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AN OPERATOR INEQUALITY RELATED TO JENSEN'S INEQUALITY

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ABSTRACT. For bounded non-negative operators A and B, Furuta showed

$$0 \le A \le B \text{ implies } A^{\frac{r}{2}} B^s A^{\frac{r}{2}} \le (A^{\frac{r}{2}} B^t A^{\frac{r}{2}})^{\frac{s+r}{t+r}} \quad (0 \le r,\ 0 \le s \le t).$$

We will extend this as follows: $0 \le A \le B \underset{\lambda}{!} C$ $(0 < \lambda < 1)$ implies

$$A^{\frac{r}{2}}(\lambda B^s + (1-\lambda)C^s)A^{\frac{r}{2}} \le \{A^{\frac{r}{2}}(\lambda B^t + (1-\lambda)C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}},$$

where $B \underset{\lambda}{!} C$ is a harmonic mean of B and C. The idea of the proof comes from Jensen's inequality for an operator convex function by Hansen-Pedersen.

1. INTRODUCTION

Throughout this article, an operator means a bounded linear operator on a Hilbert space. For selfadoint operators A, B we write $A \le B$ as usual if $B - A$ is positive semidefinite. A real continuous function f defined on an interval I is said to be *operator monotone* if f preserves this order, that is, for bounded selfadjoint operators A, B with spectra in I,

$$A \le B \quad \Rightarrow \quad f(A) \le f(B);$$

and it is said to be *operator convex* if for all selfadjoint operators A, B with spectra in I and for all λ in $[0, 1]$

$$f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda)f(B).$$

An *operator concave* function is similarly defined. In [6] Hansen and Pedersen showed that for a non-negative continuous function f on $[0, \infty)$ the following conditions are equivalent:

(i) f is operator monotone,

(ii) f is operator concave,

(iii) $T^* f(A) T \le f(T^* A T)$ for every contraction T (i.e., $||T|| \le 1$) and for every non-negative operator A,

(iv) $S^* f(A) S + T^* f(B) T \le f(S^* A S + T^* B T)$ for every pair of S, T with $S^* S + T^* T \le 1$ and for all non-negative operators A, B.

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It is well-known that $f(x) = x^a \quad (0 < a \leq 1)$ is operator monotone on $[0, \infty)$, that is,

$$0 \leq A \leq B \Rightarrow A^a \leq B^a,$$

which is called the *Löwner-Heinz* inequality. Therefore, (iv) yields

$$S^*A^aS + T^*B^aT \leq (S^*AS + T^*BT)^a \quad (0 < a \leq 1). \tag{1}$$

Related to the Löwner-Heinz inequality, Furuta showed (cf. [5]): for non-negative real numbers r, s and t such that $t \geq s$ and $(r, t) \neq (0, 0)$,

$$0 \leq A \leq B \Rightarrow A^{\frac{r}{2}}B^sA^{\frac{r}{2}} \leq (A^{\frac{r}{2}}B^tA^{\frac{r}{2}})^{\frac{s+r}{t+r}}, \quad (B^{\frac{r}{2}}A^tB^{\frac{r}{2}})^{\frac{s+r}{t+r}} \leq B^{\frac{r}{2}}A^sB^{\frac{r}{2}}. \tag{2}$$

For non-negative operators A and B and for a real number λ with $0 < \lambda < 1$, the *harmonic mean* is defined by

$$A \underset{\lambda}{!} B := (\lambda A^{-1} + (1 - \lambda)B^{-1})^{-1}$$

if A and B are invertible, and defined by the weak limit of $(A+\epsilon) \underset{\lambda}{!} (B+\epsilon)$ as $\epsilon \to +0$ if not. If non-negative operators A, B are invertible, then we have

$$\lambda A^{-1} + \mu B^{-1} - (\lambda A + \mu B)^{-1} = (A^{-1} - B^{-1})((\lambda A)^{-1} + (\mu B)^{-1})^{-1}(A^{-1} - B^{-1}),$$

where $0 \leq \lambda, \mu \leq 1$, $\lambda + \mu = 1$ (see [10], p. 117 of [2]). This shows that a function $f(x) = 1/x$ is operator convex on $(0, \infty)$ and that

$$A \underset{\lambda}{!} B \leq \lambda A + (1 - \lambda)B.$$

We need the following properties of the harmonic mean (cf. [1], [7]):

$(\alpha A) \underset{\lambda}{!} (\alpha B) = \alpha(A \underset{\lambda}{!} B)$,
$A \underset{\lambda}{!} B \leq C \underset{\lambda}{!} B$ if $A \leq C$, and
$A \underset{\lambda}{!} B + C \underset{\lambda}{!} D \leq (A + C) \underset{\lambda}{!} (B + D)$.

