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ON HOMOLOGY OF REAL ALGEBRAIC VARIETIES

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ABSTRACT. Let R be a commutative ring with unity and X an R-oriented compact nonsingular real algebraic variety of dimension n. If $i : X \to X_{\mathbb{C}}$ is any nonsingular complexification of X, then the kernel, which we will denote by $KH_k(X,R)$, of the induced homomorphism $i_* : H_k(X,R) \to H_k(X_{\mathbb{C}},R)$ is independent of the complexification. In this work, we study $KH_k(X,R)$ and give some of its applications.

1. INTRODUCTION

Let R be a commutative ring with unity and X an R-oriented compact nonsingular real algebraic variety of dimension n. The key observation of this note is the following result.

Theorem 1.1. *Let X be a compact R-oriented nonsingular real algebraic variety and $X_{\mathbb{C}}$ be a complexification of X. Let $i : X \to X_{\mathbb{C}}$ be the inclusion map and $KH_k(X,R)$ denote the kernel of the induced map*

$$i_* : H_k(X,R) \to H_k(X_{\mathbb{C}},R)$$

on homology. Then $KH_*(X,R)$ is independent of the complexification $X \subseteq X_{\mathbb{C}}$ and thus an (entire rational) isomorphism invariant of X. Dually, the image of the homomorphism

$$i^* : H^*(X_{\mathbb{C}},R) \to H^*(X,R),$$

denoted by $ImH^*(X,R)$, is also an isomorphism invariant of X.

The study of relative topology of a real algebraic variety in its complexification has been started by F. Klein ([15]) introducing (non)dividing real algebraic curves. In the seventies Rokhlin introduced complex orientations [21, 22]. In [4] (see also p. 264 of [23]) Arnold gave the following criteria for an even-dimensional real algebraic variety to bound in its complexification: Let X be an n-dimensional nonsingular compact real algebraic variety and $X_{\mathbb{C}}$ any nonsingular projective complexification with anti-holomorphic involution τ (X is the fixed point set of τ). Define the $\mathbb{Z}_2$-form of τ as follows:

$$H_n(X_{\mathbb{C}},\mathbb{Z}_2) \times H_n(X_{\mathbb{C}},\mathbb{Z}_2) \to \mathbb{Z}_2 \quad \text{by} \quad (\alpha,\beta) \mapsto \alpha \cdot \tau_*(\beta),$$

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where $\cdot$ denotes the homology intersection. Then the $\mathbb{Z}_2$ fundamental class of X is homologous to zero in $X_{\mathbb{C}}$ if and only if the $\mathbb{Z}_2$-form of τ is even.

Theorem 1.1 implies that whether the $\mathbb{Z}_2$-form of τ is even or odd is indeed an intrinsic property of X.

In [10] Bochnak and Kucharz recently gave another criteria for a real algebraic variety to be dividing (homologous to zero in its complexification) in terms of projective modules over the ring of regular (entire rational) functions: If X is a nonsingular compact connected real algebraic variety of dimension d, then X is dividing over $\mathbb{Q}$ if and only if for each nonsingular projective real algebraic variety Y of dimension e, such that $d+e$ is even and Y is orientable, every projective $R(X \times Y; \mathbb{C})$-module of rank $(d+e)/2$ splits off a free summand. (See Remark 4.3 at the end of Section 4 also.)

In the same work, Bochnak and Kucharz also generalize a criteria they had proved in [8] for real algebraic curves to be dividing to surfaces: A compact connected oriented nonsingular real algebraic surface X is dividing, if and only if for any nonsingular real algebraic surface Y, every entire rational map from $X \times Y$ into S^4 is null homotopic. Moreover, in this case X is diffeomorphic to $S^1 \times S^1$ (compare with Corollary 5.2).

In the next section, we will give another characterization and some other properties of $KH_*(X, R)$ and $ImH^*(X, R)$. Proofs of these results will be given in Section 3. In Section 4, we will show that if X admits a free algebraic S^1 action, where S^1 is the unit circle in $\mathbb{R}^2$, then $H_n(X, R) = KH_n(X, R)$, where $n = \dim X$. Some restrictions on entire rational maps of real algebraic varieties arising from $ImH^k(X, R)$ will be discussed in the last section.

