PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 129, Number 11, Pages 3275–3278
S 0002-9939(01)06144-5
Article electronically published on June 6, 2001

A GENERALIZED SCHWARZ LEMMA AT THE BOUNDARY

DOV CHELST

(Communicated by Juha M. Heinonen)

Abstract. Let ϕ be an analytic function mapping the unit disc $\mathbb{D}$ to itself. We generalize a boundary version of Schwarz's lemma proven by D. Burns and S. Krantz and provide sufficient conditions on the local behavior of ϕ near a finite set of boundary points that requires ϕ to be a finite Blaschke product. Afterwards, we supply several counterexamples to illustrate that these conditions may also be necessary.

1. Introduction and main result

Throughout this analysis, asymptotic conditions are given as z approaches $\partial\mathbb{D}$. There are a number of possible limit types: radial, non-tangential, and tangential limits. In this paper, we will avoid these distinctions and employ limits in which $z \in \mathbb{D}$ may approach a boundary point in any manner whatsoever. In addition, we use the standard limiting O and o notation.

In [1, pp. 662-664], D. Burns and S. Krantz proved the following lemma.

Lemma 1 (Burns-Krantz). *Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic function from the disc to itself such that*

$$\phi(z) = z + O((z-1)^4)$$

as $z \to 1$. Then $\phi(z) = z$ on the disc.

This lemma can be generalized in the following manner.

Theorem 2. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function from the disc to itself. In addition, let $f : \mathbb{D} \to \mathbb{D}$ be a finite Blaschke product which equals $\tau \in \partial\mathbb{D}$ on a finite set $A_f \subset \partial\mathbb{D}$. If*

(i) *for a given $\gamma_0 \in A_f$, $\phi(z) = f(z) + O((z-\gamma_0)^4)$, as $z \to \gamma_0$, and*
(ii) *for all $\gamma \in A_f - \{\gamma_0\}$, $\phi(z) = f(z) + O((z-\gamma)^{k_\gamma})$, for some $k_\gamma \geq 2$ as $z \to \gamma$,*

then $\phi(z) = f(z)$ on the disc.

Received by the editors March 10, 2000.
2000 *Mathematics Subject Classification.* Primary 30C80.
Key words and phrases. Schwarz's lemma, Schur functions, bounded analytic functions, Blaschke product.
The author would like to thank Dr. R.B. Burckel for referring him to the article by Krantz and Burns and to also thank Drs. X. Huang, S. Goldstein and B. Walsh for their advice on this article's contents.

The proof of this statement hinges on two facts regarding finite Blaschke products and on a proposition. We rely on the facts that a finite Blaschke product f is analytic throughout $\overline{\mathbb{D}}$ and that $|f| = 1$ on $\partial\mathbb{D}$. We then present a streamlined version of the Burns-Krantz argument employing Hopf's lemma.[1]

Lemma 3 (Hopf's lemma on the disc)**.** *Let u be a nonconstant real-valued harmonic function in $\mathbb{D}$. Let $\gamma \in \partial\mathbb{D}$ be such that:*

(i) *u is continuous at γ;*
(ii) *$u(\gamma) \geq u(z)$ for all $z \in \mathbb{D}$.*

Then the outer normal derivative $\frac{\partial u}{\partial \nu}$ of u at γ, if it exists, satisfies the strict inequality

$$\frac{\partial u}{\partial \nu}(\gamma) > 0.$$

We will soon construct a nonnegative harmonic function u on $\mathbb{D}$ which has a minimum at $\gamma_0 \in \partial\mathbb{D}$. It will also satisfy $\frac{\partial u}{\partial \nu}(\gamma_0) = 0$. This will force u to be identically zero. By our construction, this will lead to our theorem's conclusion.

