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HILBERT MODULAR PSEUDODIFFERENTIAL OPERATORS

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Abstract. We introduce Jacobi-like forms of several variables, and study their connections with Hilbert modular forms and pseudodifferential operators of several variables. We also construct Rankin-Cohen brackets for Hilbert modular forms using such Jacobi-like forms.

1. Introduction

One of the areas of mathematics that has been studied most actively during the past few decades is the theory of nonlinear integrable equations, also known as soliton equations, which include many well-known equations in mathematical physics such as the nonlinear Schrödinger equation, the Sine-Gordon equation, the Korteweg-de Vries (KdV) equation, and the Katomtsev-Petviashvili (KP) equation. An important tool for the systematic study of soliton equations is the algebra of pseudodifferential operators of one variable (see e.g. [3]). Soliton equations are related to various topics in mathematics, and our interest in this paper lies in their relation with modular forms, which play a major role in number theory and also appear in various other areas. For example, certain solutions of soliton equations can be expressed in terms of theta functions (cf. [1], [3], [7]), which are examples of modular forms. This indicates that there is at least an indirect relation between pseudodifferential operators and modular forms. In a recent paper [2] (see also [9]), however, Cohen, Manin and Zagier studied a more direct connection between the two objects.

In many applications modular forms often appear as Jacobi forms, which generalize theta functions and were systematically introduced by Eichler and Zagier [4]. In [9] Zagier studied Rankin-Cohen brackets for modular forms in connection with theta series, Jacobi forms, and pseudodifferential operators. In the process he introduced Jacobi-like forms which generalize Jacobi forms in some sense. Jacobi-like forms are formal power series, whose coefficients are holomorphic functions on the Poincaré upper half plane, satisfying a certain transformation formula under the action of a discrete subgroup of $SL(2,\mathbb{R})$. In [2] Cohen, Manin and Zagier extended the previous work of Zagier by establishing explicit close relations among pseudodifferential operators of one variable, modular forms, and Jacobi-like forms. The purpose of this paper is to extend the results of Cohen, Manin and Zagier further to the case of several variables.

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Pseudodifferential operators of several variables and their links to soliton equations were investigated in a recent paper by Parshin [8]. More precisely, he introduced a skew-field of formal pseudodifferential operators of several variables and studied their relations with the KP-hierarchy as well as Poisson structures. Among modular forms of several variables that are important in number theory there are Hilbert modular forms, which are holomorphic functions on the product of a finite number of copies of the Poincaré upper half plane satisfying a certain transformation formula (see e.g. [5], [6]). In this paper we introduce Jacobi-like forms of several variables, and study their connections with Hilbert modular forms and pseudodifferential operators of several variables. We also construct Rankin-Cohen brackets for Hilbert modular forms using such Jacobi-like forms.

2. Pseudodifferential operators

In this section we review pseudodifferential operators of several variables, essentially following Parshin [8], and describe linear operators which associate such pseudodifferential operators to holomorphic functions on the product of a finite number of copies of the Poincaré upper half plane.

Given a positive integer n, let $(z_1, \dots, z_n)$ be the standard coordinate system for $\mathbb{C}^n$, and let $\partial_1, \dots, \partial_n$ be the differential operators given by

$$\partial_1 = \frac{\partial}{\partial z_1}, \dots, \partial_n = \frac{\partial}{\partial z_n}.$$

Let $\mathcal{H} \subset \mathbb{C}$ be the Poincaré upper half plane, and let R be the ring of holomorphic functions $f(z_1, \dots, z_n)$ on $\mathcal{H}^n \subset \mathbb{C}^n$. Throughout this paper we shall often use multi-index notations. Thus, given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $u = (u_1, \dots, u_n) \in \mathbb{C}^n$, we have

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad u^\alpha = u_1^{\alpha_1} \cdots u_n^{\alpha_n}, \tag{2.1}$$

and for $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ we write $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for each $i = 1, \dots, n$. Furthermore, we also write $\boldsymbol{c} = (c, \dots, c) \in \mathbb{Z}^n$ if $c \in \mathbb{Z}$, and denote by $\mathbb{Z}_+$ the set of nonnegative integers. Given $\alpha \in \mathbb{Z}^n$ and $\beta \in \mathbb{Z}_+^n$, we write $\beta! = \beta_1! \cdots \beta_n!$ and

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n},$$

where for $1 \leq i \leq n$ we have $\binom{\alpha_i}{0} = 1$ and

$$\binom{\alpha_i}{\beta_i} = \frac{\alpha_i(\alpha_i - 1) \cdots (\alpha_i - \beta_i + 1)}{\beta_i!}$$

for $\beta_i > 0$.

Definition 2.1. A *pseudodifferential operator of n-variables* is a formal series of the form

$$L = \sum_{\alpha \leq \nu} f_\alpha(z) \partial^\alpha \tag{2.2}$$

for some $\nu \in \mathbb{Z}^n$, where $z = (z_1, \dots, z_n) \in \mathcal{H}_n$ and $f_\alpha \in R$ for all $\alpha = (\alpha_1, \dots, \alpha_n) \leq \nu$. We shall denote by ΨDO the complex vector space consisting of all pseudodifferential operators of n-variables.

