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ROTATION NUMBERS IN THE INFINITE ANNULUS

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ABSTRACT. Using the notion of free transverse triangulation we prove that the rotation number of a given probability measure invariant by a homeomorphism of the open annulus depends continuously on the homeomorphism under some boundedness conditions.

0. NOTATION

We give the same name to the projections $p_1 : (x, y) \mapsto x$ and $p_2 : (x, y) \mapsto y$ defined on the infinite annulus $\mathbf{A} = \mathbf{T}^1 \times \mathbf{R}$ or on the plane $\mathbf{R}^2$. We consider the projection

$$\begin{aligned} \pi \ &: \mathbf{R}^2 \to \mathbf{A} \\ &(x, y) \mapsto (x + \mathbf{Z}, y) \end{aligned}$$

of the universal covering space $\mathbf{R}^2$ and the translation

$$\begin{aligned} T \ &: \mathbf{R}^2 \to \mathbf{R}^2 \\ &(x, y) \mapsto (x + 1, y). \end{aligned}$$

We will write $\mathbf{A}_q = \mathbf{R}^2/T^q$ for the finite covering space of order $q \geq 2$ of $\mathbf{A}$ and $\pi_q : \mathbf{A}_q \to \mathbf{A}$ for the projection.

We consider the set $H(\mathbf{A})$ of homeomorphisms of $\mathbf{A}$ homotopic to the identity and the set $h(\mathbf{A})$ of lifts to $\mathbf{R}^2$ of such maps. We will put on both sets the compact-open topology. If μ is a Borelian probability measure on $\mathbf{A}$, the set $H_\mu(\mathbf{A})$ of homeomorphisms $F \in H(\mathbf{A})$ which preserve μ is closed as is the set $h_\mu(\mathbf{A})$ of lifts.

If E is a topological space, we will write respectively $\overline{X}$, $\mathrm{int}(X)$, ∂X for the closure, the interior and the frontier of $X \subset E$.

1. INTRODUCTION

If F is a homeomorphism of the compact annulus $\mathbf{T}^1 \times [0, 1]$ homotopic to the identity and f is a lift of F to $\mathbf{R} \times [0, 1]$, the *rotation number* of a point $z \in \mathbf{T}^1 \times [0, 1]$ is defined when it exists by the relation

$$\rho(z) = \lim_{n \to \pm\infty} \frac{p_1 \circ f^n(\widetilde{z})}{n}$$

where $\widetilde{z} \in \pi^{-1}(\{z\})$.

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If F preserves a probability measure μ, it is a direct consequence of the Birkhoff Ergodic Theorem that the rotation number is well defined μ-a.e., that the measurable function ρ obtained in this way is integrable and that $\int \rho \, d\mu = \int r \, d\mu$, where the map $r : \mathbf{A} \to \mathbf{R}$ is lifted to $\mathbf{R} \times [0,1]$ by $p_1 \circ f - p_1$. This integral is called the *rotation number* of μ. The rotation number is well defined for a homeomorphism F of an abstract compact annulus if we choose a generator of the first homology group. Indeed, if G is a homeomorphism of $\mathbf{T}^1 \times [0,1]$ which induces the identity on the first homology group and g is a lift of G to $\mathbf{R} \times [0,1]$, it is easy to see that a point z has a rotation number for f iff $G(z)$ has a rotation number for $g \circ f \circ g^{-1}$ and that these numbers are equal. Moreover, if μ is an invariant measure on $\mathbf{A}$, the rotation number of μ for f is equal to the rotation number of $G_*(\mu)$ for $g \circ f \circ g^{-1}$. The notion of rotation number is very classical and was introduced in the case of a general compact manifold by S. Schwartzman [S] as an element of the first group of homology.