2. Main theorem

From now on, λ and μ represent real numbers such that

$$0 \leq \lambda, \mu \leq 1 \quad \text{and} \quad \lambda + \mu = 1.$$

We start this section with a simple inequality.

Lemma 2.1. *Let S, T be contractions such that $S^*S + T^*T \leq 1$, and let A, B be non-negative operators. Then for $0 \leq r \leq s \leq t$ and $s \neq 0$,*

$$(S^*A^sS + T^*B^sT) \leq (S^*A^tS + T^*B^tT)^{\frac{s}{t}},$$
$$(S^*A^sS + T^*B^sT)^{\frac{r}{s}} \leq (S^*A^tS + T^*B^tT)^{\frac{r}{t}}.$$

Proof. The first inequality follows from (1). By using the Löwner-Heinz inequality we get the second inequality. ∎

We remark that the above implies that for fixed $r > 0$ the operator valued function $(S^*A^tS + T^*B^tT)^{\frac{r}{t}}$ is increasing on $r \leq t < \infty$.

Lemma 2.2. *Let H and K be bounded selfadjoint operators such that $0 \leq H \leq K$. Then for real numbers p, q such that $0 \leq p \leq q$*

$$H^{\frac{1}{2}}K^pH^{\frac{1}{2}} \leq (H^{\frac{1}{2}}K^qH^{\frac{1}{2}})^{\frac{p+1}{q+1}}, \quad K^{\frac{1}{2}}H^pK^{\frac{1}{2}} \geq (K^{\frac{1}{2}}H^qK^{\frac{1}{2}})^{\frac{p+1}{q+1}}.$$

Proof. To show the first inequality we may assume that K is invertible. Since $||K^{-\frac{1}{2}}H^{\frac{1}{2}}|| \leq 1$, by Lemma 2.1, we get

$$(H^{\frac{1}{2}}K^{-\frac{1}{2}}K^{p+1}K^{-\frac{1}{2}}H^{\frac{1}{2}}) \leq (H^{\frac{1}{2}}K^{-\frac{1}{2}}K^{q+1}K^{-\frac{1}{2}}H^{\frac{1}{2}})^{\frac{p+1}{q+1}}.$$

This gives the first inequality. By considering the inverse of K and H, the second inequality follows from the first one. □

Now we give the main theorem that is an extension of (2).

Theorem 2.3. *Let A, B and C be non-negative operators. Then for non-negative real numbers r, s, t such that $s \leq t$ and $(r, t) \neq (0, 0)$,*

$$A \leq B \underset{\lambda}{!} C \Longrightarrow A^{\frac{r}{2}}(\lambda B^s + \mu C^s)A^{\frac{r}{2}} \leq \{A^{\frac{r}{2}}(\lambda B^t + \mu C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}. \tag{3}$$

Proof. Suppose $r = 0$. Then, since $f(x) = x^{s/t}$ is operator concave, we obtain (3). Thus, we need to show (3) in the case of $r > 0$. Since $A \leq (B+\epsilon)\underset{\lambda}{!}(C+\epsilon)$, we may assume that B and C are invertible. We first suppose $0 < r \leq 1$. Since $f(x) = x^r$ is operator monotone and operator concave,

$$A \leq (\lambda B^{-1} + \mu C^{-1})^{-1}$$

yields

$$A^r \leq (\lambda B^{-1} + \mu C^{-1})^{-r} \leq (\lambda B^{-r} + \mu C^{-r})^{-1}.$$