In [17] the author used the group $KH_n(X, R)$, $n = \dim X$, to study entire rational maps of real algebraic varieties.

2. Basic properties of $KH_k(X, R)$

All real algebraic varieties under consideration in this report are compact and nonsingular. It is well known that real projective varieties are affine (Proposition 2.4.1 of [1] or Theorem 3.4.4 of [7]). Moreover, compact affine real algebraic varieties are projective (Corollary 2.5.14 of [1]) and therefore we will not distinguish between compact real affine varieties and real projective varieties.

For real algebraic varieties $X \subseteq \mathbb{R}^r$ and $Y \subseteq \mathbb{R}^s$ a map $F : X \to Y$ is said to be entire rational if there exist $f_i, g_i \in \mathbb{R}[x_1, \dots, x_r]$, $i = 1, \dots, s$, such that each g_i vanishes nowhere on X and $F = (f_1/g_1, \dots, f_s/g_s)$. We say X and Y are isomorphic to each other if there are entire rational maps $F : X \to Y$ and $G : Y \to X$ such that $F \circ G = id_Y$ and $G \circ F = id_X$. Isomorphic algebraic varieties will be regarded the same way. A complexification $X_{\mathbb{C}} \subseteq \mathbb{C}P^N$ of X will mean that X is embedded into some projective space $\mathbb{R}P^N$ and $X_{\mathbb{C}} \subseteq \mathbb{C}P^N$ is the complexification of the pair $X \subseteq \mathbb{R}P^N$. We also require the complexification to be nonsingular (blow up $X_{\mathbb{C}}$ along smooth centers away from X defined over reals if necessary, [13, 5]). For the basic definitions and facts about real algebraic geometry we refer the reader to [1, 7].

For a compact nonsingular real algebraic variety X of dimension n, let $H_k^A(X, \mathbb{Z}_2) \subseteq H_k(X, \mathbb{Z}_2)$ be the subgroup of classes represented by algebraic subsets of X and let $H^k_A(X, \mathbb{Z}_2)$ be the Poincaré dual of $H^A_{n-k}(X, \mathbb{Z}_2)$. These are well known and very useful in the study of real algebraic varieties.

Another useful tool of real algebraic geometry is $H^{2k}_{\mathbb{C}-alg}(X,\mathbb{Z})$, the cohomology subgroup of X generated by the pull backs of the complex algebraic cycles of its complexification. This subgroup, like $KH_k(X,R)$, is an isomorphism invariant of real algebraic variety X ([6]). Also let $H^k_A(X,\mathbb{Z}_2)^2$ be the subgroup

$$\{\alpha^2 \mid \alpha \in H^k_A(X,\mathbb{Z}_2)\} \subseteq H^{2k}_A(X,\mathbb{Z}_2)$$

(cup product preserves algebraic cycles [2]). In [3] Akbulut and King showed that $H^k_A(X,\mathbb{Z}_2)^2 \subseteq H^{2k}_{\mathbb{C}-alg}(X,\mathbb{Z}_2)$ for all k.

Let $i : X \to X_{\mathbb{C}}$ be the inclusion of X into some complexification. For any $0 \leq k \leq n$, define $H^{\mathbb{C}}_{n-k}(X,R)$ to be $i_!(H_{2n-k}(X_{\mathbb{C}},R))$, i.e., the Poincaré dual of $ImH^k(X,R)$ (see Section 3). Consider the intersection pairing

$$\cdot\ : H_{n-k}(X,R) \times H_k(X,R) \to R \quad \text{by} \quad \alpha \cdot \beta = c_*(D(\beta) \cap \alpha),$$

where $c_* : H_0(X,R) \to H_0(pt.,R) = R$ is the map induced from the constant map c of X to a point. Define $H^{\mathbb{C}}_{n-k}(X,R)^{\perp}$ to be the subgroup

$$\{\alpha \in H_k(X,R) \mid \beta \cdot \alpha = 0 \text{ for all } \beta \in H^{\mathbb{C}}_{n-k}(X,R)\}.$$

We are now ready to give another characterization of $KH_k(X,R)$ in the case that R is a field, for which the above intersection pairing becomes nondegenerate.