In the case of a homeomorphism F of the open annulus, the situation is more complicated. First of all, the existence of a rotation number for a point is not stable by conjugacy. Indeed, if z has a rotation number (in the sense given above) and if the sequence $(u_n)_{n \geq 0}$ defined by $u_n = p_2 \circ F^n(z)$ is unbounded, we can extract a strictly monotone unbounded subsequence $(u_{n_k})_{k \geq 0}$. If we consider the homeomorphism

$$\begin{aligned} g \ &: \mathbf{R}^2 \to \mathbf{R}^2 \\ &(x,y) \mapsto (x + a(y), y) \end{aligned}$$

where $a : \mathbf{R} \to \mathbf{R}$ is continous and satisfies

$$a(u_{n_k}) = -p_1 \circ f^{n_k}(\widetilde{z}) + p_1(\widetilde{z}) + k n_k \ , \ \ \widetilde{z} \in \pi^{-1}(\{z\}) \ ,$$

and the element $G \in H(\mathbf{A})$ lifted by g, then the point $G(z)$ has no rotation number for $g \circ f \circ g^{-1}$. A natural way to define the rotation number of a point in that case is to limit ourselves to the recurrent points of F. We will say that a recurrent point z has a rotation number $\rho(z)$ if for every subsequence $(F^{n_k}(z))_{k \geq 0}$ of $(F^n(z))_{n \geq 0}$ which converges to z we have

$$\lim_{k \to +\infty} \frac{p_1 \circ f^{n_k}(\widetilde{z})}{n_k} = \rho(z)$$

for every $\widetilde{z} \in \pi^{-1}(z)$ and if we have a similar result for the subsequences of $(F^n(z))_{n \leq 0}$ converging to z. For this definition it becomes clear that this notion is stable by conjugacy.

If F preserves a probability measure μ and if r is μ-integrable, we will get the same situation as in the compact case: μ-a.e. point has a rotation number (in the strong sense) and we will get the rotation number of the measure by integrating this function. It may happen that after a conjugacy the map r' lifted by $p_1 \circ g \circ f \circ g^{-1} - p_1$ is no longer integrable. However, if it is the case, the rotation number of the measure $G_*(\mu)$ for $g \circ f \circ g^{-1}$ is equal to $\int r' \, d\mu$. Indeed μ-a.e. point is recurrent for F and the rotation number in the weak sense is invariant by conjugacy and coincides with the rotation number in the strong sense. So it is natural to say that a measure μ has a rotation number if μ-a.e. recurrent point has a rotation number in the weak sense and if the measurable function ρ obtained is integrable: the rotation number will be $\int \rho \, d\mu$. We have the following immediate properties:

i) If $f' = T^k \circ f$ is another lift of F, the rotation number of a point (or an invariant measure) for f' is obtained by adding k to the rotation number for f.

ii) Fix $q \geq 2$ and consider the homeomorphism F_q of $\mathbf{A}_q$ lifted by f. If z is a recurrent point of F_q and if its image in $\mathbf{A}$, which is a recurrent point of F, has a rotation number ρ, then z has a rotation number equal to ρ/q (we use the natural generator of $H_1(A_q, \mathbf{R})$ defined by the translation T^q). So if an invariant measure μ of F has a rotation number ρ and if μ_q is the natural probability measure defined on $\mathbf{A}_q$ by μ, there exists a set of full measure of $\mathbf{A}_q$ which consists of recurrent points of F_q whose image in $\mathbf{A}$ has a rotation number. We deduce that μ_q has a rotation number which is equal to ρ/q.

There are examples of probability measures without rotation number. The homeomorphism $F \in H(\mathbf{A})$ lifted by

$$
\begin{aligned}
f : \mathbf{R}^2 &\to \mathbf{R}^2 \\
(x, y) &\mapsto (x + a(y), y)
\end{aligned}
$$

preserves the volume form $\frac{1}{\pi(1+y^2)}\, dx \wedge dy$ if a is C^1. If a is equal to $(-1)^k k^2$ on any interval $[k, k+1/2]$, the measure defined by the 2-form has no rotation number.