This implies

$$A^{\frac{r}{2}}(\lambda B^{-r} + \mu C^{-r})A^{\frac{r}{2}} \leq 1.$$

By Lemma 2.1, for $0 \leq s \leq t$ we get

$$\begin{aligned} A^{\frac{r}{2}}(\lambda B^s + \mu C^s)A^{\frac{r}{2}} &= \lambda A^{\frac{r}{2}}B^{-\frac{r}{2}}B^{s+r}B^{-\frac{r}{2}}A^{\frac{r}{2}} + \mu A^{\frac{r}{2}}C^{-\frac{r}{2}}C^{s+r}C^{-\frac{r}{2}}A^{\frac{r}{2}} \\ &\leq (\lambda A^{\frac{r}{2}}B^{-\frac{r}{2}}B^{t+r}B^{-\frac{r}{2}}A^{\frac{r}{2}} + \mu A^{\frac{r}{2}}C^{-\frac{r}{2}}C^{t+r}C^{-\frac{r}{2}}A^{\frac{r}{2}})^{\frac{s+r}{t+r}} \\ &= (\lambda A^{\frac{r}{2}}B^t A^{\frac{r}{2}} + \mu A^{\frac{r}{2}}C^t A^{\frac{r}{2}})^{\frac{s+r}{t+r}}. \end{aligned}$$

This means (3) holds for $0 < r \leq 1$ and for $0 \leq s \leq t$.

Assume (3) holds for $0 < r \leq 2^n$ and for $0 \leq s \leq t$. Take an arbitrary r in $(2^n, 2^{n+1}]$. Since $r/2 \leq 2^n$, the assumption says

$$A^{\frac{r}{4}}(\lambda B^s + \mu C^s)A^{\frac{r}{4}} \leq \{A^{\frac{r}{4}}(\lambda B^t + \mu C^t)A^{\frac{r}{4}}\}^{\frac{s+r/2}{t+r/2}} \quad (0 \leq s \leq t); \tag{4}$$

in particular,

$$A^{\frac{r}{2}} \leq \{A^{\frac{r}{4}}(\lambda B^t + (1-\lambda)C^t)A^{\frac{r}{4}}\}^{\frac{r}{2t+r}}.$$

Let us apply this to the first inequality of Lemma 2.2 with $p = (2s+r)/r$, $q = (2t+r)/r$. Then we get

$$A^{\frac{r}{4}}\{A^{\frac{r}{4}}(\lambda B^t + (1-\lambda)C^t)A^{\frac{r}{4}}\}^{\frac{2s+r}{2t+r}}A^{\frac{r}{4}} \leq [A^{\frac{r}{4}}\{A^{\frac{r}{4}}(\lambda B^t + (1-\lambda)C^t)A^{\frac{r}{4}}\}A^{\frac{r}{4}}]^{\frac{s+r}{t+r}}.$$

This in conjunction with (4) gives

$$A^{\frac{r}{2}}(\lambda B^s + (1-\lambda)C^s)A^{\frac{r}{2}} \leq \{A^{\frac{r}{2}}(\lambda B^t + (1-\lambda)C^t)A^{\frac{r}{2}}\}^{\frac{s+r}{t+r}}.$$

□

Corollary 2.4. *Let A, B and C be non-negative operators. Then*

$$A \leq B \underset{\lambda}{!} C \quad \Rightarrow \quad A^{1+r} \leq \{A^{\frac{r}{2}}(\lambda B^t + (1-\lambda)C^t)A^{\frac{r}{2}}\}^{\frac{1+r}{t+r}} \quad (1 \leq t).$$

Proof. Putting $s=1$ in (3), in virtue of $A \leq B \underset{\lambda}{!} C \leq \lambda B+\mu C$, we get the above. $\square$

Considering the previous inequality with $B=C$, we get

$$0 \leq A \leq B \Longrightarrow A^{1+r} \leq (A^{\frac{r}{2}} B^t A^{\frac{r}{2}})^{\frac{1+r}{t+r}}. \tag{5}$$

This has been found by Furuta [4] and called the *Furuta inequality*. Furuta showed (2) by using (5). Our proof seems to help us clear up the significance of the exponent $(s+r)/(t+r)$ in (2) and hence the exponent $(1+r)/(t+r)$ in (5) (cf. [8]).