Proposition 2.1. *Let R be any field and X an irreducible nonsingular compact R-oriented real algebraic variety of dimension n. Then for any $0 \leq k \leq n$, $KH_k(X,R) = H^{\mathbb{C}}_{n-k}(X,R)^{\perp}$. In case of integer coefficients we have only $KH_k(X,\mathbb{Z}) \subseteq H^{\mathbb{C}}_{n-k}(X,\mathbb{Z})^{\perp}$.*

Theorem 2.2. *Let X and Y be compact R oriented nonsingular real algebraic varieties with* $\dim(X) = n$ *and k and l nonnegative integers.*

(1) *If $f : X \to Y$ is an entire rational map, then $f_*(KH_k(X,R)) \subseteq KH_k(Y,R)$ and $f^*(ImH^k(Y,R)) \subseteq ImH^k(X,R)$.*

(2) *If $V \subseteq \mathbb{C}P^N$ is any compact nonsingular complex algebraic variety, then $KH_k(V_{\mathbb{R}},R) = 0$.*

(3) $H^{2k}_{\mathbb{C}-alg}(X,R) \subseteq ImH^{2k}(X,R)$.

(4) *Assume that R is a field or $R = \mathbb{Z}$ and X has a complexification $X_{\mathbb{C}}$ so that $H_*(X_{\mathbb{C}},\mathbb{Z})$ is torsion free. Then, for any $\alpha \in H_k(X,R)$ and $\beta \in H_l(Y,R)$, $\alpha \times \beta \in KH_{k+l}(X \times Y,R)$ if and only if $\alpha \in KH_k(X,R)$ or $\beta \in KH_l(Y,R)$.*

(5) *Assume that X is connected and the Euler characteristic $\chi(X)$ of X in R coefficients is not zero. Then $KH_n(X,R) = 0$.*

(6) *Suppose X has dimension $n \geq 3$ with a complete intersection complexification $X_{\mathbb{C}}$. Then, $KH_k(X,\mathbb{Z}) = \bar{H}_k(X,\mathbb{Z})$ for $0 \leq k \leq n-2$, where $\bar{H}$ denotes the reduced homology.*

Remark 2.3. **i**) Theorem 2.2 (1) implies the main theorem of the previous article [17] of the author after which Selman Akbulut had proposed the existence of $KH_k(X,R)$.

ii) In general $ImH^{2k}(X,R) \neq H^{2k}_{\mathbb{C}-alg}(X,R)$. By Theorem 3.3 in [9] there exists an algebraic model X of S^{2n} with $H^{2n}_{\mathbb{C}-alg}(X,\mathbb{Z}) = 0$. However, $e(X) = 2$ and thus, by Theorem 2.2 (5), $ImH^{2n}(X,\mathbb{Z}) \neq 0$.

Example 2.4. **i**) Let X be an algebraic model for the real projective space $\mathbb{R}P^{2n}$. The first Stiefel Whitney class $w_1 \in H^1_A(X,\mathbb{Z}_2)$ is nontrivial and hence by taking powers of w_1 we get $\mathbb{Z}_2 = H^k_A(X,\mathbb{Z}_2) = H^k(X,\mathbb{Z}_2)$ for $k \leq 2n$. By the

Akbulut-King result mentioned earlier, $H^{2k}(X,\mathbb{Z}_2)=H^{2k}_A(X,\mathbb{Z}_2)=H^k_A(X,\mathbb{Z}_2)^2\subseteq H^{2k}_{\mathbb{C}-alg}(X,\mathbb{Z}_2)$ and therefore $H^{2k}(X,\mathbb{Z}_2)=H^{2k}_{\mathbb{C}-alg}(X,\mathbb{Z}_2)$. By Theorem 2.2 (3) we conclude that $KH_{2k}(X,\mathbb{Z}_2)=0$. Moreover, if $X=\mathbb{R}P^{2n}$, the standard real projective space, then $X_{\mathbb{C}}=\mathbb{C}P^{2n}$ and thus $KH_{2k+1}(X,\mathbb{Z}_2)=H_{2k+1}(X,\mathbb{Z}_2)$ for all k. In particular, Theorem 2.2 (6) does not hold in $\mathbb{Z}_2$ coefficients.

