^k}]$, and therefore, uniformly in k, we have

$$\begin{aligned}\sum_{l=0}^{p^k-1}\psi_k(\frac{2\pi l}{p^k}+\frac{2\pi i}{q^k})\psi_k(\frac{2\pi l}{p^k}+\frac{2\pi j}{q^k}) &\approx \sum_{l=0}^{p^k-1}\psi_k(\frac{2\pi l}{p^k}+\frac{2\pi\beta_i}{p^k})\psi_k(\frac{2\pi l}{p^k}+\frac{2\pi\beta_j}{p^k})\\ &= \sum_{l=0}^{p^k-1}\psi_k(\frac{2\pi l}{p^k})\psi_k(\frac{2\pi l}{p^k}+\frac{2\pi(\beta_j-\beta_i)}{p^k})\\ &\approx \sum_{l=0}^{p^k-1}\psi_k(\frac{2\pi l}{p^k})\psi_k(\frac{2\pi l}{p^k}+\frac{2\pi(\beta_{j-i})}{p^k})\\ &\approx \sum_{l=0}^{p^k-1}\psi_k(\frac{2\pi l}{p^k})\psi_k(\frac{2\pi l}{p^k}+\frac{2\pi(j-i)}{q^k}).\end{aligned}$$

Notice that the equations above follow from the definition of β_i, the property of the Gauss function $[x]$ and the fact that ψ_k is 2π periodic. So far we have shown that if $\psi > 0$, then $\|S_\varphi(m,n)^k\| \approx \|S_k\|^{\frac{1}{2}} \approx \|T_k\|^{\frac{1}{2}}$ and T_k is a Toeplitz matrix on $\mathbf{C}^{q^k}$ for all k. Hence,

$$r(S_\varphi(m,n)) = \lim_{k\to\infty}\|S_\varphi(m,n)^k\|^{\frac{1}{k}} = \lim_{k\to\infty}\|S_k\|^{\frac{1}{2k}} = \lim_{k\to\infty}\|T_k\|^{\frac{1}{2k}}.$$

□

Let $\varphi \in L^\infty(\mathbf{T})$ and let P be the projection from $L^2(\mathbf{T})$ onto H^2, where H^2 is the so-called *Hardy space* on the unit disc, identified as the closed span of $\{e^{ik\theta} : k \geq 0\}$ in $L^2(\mathbf{T})$. The operator $T_\varphi : H^2 \to H^2$ defined by $T_\varphi = PM_\varphi$ is called the

Toeplitz operator with symbol φ. Now let $\varphi \sim \sum a_k e^{ik\theta}$. It is well-known that the representing matrix of T_φ with respect to the standard basis $\{e^{ik\theta} : k \geq 0\}$ on H^2 is $\{a_{j-i}\}$, $i, j \geq 0$ (which is sometimes called an "infinite" Toeplitz matrix) and we have $\|T_\varphi\| = \|\varphi\|_\infty$. Since it is easy to see that any finite Toeplitz matrix $\{a_{j-i}\}$, $0 \leq i, j \leq n-1$ is an $n \times n$ minor of the infinite Toeplitz matrix $\{a_{j-i}\}$, $i, j \geq 0$ and since all entries of T_k are positive, we have

$$\|T_k\| \leq \|T_\varphi\| = \frac{2}{p^k q^k} \sum_{i=0}^{q^k-1} \sum_{l=0}^{p^k-1} \psi_k(\frac{2\pi l}{p^k})\psi_k(\frac{2\pi(l+\beta_i)}{p^k}) - \frac{1}{p^k q^k} \sum_{l=0}^{p^k-1} \psi_k(\frac{2\pi l}{p^k})^2$$

for all k, where the symbol $\varphi \sim \sum a_j e^{ij\theta}$ for the Toeplitz operator T_φ in the above inequality is given by $a_j = \frac{1}{p^k q^k} \sum_{l=0}^{p^k-1} \psi_k(\frac{2\pi l}{p^k})\psi_k(\frac{2\pi(l+\beta_j)}{p^k})$ for $-q^k+1 \leq j \leq q^k-1$ and $a_j = 0$ if else. Here the author would like to refer the readers to [3], which is an excellent source of reference on the subjects concerning the Toeplitz operator. On the other hand, since T_k is positive definite, $\|T_k\|$ equals $\sup\{\langle T_k x, x\rangle : x \in \mathbf{C}^{q^k},\ \|x\|_2 = 1\}$. So in particular, if $x = (q^{-\frac{k}{2}}, \cdots, q^{\frac{-k}{2}})^*$, then $\|x\|_2 = 1$ and $\|T_k\| \geq \langle T_k x, x\rangle$, i.e.,

$$\|T_k\| \geq \frac{1}{p^k q^{2k}} \sum_{i,j=0}^{q^k-1} \sum_{l=0}^{p^k-1} \psi_k(\frac{2\pi(l+\beta_i)}{p^k})\psi_k(\frac{2\pi(l+\beta_j)}{p^k}).$$