If the pseudodifferential operator in (2.2) can be written in the form $L = \sum_{i\le r} a_i \partial_n^i$ with $a_r \neq 0$, then $r = \operatorname{ord}(L)$ will be called the *order* of L. The *highest term* $\mathrm{HT}(L)$ of L is then defined inductively as follows. If $L = \sum_{i\le r} a_i \partial_n^i$ with $\operatorname{ord}(L) = r$, then we set

$$\mathrm{HT}(L) = (\mathrm{HT}(a_r))\partial_n^r.$$

If the highest term of L is of the form $\mathrm{HT}(L) = f(z)\partial_1^{\eta_1}\cdots\partial_n^{\eta_n}$, then we set

$$\nu(L) = (\eta_1, \dots, \eta_n) \in \mathbb{Z}^n.$$

Given an element $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}^n$, we set

$$\Psi\mathrm{DO}_\omega = \{L \in \Psi\mathrm{DO} \mid \nu(L) \preceq \omega\}, \qquad \Psi\mathrm{DO}_\omega^* = \{L \in \Psi\mathrm{DO} \mid \nu(L) \prec \omega\},$$

where $\prec$ is the lexicographic order such that $\nu(L) = (\eta_1, \dots, \eta_n) \prec 0$ if

$$\eta_n < 0, \quad \text{or} \quad \eta_n = 0 \text{ and } \eta_{n-1} < 0, \quad \text{or } \dots, \text{ etc.},$$

and $\preceq$ means $\prec$ or $=$. Then we obtain a short exact sequence

$$0 \to \Psi\mathrm{DO}_\omega^* \to \Psi\mathrm{DO}_\omega \xrightarrow{\xi} R \to 0, \tag{2.3}$$

where ξ is the map sending L to the coefficient of its highest term, that is, $\xi(L) = f_\omega(z)$ if $\mathrm{HT}(L) = f_\omega(z)\partial^\omega$.

Given a positive integer n, the usual action of $SL(2,\mathbb{R})$ on $\mathcal{H}$ by linear fractional transformations induces an action of $SL(2,\mathbb{R})^n$ on $\mathcal{H}^n$. Thus, if $\gamma \in SL(2,\mathbb{R})^n$ and $z = (z_1, \dots, z_n) \in \mathcal{H}^n$ with

$$\gamma = (\gamma_1, \dots, \gamma_n), \quad \gamma_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in SL(2,\mathbb{R}) \quad (1 \le i \le n), \tag{2.4}$$

then we have

$$\gamma z = (\gamma_1 z_1, \dots, \gamma_n z_n) = \Big(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \dots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\Big). \tag{2.5}$$

Under this action the differential operator $\partial_i^{\alpha_i} = \partial^{\alpha_i}/\partial z_i^{\alpha_i}$ for $1 \le i \le n$ with $\alpha_i \ge 1$ is transformed to the operator

$$\widetilde{\partial}_i^{\alpha_i} = [(c_i z_i + d_i)^2 \partial_i]^{\alpha_i} = \sum_{k=0}^\infty k! \binom{\alpha_i}{k}\binom{\alpha_i - 1}{k} c_i^k (c_i z_i + d_i)^{2\alpha_i - k} \partial^{\alpha_i - k}.$$

Thus the differential operator ∂^α in (2.1) is transformed to

$$\widetilde{\partial}^\alpha = \widetilde{\partial}_1^{\alpha_1} \cdots \widetilde{\partial}_n^{\alpha_n} = \sum_{\nu \ge \mathbf{0}} \nu! \binom{\alpha}{\nu}\binom{\alpha - \mathbf{1}}{\nu} c^\nu (cz + d)^{2\alpha - \nu} \partial^{\alpha - \nu}, \tag{2.6}$$

where multi-index notations were used for $\alpha = (\alpha_1, \dots, \alpha_n)$, and

$$\sum_{\nu \ge \mathbf{0}} = \sum_{\nu_1 = 0}^\infty \cdots \sum_{\nu_n = 0}^\infty$$

with $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$.

Lemma 2.2. *Let $\xi : \Psi DO_\omega \to R$ be as in* (2.3), *and let L be an element of ΨDO_ω with $\mathrm{HT}(L) = f_\omega(z)\partial^\omega$. If $\widetilde{L}$ is the transformed operator of L under the action of an element γ of $SL(2,\mathbb{R})^n$ of the form* (2.4), *then we have*

$$\xi(\widetilde{L}) = (cz + d)^{2\omega} f_\omega(\gamma z)$$

for $z = (z_1, \dots, z_n) \in \mathcal{H}^n$, where γz is as in (2.5).