Indeed, the same argument shows that if X is a compact connected nonorientable nonsingular real algebraic surface of odd genus, then $KH_2(X,\mathbb{Z}_2)=0$. This is best possible since we know that the Klein bottle has a dividing algebraic model (Proposition 1.4 in [19]).

ii) Let X be an algebraic model for the smooth manifold $\mathbb{C}P^n$. We know that all Pontrjagin classes of X are nonzero and by the sentence preceding Corollary 2 in [3] they belong to $H^{4k}_{\mathbb{C}-alg}(X,\mathbb{Z})$. Therefore, by Theorem 2.2 (3) we see that $KH_{4k}(X,\mathbb{Z})=0$. This result is best possible: Indeed, there exists an algebraic model of the complex projective plane $\mathbb{C}P^2$ such that $KH_2(X,\mathbb{Z})\neq 0$ (Remark 1.6 in [19]).

iii) Let $X=T^n=S^1\times\cdots\times S^1$, where S^1 is the standard unit circle in $\mathbb{R}^2$. Since S^1 bounds in its complexification $S^1_{\mathbb{C}}=S^2$, for all nonzero k we have $KH_k(X,\mathbb{Z})=H_k(X,\mathbb{Z})$. Let Y be another algebraic model for T^n, where S^1 is replaced by $A=\{(x,y)\in\mathbb{R}^2 \mid x^4+y^4=1\}$ which does not bound in its complexification. (Note that $A_{\mathbb{C}}$ is a nonsingular curve of degree 4 in $\mathbb{C}P^2$ and thus of genus 3. If A is a dividing curve, then $A_{\mathbb{C}}-A$ has two connected components that are permuted under complex conjugation, which implies that the surface $A_{\mathbb{C}}$ is of even genus, a contradiction.) In this case, by Theorem 2.2 (4) we have $KH_k(Y,\mathbb{Z})=0$ for all k.

3. Proofs

To prove the above results we need some preliminaries. For any smooth map $f:N^n\to M^m$ of compact R-oriented smooth manifolds, one can define the transfer homomorphisms

$$f_!:H_{m-k}(M,R)\to H_{n-k}(N,R)\quad\text{and}\quad f^!:H^{n-k}(N,R)\to H^{m-k}(M,R)$$

via the following diagrams, where the vertical maps are the (inverses of the) Poincaré isomorphisms:

$$\begin{array}{ccc} H_{m-k}(M,R) & \xrightarrow{f_!} & H_{n-k}(N,R) \\ D\downarrow\cong & & \cong\downarrow D \\ H^k(M,R) & \xrightarrow[f^*]{} & H^k(N,R) \end{array}\qquad\qquad \begin{array}{ccc} H^{n-k}(N,R) & \xrightarrow{f^!} & H^{m-k}(M,R) \\ D^{-1}\downarrow\cong & & \cong\downarrow D^{-1} \\ H_k(N,R) & \xrightarrow[f_*]{} & H_k(M,R) \end{array}$$

Figure 1.

For any $a\in H^{n-k}(N,R)$ and $b\in H_{m-l}(M,R)$ with $\deg(f_!(b))\geq\deg(a)$, the following holds (cf. [11], p.394):

$$(*)\qquad f_*(a\cap f_!(b))=(-1)^{l\ (m-n)}\ f^!(a)\cap b.$$

Moreover, if we have a commutative diagram of smooth manifolds (see Figure 2) where the vertical maps are embeddings and f is transversal to $j(L)$ so that $f^{-1}(j(L))=i(K)$, then $f^*\circ j^!=i^!\circ g^*$ and $g_*\circ i_!=j_!\circ f_*$. This follows from

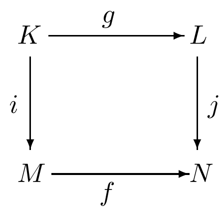

Figure 2.

the definitions, the Thom isomorphism and the fact that the Poincaré dual of an embedded submanifold is supported in any given tubular neighborhood of the submanifold so that, since f is transversal to $j(L)$, f^* pulls back the Poincaré dual of $j(L)$ to that of $i(K)$.