Now fix an i. For any $0 \leq j \leq q^k - 1$, we have

$$\begin{aligned} \sum_{l=0}^{p^k-1} \psi_k(\frac{2\pi(l+\beta_i)}{p^k})\psi_k(\frac{2\pi(l+\beta_j)}{p^k}) &= \sum_{l=0}^{p^k-1} \psi_k(\frac{2\pi(l+\beta_{i-j})}{p^k})\psi_k(\frac{2\pi l}{p^k}) \\ &= \sum_{l=0}^{p^k-1} \psi_k(\frac{2\pi l}{p^k})\psi_k(\frac{2\pi(l+\beta_{q^k-(i-j)})}{p^k}) \end{aligned}$$

since ψ_k is 2π periodic. Therefore $t_k \leq \|T_k\| \leq 2t_k$, where

$$t_k = \frac{1}{p^k} \sum_{l=0}^{p^k-1} (\frac{1}{q^k} \sum_{i=0}^{q^k-1} \psi_k(\frac{2\pi(l+\beta_i)}{p^k}))^2.$$

These observations lead to one of the main results of this paper:

Theorem 3.4. *Let* φ *be a continuous function on* $\mathbf{T}$. *If* $m > n$, *then*

$$r(S_\varphi(m,n)) = \lim_{k\to\infty} \|R_{p^k}(R_{q^k}\psi_k)^2\|_\infty^{\frac{1}{2k}}$$

if $\psi = \psi_1 > 0$, *where* p, q *and* ψ_k *are defined as before.*

Proof. As we have seen preceding the theorem, it suffices to show that $t_k \approx \|R_{p^k}(R_{q^k}\psi_k)^2\|_\infty$ by Theorem 3.3. Now since C_{q^k} is also an isometry on $C(\mathbf{T})$, we have, with the help of Lemma 2.1, that $\|R_{p^k}(R_{q^k}\psi_k)^2\|_\infty = \|R_{p^k}(C_{q^k}R_{q^k}\psi_k)^2\|_\infty$. On the other hand, since $R_{p^k}(C_{q^k}R_{q^k}\psi_k)^2 \in C(\mathbf{T})$ and $\mathbf{T}$ is compact, there exists $\theta_k \in [0, 2\pi)$ such that $\|R_{p^k}(C_{q^k}R_{q^k}\psi_k)^2\|_\infty = (R_{p^k}(C_{q^k}R_{q^k}\psi_k)^2)(\theta_k)$, i.e. for each k, we have

$$\|R_{p^k}(R_{q^k}\psi_k)^2\|_\infty = \frac{1}{p^k} \sum_{l=0}^{p^k-1} (\frac{1}{q^k} \sum_{i=0}^{q^k-1} \psi_k(\frac{\theta_k + 2\pi l}{p^k} + \frac{i}{q^k}))^2$$

by (2.2). Hence, $t_k \approx \|R_{p^k}(R_{q^k}\psi_k)^2\|_\infty$ follows from the fact

$$\psi_k(\frac{2\pi(l+\beta_i)}{p^k}) \approx \psi_k(\frac{2\pi l}{p^k} + \frac{i}{q^k}) \approx \psi_k(\frac{\theta_k + 2\pi l}{p^k} + \frac{i}{q^k})$$

uniformly in k by Lemma 3.1. □

Remark. When $q = 1$, $S_\varphi(m,n)$ reduces to $S_\psi(p,1) = R_p M_\psi$. In this special case, the spectral radius formula of $S_\varphi(m,n)$ in the form given in Theorem 3.4 can be derived directly from repeated applications of (2.4) and Lemma 2.1. The author believes that the formula presented in Theorem 3.3 is true for all nonnegative ψ, but has not been able prove it at this stage. On the other hand, the author has also come to believe that the possibility of the strict inequality $r(S_\varphi(m,n)) < r(S_{|\varphi|}(m,n))$ exists (i.e., Theorem 3.4 may be invalid) if ψ is no longer nonnegative. The reason being that since if $q > 1$, then C_{q^k} does not map $L^\infty(\mathbf{T})$ onto itself and, consequently, there may not exist $f \in L^2(\mathbf{T})$ such that

$$R_{p^k} M_{\psi_k} C_{q^k} f = R_{p^k}(\psi_k C_{q^k} f) = R_{p^k}(|\psi_k| C_{q^k}|f|) = R_{p^k} M_{|\psi_k|} C_{q^k}|f|.$$

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