Proposition 4. *Let $f : \mathbb{D} \to \mathbb{D}$ be analytic and let f have a continuous limit at some $\gamma \in \partial\mathbb{D}$ and let $f(\gamma) = 1$. Then f is not $o(z - \gamma)$.*

Proof. Assume otherwise, namely that $f - 1 = o(z - \gamma)$. Define the positive harmonic function u by

$$u(z) = \Re(1 - f(z)).$$

By the properties of f, u is $o(z - \gamma)$ around γ. However, u is a positive harmonic function which takes a minimum at γ. By Hopf's lemma, its first derivative at γ cannot be zero, yielding a contradiction. □

Proof of Theorem 2. Without loss of generality, we may assume that $\tau = 1$ and that $\gamma_0 = 1$. Consider the harmonic function g defined as

$$g(z) = \Re\left(\frac{1 + \phi(z)}{1 - \phi(z)}\right) - \Re\left(\frac{1 + f(z)}{1 - f(z)}\right). \tag{1}$$

Note that the second term of g is zero almost everywhere on $\partial\mathbb{D}$, i.e., on $\partial\mathbb{D} - A_f$. Of course, the first term of g is nonnegative. Consequently, when taking liminfs to any boundary point in $\partial\mathbb{D} - A_f$, one always obtains a nonnegative (possibly infinite) value.

In addition, define h as $\phi - f$. Then, h has all of the limiting properties described by conditions (i) and (ii) of Theorem 2. In addition, one may write

$$g(z) = \Re\left(\frac{2h(z)}{(1 - f(z) - h(z))(1 - f(z))}\right). \tag{2}$$

For any $\gamma \in A_f - \{\gamma_0\}$, the numerator of g approaches 0 as $O((z - \gamma)^k)$ for some $k \geq 2$, by condition (ii) of Theorem 2. In addition, by Lemma 4, the denominator can decrease no more quickly than $O((z - \gamma)^2)$. Thus, g must have a liminf at γ. Moreover, condition (i) of Theorem 2 and the same reasoning indicates that g is $O((z - \gamma_0)^2)$ in some neighborhood of γ_0. Finally, since A_f is finite, g must be bounded below through any approach to the boundary $\partial\mathbb{D}$.

The fact that $g(z)$ is nonnegative almost everywhere and has a finite lower bound everywhere on $\partial\mathbb{D}$ implies that $g(z)$ is in fact nonnegative for all $z \in D$. We argue

[1] See [2, p. 34].

as follows. First, by the continuity of f off of A_f, the behavior of g near A_f, and the compactness of $\partial\mathbb{D}$, there exists a lower bound M and a radius $r < 1$ so that $g(w) > M$ for every complex w so that $r < |w| < 1$. Using the Poisson kernel with $r < R < 1$, one may write $g(\rho e^{i\theta}) = \int_0^{2\pi} P_{\frac{\rho}{R}}(t,\theta) g(Re^{it})dt$. Then, taking a liminf as R approaches 1 yields

$$\begin{aligned} g(\rho e^{i\theta}) &= \liminf_{R\to 1^-} \int_0^{2\pi} P_{\frac{\rho}{R}}(t,\theta) g(Re^{it})dt \\ &\geq \int_0^{2\pi} P_\rho(t,\theta) \liminf_{R\to 1^-}(g(Re^{it}))dt \geq 0. \end{aligned}$$

The second-to-last inequality follows from Fatou's lemma, while the last inequality uses the fact that g is nonnegative almost everywhere on $\partial\mathbb{D}$, i.e., it follows from the fact that A_f is finite.

Thus, g is a nonnegative harmonic function on $\mathbb{D}$. In addition, it equals 0 at γ_0. By Hopf's lemma, if g is nonconstant, it must have a nonzero derivative at γ_0 which is contradicted by the assumption that g is $O((z-\gamma_0)^2)$ as $z \to \gamma_0$. Thus, $g \equiv 0$, which implies that $\phi \equiv f$. □

2. Examples and conclusion

Here are some examples to convince the reader that this result is optimal. Burns and Krantz demonstrated that one needs the first condition of Theorem 2. They pointed out that $\phi(z) = z - \frac{1}{10}(z-1)^3$ maps the unit disc to itself and thus the degree of asymptotic agreement between ϕ and z cannot be lowered to reach the conclusions of Lemma 1. Consequently, take any Blaschke product f and consider the function $\psi(z) = \phi(f(z))$. We can see that f and ψ agree with each other to second order at every point in A_f and yet they are not equal.