If V is a compact nonsingular complex algebraic variety, then we can view V as a real algebraic variety which we will denote by $V_{\mathbb{R}}$. Indeed, $V_{\mathbb{R}}$ is just the fixed point set of the antiholomorphic involution $\sigma : V \times \bar{V} \to V \times \bar{V}$ given by $\sigma(x, y) = (\bar{y}, \bar{x})$, where $\bar{V}$ is the complex conjugate of V. It is well known that there is a complex algebraic subvariety Z of some projective space $\mathbb{C}P^N$ defined by real polynomials which is biregularly isomorphic to $V \times \bar{V}$. Moreover, the real part $Z \cap \mathbb{R}P^N$ is isomorphic to $V_{\mathbb{R}}$. However, any projective real algebraic variety is affine (Proposition 3.4.4 in [7]) and hence $V_{\mathbb{R}}$ can be viewed as an affine real algebraic variety. For more details, we refer the reader to Section 1 and 2 of [14].

Proof of Theorem 1.1. Let Z_1 and Z_2 be two nonsingular complexifications of the nonsingular variety X and $i : X \to Z_1$ and $j : X \to Z_2$ be the respective inclusion maps. Assume that the homology class $i_*(\alpha)$ is zero in $H_k(Z_1, R)$ for some $\alpha \in H_k(X, R)$. It suffices to show that the homology class $j_*(\alpha)$ is zero in $H_k(Z_2, R)$. There exists a complex birational map $T : Z_1 \to Z_2$, which may not be well defined on all of Z_1, so that $j = T \circ i$. Using Hironaka's theorem ([13, 5]), we can make T well defined everywhere by blowing up Z_1 along smooth centers, defined over reals and away from its real part. Let $\pi : \tilde{Z}_1 \to Z_1$ be this sequence of blow ups. Now by Figure 2 we obtain the following commutative diagrams:

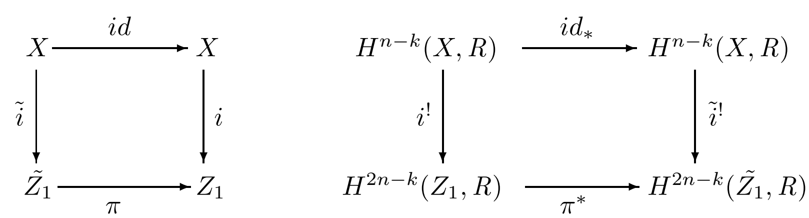

Figure 3.

Let $a = D(\alpha) \in H^{n-k}(X, R)$. Since $i_*(\alpha) = 0$, by diagram chasing, we see that $\tilde{i}^!(a) = 0$ which implies that $\tilde{i}_*(\alpha) = 0$. Hence by replacing $\tilde{Z}_1$ with Z_1, we can assume that T is well defined on all of Z_1. This implies that $j_*(\alpha) = T_*(i_*(\alpha)) = 0$.

The proof of the second statement is similar and left as an exercise. (Just use a similar diagram for homology groups.) □

Proof of Proposition 2.1. First assume that R is a field. Let $a \in H_k(X, R)$ and $c \in H^{\mathbb{C}}_{n-k}(X, R)$. Then $c = i_!(b)$ for some $b \in H_{2n-k}(X_{\mathbb{C}}, R)$. Rewriting the formula $(*)$ above, we get

$$i_*(D(a) \cap i_!(b)) = (-1)^{n(2n-\deg(b))} \;\; i^!(D(a)) \cap b.$$

From the definition of our intersection pairing and the fact that complex irreducible algebraic sets are connected, $i_!(b) \cdot a$ is zero if and only if $i_*(D(a) \cap i_!(b))$ is zero. Now, $a \in H^{\mathbb{C}}_{n-k}(X,R)^{\perp}$ if and only if $i_!(b) \cdot a = 0$ for all $b \in H_{2n-k}(X_{\mathbb{C}}, R)$, and thus if and only if $i^!(D(a)) \cap b = 0$ for all $b \in H_{2n-k}(X_{\mathbb{C}}, R)$. Since the intersection pairing is nondegenerate the latter is equivalent to $i_*(a) = 0$ and hence $a \in KH_k(X,R)$.

The proof of the second statement is the same as the first one except for the last line, which we do not need. Just note that $i_*(a) = 0$ gives us $i^!(D(a)) \cap b = 0$ for all $b \in H_{2n-k}(X_{\mathbb{C}}, R)$. □

Proof of Theorem 2.2. (1) Let $f_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ be any complexification of $f : X \to Y$. By blowing up $X_{\mathbb{C}}$ along smooth centers away from X we may assume that the complexification map is well defined on the whole $X_{\mathbb{C}}$. Now Theorem 1.1 finishes the proof.