If $A \leq B$ and $A \leq C$, then $A \leq B \underset{\lambda}{!} C$ for every λ; therefore, the inequality in (3) holds. We remark that in this case we can show it by (2) and the concavity of $f(x)=x^{(s+r)/(t+r)}$.

We proved that Theorem 2.3 and Corollary 2.4 hold for just the harmonic mean of two operators B and C; however, it is easy to see that they do even for the harmonic mean of a finite number of operators, too.

Proposition 2.5. *Let B, C and D be non-negative operators. Then for non-negative real numbers r, s, t such that $s \leq t$ and $(r,t) \neq (0,0)$,*

$$\lambda B+\mu C \leq D \quad \Rightarrow \quad \{D^{\frac{r}{2}}(B^t \underset{\lambda}{!} C^t) D^{\frac{r}{2}}\}^{\frac{s+r}{t+r}} \leq D^{\frac{r}{2}}(B^s \underset{\lambda}{!} C^s) D^{\frac{r}{2}}.$$

Proof. By the continuity of harmonic mean in the norm sense, we may assume that B, C and D are all invertible. The condition $\lambda B+\mu C \leq D$ implies

$$D^{-1} \leq B^{-1} \underset{\lambda}{!} C^{-1}.$$

By Theorem 2.3, this yields

$$D^{-\frac{r}{2}}(\lambda B^{-s}+\mu C^{-s}) D^{-\frac{r}{2}} \leq \{D^{-\frac{r}{2}}(\lambda B^{-t}+\mu C^{-t}) D^{-\frac{r}{2}}\}^{\frac{s+r}{t+r}}.$$

Take the inverse of both sides of this inequality to get the required inequality. $\square$

Put $s=1$ in Proposition 2.5. Since

$$B \underset{\lambda}{!} C \leq \lambda B+(1-\lambda) C \leq D,$$

we get

Corollary 2.6. *Let B, C and D be non-negative operators. Then for $0 \leq r$ and $1 \leq t$,*

$$\lambda B+\mu C \leq D \quad \Rightarrow \quad \{D^{\frac{r}{2}}(B^t \underset{\lambda}{!} C^t) D^{\frac{r}{2}}\}^{\frac{1+r}{t+r}} \leq D^{1+r}.$$

By setting $B=C$ in the above, we get an alternate Furuta inequality:

$$0 \leq C \leq D \Rightarrow (D^{\frac{r}{2}} C^t D^{\frac{r}{2}})^{\frac{1+r}{t+r}} \leq D^{1+r}. \tag{5$'$}$$

By combining (5) and (5)$'$, we get, for $0 \leq r$, $1 \leq t$,

$$0 \leq A \leq B \leq C \Rightarrow (B^{\frac{r}{2}} A^t B^{\frac{r}{2}})^{\frac{1+r}{t+r}} \leq B^{1+r} \leq (B^{\frac{r}{2}} C^t B^{\frac{r}{2}})^{\frac{1+r}{t+r}}.$$

Suppose that B and C are non-negative operators. Replace A with $B \underset{\lambda}{!} C$ in Theorem 2.3 and D with $\lambda B+\mu C$ in Proposition 2.5. Then we get

Corollary 2.7. *Let B, C be non-negative operators. Then for non-negative numbers r, s, t such that $s \le t$ and $(r, t) \ne (0, 0)$,*

$$(B \mathop{!}_{\lambda} C)^{\frac{r}{2}}(\lambda B^s + \mu C^s)(B \mathop{!}_{\lambda} C)^{\frac{r}{2}} \le \{(B \mathop{!}_{\lambda} C)^{\frac{r}{2}}(\lambda B^t + \mu C^t)(B \mathop{!}_{\lambda} C)^{\frac{r}{2}}\}^{\frac{s+r}{t+r}},$$
$$\{(\lambda B + \mu C)^{\frac{r}{2}}(B^t \mathop{!}_{\lambda} C^t)(\lambda B + \mu C)^{\frac{r}{2}}\}^{\frac{s+r}{t+r}} \le (\lambda B + \mu C)^{\frac{r}{2}}(B^s \mathop{!}_{\lambda} C^s)(\lambda B + \mu C)^{\frac{r}{2}}.$$