If z is a fixed point of F, there exists an integer $p \in \mathbf{Z}$ such that $f(z) = T^p(z)$. This integer is the rotation number of the point z and the rotation number of the Dirac measure concentrated on z. In the previous example there were a lot of fixed points with rotation numbers arbitrarily large. In fact J. Franks [Fr2] proved that this is always the case when the measure has no rotation number and a support equal to $\mathbf{A}$. We will recall the proof of this result (Theorem 1), expressing it in terms of *free transverse triangulation* (notion introduced by M. Flucher [Fl]). This will permit us to show a continuity result (Theorem 2) for the rotation number of a measure. It is easy to see that it is optimal because the compact-open topology permits us to perturb a given map by changing it arbitrarily outside of a compact set. We denote by $\mathrm{Fix}(F)$ the set of fixed points of F, by $R(f)$ the smallest interval of $\mathbf{Z}$ which contains all the rotation numbers of the fixed points of F and by $|R(f)|$ the cardinality of $R(f)$.

Theorem 1. *Let μ be a probability measure, F an element of $H_\mu(\mathbf{A})$ and $f \in h_\mu(\mathbf{A})$ a lift of F. If $R(f)$ is finite, then μ-a.e. recurrent point of F has a rotation number. Moreover if the support of μ is $\mathbf{A}$, the rotation numbers are bounded and the measure μ has a rotation number.*

Theorem 2. *Let μ be a probability measure whose support is $\mathbf{A}$. If $(F_n)_{n\geq 0}$ is a sequence in $H_\mu(\mathbf{A})$ converging to $F \in H_\mu(\mathbf{A})$, if $(f_n)_{n\geq 0}$ is a sequence of lifts converging to a lift f of F and if the sequence $(|R(f_n)|)_{n\geq 0}$ is uniformly bounded, then $R(f)$ is finite and the sequence of rotation numbers of μ for f_n converges to the rotation number of μ for f.*

2. Free disk chains

A *free disk* of a homeomorphism F of a surface S is a set homeomorphic to an open disk which doesn't meet its image by F. A *free disk chain* C is given by a family $(U_i)_{1\leq i\leq n}$ of free disks and a family $(p_i)_{1\leq i\leq n-1}$ of positive integers such that:

i) two disks U_i and U_j are equal or disjoint;

ii) for every $i \in \{1, \ldots, n-1\}$ the set $F^{p_i}(U_i)$ meets U_{i+1}.

We say that the chain *goes from* U_1 *to* U_n and will define the *length of the chain* C to be the integer $l(C)=\sum_{i=1}^{n-1}p_i$; we say that the chain is *closed* if $U_1=U_n$. If $\mathcal{D}$ is a family of disjoint free disks of F, a $\mathcal{D}$*-chain* is a chain of disks chosen in $\mathcal{D}$. We say that $\mathcal{D}$ is *transitive* if for every pair of disks U_1 and U_2 in $\mathcal{D}$ we can find a $\mathcal{D}$-chain from U_1 to U_2.

We have the following fundamental result, due to J. Franks [Fr1], which is a consequence of the Brouwer Lemma on Arcs of Translation.

Proposition 1. *A fixed point free and orientation preserving homeomorphism of* $\mathbf{R}^2$ *has no closed free disk chain.*

Let F be a fixed point free homeomorphism of a surface S and consider a triangulation of S. We can construct a subtriangulation sufficiently small such that the closure of any 2-cell is free. Such a triangulation will be called a *free triangulation.* If we perturb slightly a free triangulation we still get a free triangulation. We can do it in such a way that we have the following property: if γ and γ' are two edges such that $F(\gamma)$ (resp. $F^{-1}(\gamma)$) meets γ', then $F(\gamma)$ (resp. $F^{-1}(\gamma)$) meets the two 2-cells adjacent to γ'. Such a triangulation will be called a *free transverse triangulation.* If μ is a probability measure, we can perturb slightly a free transverse triangulation to get another free transverse triangulation such that each edge of the triangulation is a null-set. Such a triangulation will be called a *full free transverse triangulation* (for μ). We will denote by the same letter a triangulation $\mathcal{D}$ and the family of 2-cells of this triangulation. We have the following result (see [Fl] or [LS] for more details).