(2) Since the composition of the inclusion map $V_{\mathbb{R}}$ into $V \times \bar{V}$, $p \mapsto (p, \bar{p})$, with the projection of $V \times \bar{V}$ onto $V = V_{\mathbb{R}}$, is a diffeomorphism of the underlying smooth manifold $V_{\mathbb{R}}$ we get $KH_k(V_{\mathbb{R}}, R) = 0$ for all k.

(3) This follows from the definitions.

(4) This follows from the Künneth formulas and the fact that if $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are complexifications for X and Y respectively, then so is $X_{\mathbb{C}} \times Y_{\mathbb{C}}$ for $X \times Y$.

(5) This is classical but we will produce the argument for completeness. Multiplying the tangent vectors by i we see that the normal bundle of a nonsingular real algebraic set in its complexification is isomorphic to its tangent bundle, possibly with reversed orientation, so that the self-intersection number of X in its complexification is equal, up to a sign, to its Euler characteristic.

(6) If k is odd, then $H_k(X_{\mathbb{C}}, \mathbb{Z}) = 0$ and hence $KH_k(X, \mathbb{Z}) = H_k(X, \mathbb{Z})$. If k is even, then $H_k(X_{\mathbb{C}}, \mathbb{Z}) = \mathbb{Z}$ on which the cohomology class $\omega^{k/2}$ is nonzero, where ω is the Kähler form on $X_{\mathbb{C}}$. However, ω is identically zero on X and thus $KH_k(X, \mathbb{Z}) = \bar{H}_k(X, \mathbb{Z})$. □

4. S^1 actions on algebraic sets

Let S^1 denote the unit circle in $\mathbb{R}^2$. We say that S^1 acts algebraically on a real algebraic variety X if the action is given by some entire rational map $\theta : S^1 \times X \to X$.

Theorem 4.1. *Let X be a compact connected nonsingular R-oriented real algebraic variety of dimension n on which S^1 acts freely and algebraically, and $\pi : X \to X/S^1 = B$ be the smooth quotient map. Let $0 \leq k \leq n-1$. Assume that R is either a field or $R = \mathbb{Z}$ and $H_{k+1}(B, \mathbb{Z})$ is torsion free. Then $\pi_!(H_k(B,R)) \subseteq KH_{k+1}(X,R)$.*

Corollary 4.2. *Let S^1 act freely and algebraically on a compact connected nonsingular R oriented real algebraic variety X of dimension n and B be the smooth quotient X/S^1. Then $KH_n(X,R) = H_n(X,R)$. Moreover, if the associated S^1 bundle $\pi : X \to B$ has nontorsion Euler class, then $KH_{n-1}(X,R) = H_{n-1}(X,R)$.*

Theorem 4.1 appeared in [18] without proof.

Proof of Theorem 4.1. The algebraic S^1 action on X is given by some entire rational map $\theta : S^1 \times X \to X$. Complexifying this we get a rational map $\theta_{\mathbb{C}}$:

$S^1_{\mathbb{C}} \times X_{\mathbb{C}} \to X_{\mathbb{C}}$ which may not be well defined on all of $S^1_{\mathbb{C}} \times X_{\mathbb{C}}$. However, this map can be made well defined by blowing up some smooth complex algebraic centers away from the real part $S^1 \times X$ ([13, 5]). Let $L \subseteq S^1_{\mathbb{C}} \times X_{\mathbb{C}}$ be such a smooth center. Since $L \cap (S^1 \times X) = \emptyset$, L meets each $S^1_{\mathbb{C}} \times \{x_0\}$, $x_0 \in X$, in finitely many points. Therefore blowing up L will not affect $S^1_{\mathbb{C}} \times \{x_0\}$ or $S^1_{\mathbb{C}} \times X$. Hence we may assume that $\theta_{\mathbb{C}}$ is well defined on $S^1_{\mathbb{C}} \times X$. Note that $S^1_{\mathbb{C}}$ is S^2 and S^1 acts on $S^1_{\mathbb{C}}$ by rotations. Moreover, S^1 acts freely on $S^1_{\mathbb{C}} \times X$: For any $z \in S^1$ and $(w, x) \in S^1_{\mathbb{C}} \times X$, let $z \cdot (w, x) = (w \cdot z^{-1}, \theta_{\mathbb{C}}(z, x))$, so that $\theta_{\mathbb{C}}$ sends each orbit of this action to a point of $X_{\mathbb{C}}$. Let D^2 denote the closure of one of the two components of $S^1_{\mathbb{C}} - S^1$ and T denote the restriction of $\theta_{\mathbb{C}}$ to $D^2 \times X$. From now on regard $T : D^2 \times X \to X_{\mathbb{C}}$ as a smooth map. This map descends to a map $\tilde{T} : W \to X_{\mathbb{C}}$, where W is the smooth quotient $(D^2 \times X)/S^1$ with boundary $\partial W = X$. Note that W can be identified with the mapping cylinder