In addition, we can show that the second condition is also optimal in some cases. The following three functions map the unit disc into itself:

1. $\phi_1(z) = z^2 - \frac{1}{16}(z+1)(z-1)^4$,
2. $\phi_2(z) = z^4 - \frac{1}{64}(z+1)(z^2+1)^2(z-1)^4$, and
3. $\phi_2(z) = z^8 - \frac{1}{256}(z+1)\left[(z^2+1)(z^4+1)\right]^2(z-1)^4$.

These examples illustrate that one cannot lower the degree of radial asymptotics at the other points of A_f and obtain the same result. When considering $f(z) = z^2$, $A_f = \{\pm 1\}$. While ϕ_1 agrees with z^2 up to third order at 1, because it does not agree with z^2 up to first order at -1, ϕ_1 may differ from z^2. ϕ_2 and ϕ_3 illustrate the same point when compared with z^4 ($A_f = \{\pm 1, \pm i\}$) and z^8 ($A_f = \{\frac{1+i}{\sqrt{2}}^n, n = 1, 2, ..., 8\}$) respectively. Both ϕ_2 and ϕ_3 satisfy the second criterion at all but the single point -1 of A_f which is nonetheless enough to allow them to differ from their respective Blaschke products.

For completeness, we will prove that ϕ_1 is a self-map of the unit disc. ϕ_1 is clearly an entire function. Therefore, if we can show that $\Phi(t) \equiv |\phi_1(e^{it})|^2 \leq 1$ for all $-\pi \leq t < \pi$, then we are finished. As ϕ_1 is a polynomial with real coefficients, we may use Chebyshev polynomials to expand $\Phi(t)$ in powers of $\cos t$ and then set

$x = \cos t$ as follows:

$$\begin{aligned}\Phi(t) &= (e^{2it} - \frac{1}{16}(e^{it}+1)(e^{it}-1)^4)(e^{-2it} - \frac{1}{16}(e^{-it}+1)(e^{-it}-1)^4) \\ &= \frac{1}{8}(5 + \cos t + 6\cos^2 t - 2\cos^3 t - 3\cos^4 t + \cos^5 t) \\ &= \frac{1}{8}(5 + x + 6x^2 - 2x^3 - 3x^4 + x^5) \equiv p(x).\end{aligned}$$

We now maximize $p(x)$ over $[-1,1]$ using standard calculus:

$$p'(x) = \frac{1}{8}(5x^4 - 12x^3 - 6x^2 + 12x + 1) = \frac{1}{8}(x-1)(x+1)(5x^2 - 12x - 1).$$

Since $p(1) = p(-1) = 1$ and $p(\frac{6-\sqrt{41}}{5}) \approx 0.62$, ϕ_1 maps the unit disc to itself.

We suspect that this result is not the final generalization of the theorem by Burns and Krantz. It is likely possible to extend the result to more functions such as infinite Blaschke products, inner functions or extreme functions. However, we can show that f in Theorem 2 must have at least some special properties. While $f(z) = \frac{1}{2-z}$ maps the unit disc to itself and equals 1 at 1, it is not a Blaschke product nor is it even an extreme function. Moreover, $\phi_4(z) = \frac{1}{2-z} - \frac{1}{15}(z-1)^4$ also maps the unit disc to itself and thus the consequences of Theorem 2 do not hold in this instance.

We have shown how the arguments originally proposed by D. Burns and S. Krantz can be extended to included finite Blaschke products. Toward this end, the hypotheses of their original theorem required a modification. In addition, we showed that such a modification cannot be avoided. Furthermore, we have shown that a similar statement cannot be made for arbitrary self-maps of the unit disc.

References

[1] Daniel M. Burns and Steven G. Krantz, *Rigidity of holomorphic mappings and a new schwarz lemma at the boundary*, Journal of the AMS **7** (1994), no. 3, 661–676. MR **94j:**32016

[2] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin, 1983. MR **86c:**35035

Department of Mathematics, Hill Center, Rutgers, The State University of New Jersey, 110 Frelinghuysen Rd., Piscataway, New Jersey 08854-8019

E-mail address: chelst@math.rutgers.edu