3. Exponential inequality

Theorem 3.1. *Let A, B and C be non-negative operators. Then for non-negative real numbers r, s, t such that $s \le t$ and $(r, t) \ne (0, 0)$,*

$$A \le B \mathop{!}_{\lambda} C \quad \Longrightarrow \quad e^{\frac{r}{2}A}(\lambda e^{sB} + \mu e^{sC})e^{\frac{r}{2}A} \le \{e^{\frac{r}{2}A}(\lambda e^{tB} + \mu e^{tC})e^{\frac{r}{2}A}\}^{\frac{s+r}{t+r}}.$$

Proof. For every natural number n, $1 + A/n \le (1 + B/n) \mathop{!}_{\lambda} (1 + C/n)$ because of the properties of harmonic mean given in Section 1.

Therefore, (3) gives

$$\begin{aligned} &(1 + A/n)^{\frac{nr}{2}}\{\lambda(1 + B/n)^{ns} + \mu(1 + C/n)^{ns}\}(1 + A/n)^{\frac{nr}{2}} \\ \le\ &[(1 + A/n)^{\frac{nr}{2}}\{\lambda(1 + B/n)^{nt} + \mu(1 + C/n)^{nt}\}(1 + A/n)^{\frac{nr}{2}}]^{\frac{ns+nr}{nt+nr}}. \end{aligned}$$

Letting $n \to \infty$, we get the required inequality. □

Remark. Harmonic mean is just defined for non-negative operators. However, it is clear that the above inequality holds for general selfadjoint operators A, B, C if there is a real number a so that $A + a, B + a, C + a \ge 0$ and $A + a \le (B + a) \mathop{!}_{\lambda} (C + a)$. In particular, putting $B = C$ we can obtain:

Let A, B be (not necessarily non-negative) selfadjoint operators. Then

$$A \le B \Rightarrow e^{\frac{r}{2}A}e^{sB}e^{\frac{r}{2}A} \le (e^{\frac{r}{2}A}e^{tB}e^{\frac{r}{2}A})^{\frac{s+r}{t+r}}.$$

Proposition 3.2. *Let B, C and D be (not necessarily non-negative) selfadjoint operators. Then for non-negative real numbers r, s, t such that $s \le t$ and $(r, t) \ne (0, 0)$,*

$$\lambda B + \mu C \le D \quad \Rightarrow \quad \{e^{\frac{r}{2}D}(e^{tB} \mathop{!}_{\lambda} e^{tC})e^{\frac{r}{2}D}\}^{\frac{s+r}{t+r}} \le e^{\frac{r}{2}D}(e^{sB} \mathop{!}_{\lambda} e^{sC})e^{\frac{r}{2}D}.$$

Proof. For sufficiently large n, $(1 + B/n)$, $(1 + C/n)$ and $(1 + D/n)$ are all non-negative. Since $\lambda(1 + B/n) + \mu(1 + C/n) \le (1 + D/n)$, taking notice that

$$(1 + B/n)^{nt} \mathop{!}_{\lambda} (1 + C/n)^{nt} \to e^{tB} \mathop{!}_{\lambda} e^{tC} \quad \text{as} \quad n \to \infty,$$

we can derive the desired inequality from Proposition 2.5. □

By putting $s = 0$ in the above remark, we get

$$A \le B \Rightarrow e^{rA} \le (e^{\frac{r}{2}A}e^{tB}e^{\frac{r}{2}A})^{\frac{r}{t+r}}.$$

Similarly, by putting $B = C$ and $s = 0$ in Proposition 3.2 we get

$$C \le D \Rightarrow (e^{\frac{r}{2}D}e^{tC}e^{\frac{r}{2}D})^{\frac{r}{t+r}} \le e^{rD}.$$

These are known inequalities (cf. [3], [9]). Combining them, we see:

For $r, t \ge 0$ with $(r, t) \ne (0, 0)$,

$$A \le B \le C \Rightarrow (e^{\frac{r}{2}B}e^{tA}e^{\frac{r}{2}B})^{\frac{r}{t+r}} \le e^{rB} \le (e^{\frac{r}{2}B}e^{tC}e^{\frac{r}{2}B})^{\frac{r}{t+r}}.$$

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