$$X \times [0, 1] \cup_{(x,0) \sim \pi(x)} B$$

of the quotient map $\pi : X \to B$. The restriction of $\tilde{T}$ to its boundary is the inclusion $i : X \to X_{\mathbb{C}}$. To finish the proof we need to show that $i_*(\pi_!(\alpha)) = 0$ for any $\alpha \in H_k(B, R)$. To see this consider the Gysin sequence associated to the S^1 bundle $\pi : X \to B$:

$$\cdots \to H^{k-1}(B, R) \xrightarrow{\cup\chi} H^{k+1}(B, R) \xrightarrow{\pi^*} H^{k+1}(X, R) \xrightarrow{\pi^!} H^k(B, R) \to \cdots,$$

where χ is the Euler class of the bundle. (This may not be the standard Gysin sequence. For a proof one may look at Theorem 9.2, 11.3 and (the proof of) Theorem 13.2 of [11].)

Claim. $\pi_* \circ \pi_! = 0$.

Proof of the Claim. Let $\alpha \in H_k(B, R)$ and $a \in H^{k+1}(B, R)$. Now we have $a(\pi_*(\pi_!(\alpha))) = \pi_*(\pi^*(a)(\pi_!(\alpha)))$ and by the identity $(*)$ in Section 3 this is equal to $\pm((\pi^! \circ \pi^*)(a))(\alpha)$. But the latter is zero since the composition $\pi^! \circ \pi^* = 0$ in the above Gysin exact sequence. So we obtain $a(\pi_*(\pi_!(\alpha))) = 0$ for all $a \in H^{k+1}(B, R)$. Now the Universal Coefficient Theorem finishes the proof of the claim because R is either a field or $R = \mathbb{Z}$ and $H_{k+1}(B, \mathbb{Z})$ is torsion free.

Finally, since W is the mapping cylinder of the quotient map $\pi : X \to B$ the composition $i_* \circ \pi_!$ is the same as $\tilde{T}_* \circ \pi_* \circ \pi_!$ which is zero by the claim. □

Proof of Corollary 4.2. Let us prove the second statement first. Consider the Gysin sequence with integer coefficients:

$$\cdots \to H^{n-3}(B, \mathbb{Z}) \xrightarrow{\cup\chi} H^{n-1}(B, \mathbb{Z}) \xrightarrow{\pi^*} H^{n-1}(X, \mathbb{Z}) \xrightarrow{\pi^!} H^{n-2}(B, \mathbb{Z}) \to 0.$$

Since the Euler class is not torsion this sequence descends to the following short exact sequence:

$$0 \to \mathbb{Z}_d \xrightarrow{\pi^*} H^{n-1}(X, \mathbb{Z}) \xrightarrow{\pi^!} H^{n-2}(B, \mathbb{Z}) \to 0$$

for some nonnegative integer d. X and B are orientable and hence the groups $H_{n-1}(X, \mathbb{Z})$ and $H_{n-2}(B, \mathbb{Z})$ are torsion free. The identity $(*)$ in Section 3 becomes $\pi_*(a \cap \pi_!(b)) = -\pi^!(a) \cap b$. Now this identity together with the Universal Coefficient Theorem gives the result $KH_{n-1}(X, \mathbb{Z}) = H_{n-1}(X, \mathbb{Z})$. In the case that R is a field the proof is similar.