Proof. Since $\mathrm{HT}(L) = f_\omega(z)\partial^\omega$, by (2.6) we see that

$$\mathrm{HT}(\widetilde{L}) = f_\omega(\gamma z)\widetilde{\partial}^\omega = f_\omega(\gamma z)(cz+d)^{2\omega}\partial^\omega;$$

hence the lemma follows. □

Given $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{Z}^n$ and $f \in R$, we set

$$(f \mid_{2\eta} \gamma)(z) = (cz+d)^{-2\eta} f(\gamma z), \tag{2.7}$$

$$\mathcal{L}_\eta(f) = \sum_{\nu \geq \mathbf{0}} (-1)^\nu \frac{(\nu+\eta)!(\nu+\eta-1)!}{\nu!(\nu+2\eta-1)!}(\partial^\nu f)\partial^{-\eta-\nu} \tag{2.8}$$

for all $z \in \mathcal{H}^n$ if $\gamma \in SL(2,\mathbb{R})^n$ is as in Lemma 2.2.

Proposition 2.3. *For each $\gamma \in SL(2,\mathbb{R})^n$ of the form* (2.4) *and $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}_+^n$ we have*

$$\mathcal{L}_\omega(f \mid_{2\omega} \gamma) = \mathcal{L}_\omega(f) \circ \gamma$$

for all $f \in R$.

Proof. Given $z = (z_1, \dots, z_n)$ and $1 \leq i \leq n$, let $q_i : \mathcal{H} \to \mathcal{H}^n$ be the map defined by

$$\mathrm{pr}_j \circ q_i(x) = \begin{cases} z_j & \text{for } j \neq i, \\ x & \text{for } j = i, \end{cases}$$

where $\mathrm{pr}_j : \mathcal{H}^n \to \mathcal{H}$ denotes the natural projection onto the j-th component. Given $i \in \{1, \dots, n\}$ and a holomorphic function $f : \mathcal{H}^n \to \mathbb{C}$, let $f_i = f \circ q_i$, and set

$$\mathcal{L}^{(i)}_{\omega_i}(f_i) = \sum_{\nu_i \geq 0} (-1)^{\nu_i} \frac{(\nu_i+\omega_i)!(\nu_i+\omega_i-1)!}{\nu_i!(\nu_i+2\omega_i-1)!} f_i^{(\nu_i)} \partial^{-\omega_i-\nu_i},$$

where $f_i^{(\nu_i)}$ denotes the ν_i-th derivative of $f_i : \mathcal{H} \to \mathbb{C}$. Then it follows from Proposition 1 in [2] that

$$\mathcal{L}^{(i)}_{\omega_i}(f_i) \circ \gamma = \mathcal{L}^{(i)}_{\omega_i}(f_i \mid_{2\omega_i} \gamma) \tag{2.9}$$

for all $\gamma \in SL(2,\mathbb{R})^n$. Since $\partial^\nu f = \partial_1^{\nu_1} \cdots \partial_n^{\nu_n} f = \partial_1^{\nu_1} \cdots \partial_{n-1}^{\nu_{n-1}} f_n^{(\nu_n)}$, we have

$$\begin{aligned}\mathcal{L}_\omega(f) = \sum_{\nu_1=0}^{\infty} (-1)^{\nu_1} \frac{(\nu_1+\omega_1)!(\nu_1+\omega_1-1)!}{\nu_1!(\nu_1+2\omega_1-1)!} \\ \cdots \sum_{\nu_{n-1}=0}^{\infty} (-1)^{\nu_{n-1}} \frac{(\nu_{n-1}+\omega_{n-1})!(\nu_{n-1}+\omega_{n-1}-1)!}{\nu_{n-1}!(\nu_{n-1}+2\omega_{n-1}-1)!} \\ \times \partial_1^{\nu_1} \cdots \partial_{n-1}^{\nu_{n-1}} \mathcal{L}^{(n)}_{\omega_n}(f_n) \partial_1^{-\omega_1-\nu_1} \cdots \partial_{n-1}^{-\omega_{n-1}-\nu_{n-1}}.\end{aligned}$$

Therefore by induction we see that

$$\mathcal{L}_\omega(f) = \mathcal{L}^{(1)}_{\omega_1}(\mathcal{L}^{(2)}_{\omega_2}(\cdots \mathcal{L}^{(n-1)}_{\omega_{n-1}}(\mathcal{L}^{(n)}_{\omega_n}(f_n)_{n-1})_{n-2} \cdots)_1).$$

Thus for $\gamma = (\gamma_1, \dots, \gamma_n) \in SL(2,\mathbb{R})^n$, using (2.9), we obtain

$$\begin{aligned}\mathcal{L}_\omega(f) \circ \gamma &= \mathcal{L}^{(1)}_{\omega_1}(\mathcal{L}^{(2)}_{\omega_2}(\cdots \mathcal{L}^{(n-1)}_{\omega_{n-1}}(\mathcal{L}^{(n)}_{\omega_n}(f_n \mid_{2\omega_n} \gamma_n)_{n-1} \mid_{2\omega_{n-1}} \gamma_{n-1})_{n-2} \cdots)_1 \mid_{2\omega_1} \gamma_1) \\ &= \mathcal{L}_\omega(f \mid_{2\omega} \gamma).\end{aligned}$$

Hence the proposition follows. □

3. Hilbert modular forms

Let Γ be a discrete subgroup of $SL(2,\mathbb{R})^n$, and let $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}_+^n$. The action of $SL(2,\mathbb{R})^n$ on $\mathcal{H}^n$ given by (2.5) induces an action of Γ on $\mathcal{H}^n$.