For the first statement observe that the Gysin sequence induces an isomorphism

$$0 \to R = H^n(X, R) \xrightarrow{\pi^!} H^{n-1}(B, R) = R \to 0.$$

The rest is similar to that of the second statement. □

Remark 4.3. **i**) Note that if R is a field of positive characteristic, say p, then the Euler class can be zero mod p even if the bundle $\pi : M \to B$ is nontrivial.

ii) Note that the algebraic set X in Theorem 4.1 has necessarily zero Euler characteristic. In fact, we conjecture that any compact connected smooth boundary $M = \partial W$ with zero Euler characteristic has an algebraic model X with torsion $[X]$ in $H_n(X_{\mathbb{C}}, \mathbb{Z})$. By Theorem 2.2 (5) and Theorem 2.1 of [19] the assumptions are necessary. We have to mention the result of R. S. Kulkarni that for compact homogeneous manifolds this conjecture is true. In other words, a compact homogeneous manifold M has an algebraic model X with $[X]$ torsion in $H_n(X_{\mathbb{C}}, \mathbb{Z})$ if and only if $e(M) = 0$ (Corollary 4.6 and Theorem 5.1 in [16]). (See also Corollary 2.3 of [18].)

iii) Following the referee's suggestion the author obtained some results about algebraic K-theory of varieties with algebraic circle action correlating the Bochnak-Kucharz result, mentioned in the introduction, with Theorem 4.1 ([20]).

5. Restrictions on entire rational maps

Although the relative topology of the pair $(X_{\mathbb{C}}, X)$ is interesting in its own right, another motivation for defining $KH_k(X, R)$ $(ImH^k(X, R))$ comes from the study of the entire rational maps between real algebraic varieties. The following theorem is a corollary of Theorem 2.2 (1).

Theorem 5.1. *Suppose that $f : X \to Y$ is an entire rational map of compact connected nonsingular real algebraic varieties of the same dimension n, where $R = \mathbb{Q}$ if they are both orientable and $R = \mathbb{Z}_2$ otherwise. If f has nonzero R-degree, then for any integer $0 \leq k \leq n$, we have*

$$\dim_R(ImH^k(X, R)) \geq \dim_R(ImH^k(Y, R)).$$

Corollary 5.2. *Let M be a compact connected smooth manifold of dimension $2n$ and S^1 acts on it freely. Then, M has an algebraic model X, so that for any compact nonsingular real algebraic set Y, homotopy equivalent to S^{2n}, any entire rational map $f : X \to Y$ is null homotopic.*

Proof. By [12] there is an algebraic model X of M on which the S^1 action is algebraic. First assume that M is orientable. Theorem 4.1 implies that $KH_{2n}(X, \mathbb{Q}) = H_{2n}(X, \mathbb{Q})$ and hence $ImH^{2n}(X, \mathbb{Q}) = 0$. On the other hand, Euler characteristic of Y is 2 and therefore by Theorem 2.2 (5) $ImH^{2n}(Y, \mathbb{Q}) \neq 0$. Now by the above theorem, any entire rational map $f : X \to Y$ induces the zero map in the top homology, and thus has degree zero. Hopf's Theorem implies that f is null homotopic.

Now assume that M is nonorientable. We need to show that mod 2 degree of $f : X \to Y$ is zero. We claim that $KH_{2n}(Y, \mathbb{Z}_2) = 0$ also. To see this just consider the Bockstein homology sequence

$$\cdots \to H_{2n}(Y_{\mathbb{C}}, \mathbb{Z}) \xrightarrow{\times 2} H_{2n}(Y_{\mathbb{C}}, \mathbb{Z}) \to H_{2n}(Y_{\mathbb{C}}, \mathbb{Z}_2) \xrightarrow{\partial} H_{2n-1}(Y_{\mathbb{C}}, \mathbb{Z}) \to \cdots$$

and note that since ± 2 (the self intersection number of Y in its complexification) is not divisible by 4 the integer fundamental class $[Y]$ is not in the image of the first map. So $[Y]$ is not zero in $H_{2n}(Y_{\mathbb{C}}, \mathbb{Z}_2)$. Again by the above theorem f has zero mod 2 degree and hence is null homotopic. □

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