Definition 3.1. A *Hilbert modular form of weight* ω *for* Γ is a holomorphic function $f : \mathcal{H}^n \to \mathbb{C}$ such that

$$f \mid_\omega \gamma = f$$

for all $\gamma \in \Gamma$, where $f \mid_\omega \gamma$ is as in (2.7). We shall denote by $\mathcal{M}_\omega(\Gamma)$ the space of all Hilbert modular forms of weight ω for Γ.

Remark 3.2. The usual definition of Hilbert modular forms also includes the regularity condition at the cusps, which is satisfied automatically for $n > 1$ by Koecher's principle (cf. [5], [6]).

Let $\Psi\mathrm{DO}_\omega$ and $\Psi\mathrm{DO}^*_\omega$ be as in (2.3), and let $\Psi\mathrm{DO}^\Gamma_\omega$ and $\Psi\mathrm{DO}^{*\Gamma}_\omega$ be their Γ-invariant subspaces, respectively.

Theorem 3.3. *The short exact sequence in* (2.3) *induces the short exact sequence*

$$0 \to \Psi DO^{*\Gamma}_\omega \to \Psi DO^\Gamma_\omega \xrightarrow{\xi} \mathcal{M}_{2\omega}(\Gamma) \to 0,$$

which splits.

Proof. Let L be a Γ-invariant element of $\Psi\mathrm{DO}_\omega$ with $\mathrm{HT}(L) = f_\omega(z)\partial^\omega$. Then for each $\gamma \in \Gamma$ the transformed operator $\widetilde{L}$ under the action of γ coincides with L. Thus, if $\xi : \Psi\mathrm{DO}_\omega \to R$ is as in (2.3), by Lemma 2.2 we have

$$f_\omega(z) = \xi(L) = \xi(\widetilde{L}) = (cz+d)^{2\omega} f(\gamma z)$$

for all $z \in \mathcal{H}^n$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subset SL(2,\mathbb{R})^n$. Hence it follows that

$$\xi(\Psi\mathrm{DO}^\Gamma_\omega) \subset \mathcal{M}_{2\omega}(\Gamma).$$

On the other hand, if $f \in \mathcal{M}_{2\omega}(\Gamma)$, by Proposition 2.3 we see that

$$\mathcal{L}_\omega(f) \circ \gamma = \mathcal{L}_\omega(f \mid_{2\omega} \gamma) = \mathcal{L}_\omega(f)$$

for all $\gamma \in \Gamma$. Thus we have $\mathcal{L}_\omega(f) \in \Psi\mathrm{DO}^\Gamma_\omega$, and hence we obtain a map $\mathcal{L}_\omega : \mathcal{M}_{2\omega}(\Gamma) \to \Psi\mathrm{DO}^\Gamma_\omega$. Furthermore, using (2.8), we see that

$$\xi(\mathcal{L}_\omega(f)) = f.$$

Since the exactness at $\Psi\mathrm{DO}^{*\Gamma}_\omega$ is clear, the proof of the theorem is complete. □

4. Jacobi-like forms

Let R be the ring of holomorphic functions on $\mathcal{H}^n$ as in Section 2, and let $R[[X]] = R[[X_1, \dots, X_n]]$ be the set of all formal power series in $X_1, \dots, X_n$ with coefficients in R. Thus, using multi-index notations, an element of $R[[X]]$ can be written in the form

$$\Phi(z, X) = \sum_{\alpha \geq 0} f_\alpha(z) X^\alpha$$

with $z = (z_1, \dots, z_n) \in \mathcal{H}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ and $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Let Γ be a discrete subgroup of $SL(2,\mathbb{R})^n$.

Definition 4.1. A *Jacobi-like form of n variables for Γ* is an element

$$\Phi(z, X) = \Phi(z, X_1, \dots, X_n)$$

of $R[[X]]$ such that

$$\Phi(\gamma z, (cz+d)^{-2} X) = \exp\biggl(\sum_{i=1}^{n} c_i (c_i z_i + d_i)^{-1} X_i\biggr) \cdot \Phi(z, X) \tag{4.1}$$

for all $\gamma \in \Gamma$ as in (2.4) and $z = (z_1, \dots, z_n) \in \mathcal{H}^n$, where

$$(cz+d)^{-2} X = ((c_1 z_1 + d_1)^{-2} X_1, \dots, (c_n z_n + d_n)^{-2} X_n).$$

We shall denote by $\mathcal{J}(\Gamma)$ the space of all Jacobi-like forms of n variables for Γ.

Lemma 4.2. *The formal power series $\Phi(z, X) = \sum_{\alpha \geq \mathbf{1}} \phi_\alpha(z) X^\alpha \in R[[X]]$ is an element of $\mathcal{J}(\Gamma)$ if and only if*

$$(\phi_\alpha \mid_{2\alpha} \gamma)(z) = \sum_{\delta=\mathbf{0}}^{\alpha-\mathbf{1}} \frac{1}{\delta!} \Bigl(\frac{c}{cz+d}\Bigr)^{\delta} \phi_{\alpha-\delta}(z) \tag{4.2}$$

for all $\gamma \in \Gamma$ as in (2.4), $z \in \mathcal{H}^n$ and $\alpha \geq \mathbf{1}$.

Proof. Given $\gamma \in \Gamma$ as in (2.4), by (4.1) we have

$$\begin{aligned}\sum_{\alpha \geq \mathbf{1}} \phi_\alpha(\gamma z)(cz+d)^{-2\alpha} X^\alpha &= \prod_{i=1}^{n} \biggl(\sum_{\mu_i=0}^{\infty} \frac{1}{\mu_i!} \frac{c_i^{\mu_i} X_i^{\mu_i}}{(c_i z_i + d_i)^{\mu_i}}\biggr) \cdot \sum_{\nu \geq \mathbf{1}} \phi_\nu(z) X^\nu \\ &= \sum_{\mu \geq \mathbf{0}} \sum_{\nu \geq \mathbf{1}} \frac{1}{\mu!} \biggl(\frac{c}{cz+d}\biggr)^{\mu} \phi_\nu(z) X^{\mu+\nu}.\end{aligned}$$

Comparing the coefficients of X^α, we obtain

$$\phi_\alpha(\gamma z)(cz+d)^{-2\alpha} = \sum_{\delta=\mathbf{0}}^{\alpha-\mathbf{1}} \frac{1}{\delta!} \Bigl(\frac{c}{cz+d}\Bigr)^{\delta} \phi_{\alpha-\delta}(z)$$

for all $z \in \mathcal{H}^n$; hence the lemma follows. □

Given $\omega \in \mathbb{Z}^n$ with $\omega \geq \mathbf{1}$, we set

$$\mathcal{J}(\Gamma)_\omega = \mathcal{J}(\Gamma) \cap X^\omega \mathcal{J}(\Gamma).$$

If $\Phi(z, X) = \sum_{\alpha \geq \omega} \phi_\alpha(z) X^\alpha \in \mathcal{J}(\Gamma)_\omega$, then by Lemma 4.2 we have

$$(\phi_\omega \mid_{2\omega} \gamma)(z) = \phi(z)$$

for all $z \in \mathcal{H}^n$ and $\gamma \in \Gamma$; hence the initial coefficient $\phi_\omega(z)$ is a Hilbert modular form of weight 2ω for Γ. Thus we obtain a map

$$\mathfrak{F} : \mathcal{J}(\Gamma)_\omega \to \mathcal{M}_{2\omega}(\Gamma) \tag{4.3}$$

sending an element of $\mathcal{J}(\Gamma)_\omega$ to its coefficient of X^ω.

Lemma 4.3. *If $f \in R$ and $\nu, \mu \in \mathbb{Z}_+^n$, then we have*

$$\partial^\nu (f \mid_\mu \gamma)(z) = \sum_{\alpha=\mathbf{0}}^{\nu} \frac{\nu!}{\alpha!} \binom{\mu+\nu-\mathbf{1}}{\nu-\alpha} \frac{(-c)^{\nu-\alpha}}{(cz+d)^{\mu+\nu+\alpha}} \partial^\alpha f(\gamma z) \tag{4.4}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$ as in (2.4).

Proof. If $\nu = (\nu_1, \dots, \nu_n)$ and $\mu = (\mu_1, \dots, \mu_n)$, then we have

$$\begin{aligned}\partial^\nu (f\,|_\mu\,\gamma)(z) &= \partial^\nu (f(\gamma z)(cz+d)^{-\mu})\\ &= \partial_1^{\nu_1}\cdots\partial_n^{\nu_n}(f(\gamma_1 z_1,\dots,\gamma_n z_n)(c_1z_1+d_1)^{-\mu_1}\cdots(c_nz_n+d_n)^{-\mu_n}).\end{aligned}$$

Thus, using [2, (1.9)], we have

$$\begin{aligned}\partial^\nu (f\,|_\mu\,\gamma)(z) = \partial^{\nu_1}\cdots\partial^{\nu_{n-1}}\bigg(\sum_{\alpha_n=\mathbf{0}}^{\nu_n}\frac{\nu_n!}{\alpha_n!}\binom{\mu_n+\nu_n-1}{\nu_n-\alpha_n}\frac{(-c_n)^{\nu_n-\alpha_n}}{(c_nz_n+d_n)^{\mu_n+\nu_n+\alpha_n}}\\ \times\,\partial_n^{\nu_n} f(\gamma_1z_1,\dots,\gamma_nz_n)(c_1z_1+d_1)^{-\mu_1}\cdots(c_{n-1}z_{n-1}+d_{n-1})^{-\mu_{n-1}}\bigg).\end{aligned}$$

Thus the formula (4.4) can be obtained by induction. □

Theorem 4.4. *The formal power series $\Phi(z,X) = \sum_{\alpha\geq\mathbf{1}}\phi_\alpha(z)X^\alpha \in R[[X]]$ is an element of $\mathcal{J}(\Gamma)$ if and only if*

$$\phi_\alpha(z) = \sum_{\beta=\mathbf{0}}^{\alpha-\mathbf{1}}\frac{1}{\beta!(2\alpha-\beta-\mathbf{1})!}\partial^\beta f_{\alpha-\beta}(z) \tag{4.5}$$

for all $\alpha\geq\mathbf{1}$ with $f_\nu\in\mathcal{M}_{2\nu}(\Gamma)$ for all $\nu\geq\mathbf{1}$.

Proof. By Lemma 4.2 it suffices to derive the formula (4.2) assuming that (4.5) holds. Given $\gamma\in\Gamma$, $\alpha\in\mathbb{Z}_+^n$ and $z\in\mathcal{H}^n$, using (4.5), we obtain the left-hand side of (4.2) in the form

$$(\phi_\alpha\,|_{2\alpha}\,\gamma)(z) = \sum_{\beta=\mathbf{0}}^{\alpha-\mathbf{1}}\frac{1}{\beta!(2\alpha-\beta-\mathbf{1})!}((\partial^\beta f_{\alpha-\beta})\,|_{2\alpha}\,\gamma)(z). \tag{4.6}$$

Let $\gamma\in\Gamma$ be as in (2.4) so that $\gamma^{-1} = \left(\begin{smallmatrix} d & -b\\ -c & a\end{smallmatrix}\right)\in\Gamma$. Since $f_{\alpha-\beta}\in\mathcal{M}_{2\alpha-2\beta}(\Gamma)$, by (4.4) we have

$$\begin{aligned}\partial^\beta f_{\alpha-\beta}(z) &= \partial^\beta (f_{\alpha-\beta}\,|_{2\alpha-2\beta}\,\gamma^{-1})(z)\\ &= \sum_{\mu=\mathbf{0}}^{\beta}\frac{\beta!}{\mu!}\binom{2\alpha-\beta-\mathbf{1}}{\beta-\mu}\frac{c^{\beta-\mu}}{(-cz+a)^{2\alpha-\beta+\mu}}\partial^\mu f_{\alpha-\beta}(\gamma^{-1}z).\end{aligned}$$

Thus, using the relation

$$-c(\gamma z)+a = -c\Big(\frac{az+b}{cz+d}\Big)+a = (cz+d)^{-1},$$

we obtain

$$\partial^\beta f_{\alpha-\beta}(\gamma z) = \sum_{\mu=\mathbf{0}}^{\beta}\frac{\beta!}{\mu!}\binom{2\alpha-\beta-\mathbf{1}}{\beta-\mu}c^{\beta-\mu}(cz+d)^{2\alpha-\beta+\mu}\partial^\mu f_{\alpha-\beta}(z).$$

Using this and (4.6), we have

(4.7)

$$(\phi_\alpha\,|_{2\alpha}\,\gamma)(z) = \sum_{\beta=\mathbf{0}}^{\alpha-\mathbf{1}}\sum_{\mu=0}^{\beta}\frac{1}{\mu!(\beta-\mu)!(2\alpha-2\beta+\mu-\mathbf{1})!}\Big(\frac{c}{cz+d}\Big)^{\beta-\mu}\cdot\partial^\mu f_{\alpha-\beta}(z).$$

On the other hand, using (4.5), the right-hand side of (4.2) is equal to

$$\sum_{\delta=\mathbf{0}}^{\alpha-\mathbf{1}} \sum_{\mu=\mathbf{0}}^{\alpha-\mathbf{1}-\delta} \frac{1}{\mu!\delta!(2\alpha-2\delta-\mu-\mathbf{1})!}\Big(\frac{c}{cz+d}\Big)^{\delta}\partial^{\mu} f_{\alpha-\delta-\mu}(z). \tag{4.8}$$

Replacing $\sum_{\delta=\mathbf{0}}^{\alpha-\mathbf{1}} \sum_{\mu=\mathbf{0}}^{\alpha-\mathbf{1}-\delta}$ by $\sum_{\mu=\mathbf{0}}^{\alpha-\mathbf{1}} \sum_{\delta=\mathbf{0}}^{\alpha-\mathbf{1}-\mu}$ and δ by $\beta-\mu$, we can write (4.8) in the form

$$\sum_{\mu=\mathbf{0}}^{\alpha-\mathbf{1}} \sum_{\beta=\mu}^{\alpha-\mathbf{1}} \frac{1}{\mu!(\beta-\mu)!(2\alpha-2\beta+\mu-\mathbf{1})!}\Big(\frac{c}{cz+d}\Big)^{\beta-\mu}\cdot\partial^{\mu} f_{\alpha-\beta}(z),$$

which reduces to the right-hand side of equation (4.7) after replacing $\sum_{\mu=\mathbf{0}}^{\alpha-\mathbf{1}} \sum_{\beta=\mu}^{\alpha-\mathbf{1}}$ by $\sum_{\beta=\mathbf{1}}^{\alpha-\mathbf{1}} \sum_{\mu=\mathbf{0}}^{\beta}$. Thus we obtain (4.2), and the proof of the theorem is complete. □

We now construct a split exact sequence involving Jacobi-like forms of n variables and Hilbert modular forms. For each $i \in \{1,\dots,n\}$ let $\varepsilon_i \in \mathbb{Z}^n$ denote the vector which has 1 in the i-th entry and 0 elsewhere.

Theorem 4.5. *If $\mathfrak{F} : \mathcal{J}(\Gamma)_\omega \to \mathcal{M}_{2\omega}(\Gamma)$ is as in* (4.3)*, then there is a short exact sequence*

$$0 \to \sum_{i=1}^{n} \mathcal{J}(\Gamma)_{\omega+\varepsilon_i} \to \mathcal{J}(\Gamma)_\omega \xrightarrow{\mathfrak{F}} \mathcal{M}_{2\omega}(\Gamma) \to 0$$

which splits.

Proof. If $\mathfrak{F} : \mathcal{J}(\Gamma)_\omega \to \mathcal{M}_{2\omega}(\Gamma)$ is as in (4.3), we see easily that

$$\ker \mathfrak{F} = \sum_{i=1}^{n} \mathcal{J}(\Gamma)_{\omega+\varepsilon_i}.$$

Hence it suffices to show that $\mathfrak{F}$ is surjective and that there is a linear map $\mathfrak{G} : \mathcal{M}_{2\omega}(\Gamma) \to \mathcal{J}(\Gamma)_\omega$ with $\mathfrak{F}\circ\mathfrak{G} = \mathrm{id}$. Let $f \in \mathcal{M}_{2\omega}(\Gamma)$, and consider the sequence $(f_\nu)_{\nu\geq 1}$ such that

$$f_\nu = \begin{cases} f & \text{if } \nu = \omega, \\ 0 & \text{if } \nu \neq \omega. \end{cases}$$

Then we have $f_\nu \in \mathcal{M}_{2\nu}(\Gamma)$ for all $\nu \geq \mathbf{1}$. If we set

$$\begin{aligned}\phi_\alpha(z) &= \sum_{\beta=0}^{\alpha-\mathbf{1}} \frac{1}{\beta!(2\alpha-\beta-\mathbf{1})!}\partial^{\beta} f_{\alpha-\beta}(z) \\ &= \frac{1}{(\alpha-\omega)!(\alpha+\omega-\mathbf{1})!}\partial^{\alpha-\omega} f(z)\end{aligned}$$

for each $\alpha \geq \mathbf{1}$, then by Theorem 4.4 we see that the associated formal power series

$$\Phi_f(z,X) = \sum_{\alpha\geq\omega} \phi_\alpha(z) X^\alpha \in R[[X]]$$

is an element of $\mathcal{J}_\omega(\Gamma)$. Thus we have a linear map

$$\mathfrak{G} : \mathcal{M}_{2\omega}(\Gamma) \to \mathcal{J}(\Gamma)_\omega, \quad f(z) \mapsto \Phi_f(z,X),$$

which satisfies $\mathfrak{F}(\mathfrak{G}(f)) = f$; hence the proof of the theorem is complete. □

5. Rankin-Cohen brackets

In this section we use Jacobi-like forms of several variables to construct a bilinear map that associates to each pair of Hilbert modular forms another Hilbert modular form, which is the higher dimensional analogue of the Rankin-Cohen bracket described in [2] and [9].

Let Γ be a discrete subgroup of $SL(2,\mathbb{R})^n$, and for each $\omega \in \mathbb{Z}^n$ let $\mathcal{M}_\omega(\Gamma)$ denote the space of all Hilbert modular forms of weight ω for Γ in the sense of Definition 3.1. Given Hilbert modular forms $f \in \mathcal{M}_{2\alpha}(\Gamma)$ and $g \in \mathcal{M}_{2\beta}(\Gamma)$ for some $\alpha, \beta \in \mathbb{Z}^n_+$, we set

$$[f,g]^{(\alpha,\beta)}_\nu(z) = \sum_{\lambda=0}^{\nu} (-1)^{|\lambda|} \binom{2\alpha+\nu-\mathbf{1}}{\nu-\lambda} \binom{2\beta+\nu-\mathbf{1}}{\lambda} (\partial^\lambda f(z))(\partial^{\nu-\lambda} g(z)) \tag{5.1}$$

for all $z \in \mathcal{H}^n$ and $\nu \in \mathbb{Z}^n_+$, where $|\lambda| = \lambda_1 + \cdots + \lambda_n$ for $\lambda = (\lambda_1, \dots, \lambda_n)$.

Theorem 5.1. *The function $[f,g]^{(\alpha,\beta)}_\nu : \mathcal{H}^n \to \mathbb{C}$ given by* (5.1) *is a Hilbert modular form of weight $2\alpha+2\beta+2\nu$ for Γ.*

Proof. Given $f \in \mathcal{M}_{2\alpha}(\Gamma)$ and $g \in \mathcal{M}_{2\beta}(\Gamma)$, let $\Phi_f(z,X)$ and $\Phi_g(z,X)$ be their liftings in $\mathcal{J}(\Gamma)_\alpha$ and $\mathcal{J}(\Gamma)_\beta$, respectively, described in the proof of Theorem 4.5. Thus we have

$$\Phi_f(z,X) = \sum_{\lambda \geq \alpha} \phi_{f,\lambda}(z) X^\lambda, \quad \Phi_g(z,X) = \sum_{\mu \geq \beta} \phi_{g,\mu}(z) X^\mu,$$

where the coefficients are holomorphic functions on $\mathcal{H}^n$ given by

$$\phi_{f,\lambda} = \frac{\partial^{\lambda-\alpha} f}{(\lambda-\alpha)!(\lambda+\alpha-\mathbf{1})!}, \quad \phi_{g,\mu} = \frac{\partial^{\mu-\beta} g}{(\mu-\beta)!(\mu+\beta-\mathbf{1})!} \tag{5.2}$$

for $\alpha \leq \lambda \in \mathbb{Z}^n$ and $\beta \leq \mu \in \mathbb{Z}^n$. Using (4.1), we see that the formal power series

$$\Psi(z,X) = \Phi_f(z,-X) \cdot \Phi_g(z,X)$$

is invariant under the transformation

$$(z,X) \mapsto (\gamma z, (cz+d)^{-2} X)$$

for all $\gamma \in \Gamma$ of the form (2.4). Thus, if $\psi_{\alpha+\beta+\nu}(z)$ is the coefficient of $X^{\alpha+\beta+\nu}$ in $\Psi(z,X)$, we have

$$\begin{aligned} \psi_{\alpha+\beta+\nu}(z) X^{\alpha+\beta+\nu} &= \psi_{\alpha+\beta+\nu}(\gamma z)((cz+d)^{-2} X)^{\alpha+\beta+\nu} \\ &= (\psi_{\alpha+\beta+\nu} \mid_{2\alpha+2\beta+2\nu} \gamma)(z) X^{\alpha+\beta+\nu} \end{aligned}$$

for all $z \in \mathcal{H}^n$. Hence we see that $\psi_{\alpha+\beta+\nu}(z)$ is a Hilbert modular form of weight $2\alpha+2\beta+2\nu$ for Γ. On the other hand, by (5.2) we obtain

$$\begin{aligned} \psi_{\alpha+\beta+\nu} &= \sum_{\substack{\lambda+\mu=\alpha+\beta+\mu \\ \lambda\geq\alpha, \mu\geq\beta}} \frac{(-1)^{|\lambda|}(\partial^{\lambda-\alpha} f)(\partial^{\mu-\beta} g)}{(\lambda-\alpha)!(\lambda+\alpha-\mathbf{1})!(\mu-\beta)!(\mu+\beta-\mathbf{1})!} \\ &= \sum_{\substack{\lambda+\mu=\nu \\ \lambda\geq\mathbf{0}, \mu\geq\mathbf{0}}} \frac{(-1)^{|\lambda+\alpha|}(\partial^\lambda f)(\partial^\mu g)}{\lambda!(\lambda+2\alpha-\mathbf{1})!\mu!(\mu+2\beta-\mathbf{1})!} \\ &= \sum_{\lambda=\mathbf{0}}^{\nu} \frac{(-1)^{|\lambda|}(\partial^\lambda f)(\partial^\mu g)}{\lambda!(\lambda+2\alpha-1)!(\nu-\lambda)!(\nu-\lambda+2\beta-1)!}. \end{aligned}$$

Using (5.1) and and the relation

$$\binom{2\alpha+\nu-\mathbf{1}}{\nu-\lambda}\binom{2\beta+\nu-\mathbf{1}}{\lambda} = \frac{(2\alpha+\nu-\mathbf{1})!(2\beta+\nu-\mathbf{1})!}{\lambda!(\lambda+2\alpha-\mathbf{1})!(\nu-\lambda)!(\nu-\lambda+2\beta-\mathbf{1})!},$$

we see that

$$[f,g]_\nu^{(\alpha,\beta)} = (2\alpha+\nu-\mathbf{1})!(2\beta+\nu-\mathbf{1})!\psi_{\alpha+\beta+\nu},$$

and therefore $[f,g]_\nu^{(\alpha,\beta)}$ is a Hilbert modular form of weight $2\alpha+2\beta+2\nu$ for Γ. □

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