Proposition 2. *If F is a fixed point free homeomorphism of a connected surface S which preserves a probability measure μ whose support is S, then the family of 2-cells of a free transverse triangulation is transitive.*

Proof. Let $\mathcal{D}=(V_\alpha)_{\alpha\in A}$ be the family of 2-cells of this triangulation. Fix $\alpha_0\in A$ and consider the set A' of elements $\alpha\in A$ such that there exists a $\mathcal{D}$-chain from V_{α_0} to V_α. It is easy to prove that the set $W=\operatorname{int}\left(\overline{\bigcup_{\alpha\in A'}V_\alpha}\right)$ is positively invariant and by the transversality condition that it satisfies $F(\partial W)\subset W$. From the fact that the support of the measure is S we deduce that $\partial W=\emptyset$, and from the connectedness of S we deduce that $W=S$ and that $A'=A$. □

Consider $F\in H(\mathbf{A})$, a lift $f\in h(\mathbf{A})$ and a family $\mathcal{D}$ of disjoint free disks of F, and write $\widetilde{\mathcal{D}}=(V_\alpha)_{\alpha\in A}$ the lifted family (i.e. the family of connected components of the preimages by π of the disks) which consists of free disks of f. For $\alpha\in A$ and $k\in\mathbf{Z}$ we will write $\beta=\alpha+k$ if $V_\beta=T^k(V_\alpha)$.

For every $\alpha\in A$, the set $Z(\alpha)$ of integers $k\in\mathbf{Z}$ such that there exists a $\widetilde{\mathcal{D}}$-chain of f from V_α to $V_{\alpha+k}$ is stable by addition. Indeed, if k and k' belong to $Z(\alpha)$ we can concatenate a $\widetilde{\mathcal{D}}$-chain from V_α to $V_{\alpha+k}$ and a $\widetilde{\mathcal{D}}$-chain, translated by T^k, from V_α to $V_{\alpha+k'}$ to get a $\widetilde{\mathcal{D}}$-chain from V_α to $V_{\alpha+k+k'}$. We write

$$Z(\mathcal{D})=\bigcup_{\alpha\in A}Z(\alpha).$$

For every $\widetilde{\mathcal{D}}$-chain C from V_α to a translated disk $V_{\alpha+k}$, consider the rational number $k/l(C)$, where $l(C)$ is the length of C. Write $Q(\alpha)$ for the set of rational numbers obtained in this way. If we change the lift f by the lift $T^s\circ f$ we change the set $Q(\alpha)$ by $Q(\alpha)+s$. We define

$$Q(\mathcal{D})=\bigcup_{\alpha\in A} Q(\alpha).$$

If f is fixed point free, the set $Z(\alpha)$ doesn't contain 0 by Proposition 1: we deduce that all the integers $k\in Z(\alpha)$ have the same sign using the additive property. If F is fixed point free, for every $k\in\mathbf{Z}$, all the rational numbers in $Q(\alpha)+k$ have the same sign: we deduce that there exists an integer m such that $Q(\alpha)\subset]m,m+1[$.

Suppose now that $\mathcal{D}$ is transitive and fix α_1 and α_2 in A. By the transitivity of $\mathcal{D}$ we know that there exist two integers l_1 and l_2 , a $\widetilde{\mathcal{D}}$-chain from V_{α_1} to $V_{\alpha_2+l_2}$ and a $\widetilde{\mathcal{D}}$-chain from V_{α_2} to $V_{\alpha_1+l_1}$. Using concatenations and translations we deduce that for every $n\geq 0$ and for every $k_1\in Z(\alpha_1)$ the number $l_2+nk_1+l_1$ belongs to $Z(\alpha_2)$. If f is fixed point free we deduce that all the integers $k\in Z(\mathcal{D})$ have the same sign, and if F is fixed point free that there exists an integer m such that $Q(\mathcal{D})\subset]m,m+1[$.

Proposition 3. *Fix $F\in H_\mu(\mathbf{A})$ and a lift $f\in h_\mu(\mathbf{A})$, where μ is a probability measure whose support is $\mathbf{A}$. Consider a free triangulation $\mathcal{D}$ of $\mathbf{A}\setminus\mathrm{Fix}(F)$ and a finite set $X\subset Q(\mathcal{D})$. If $f'\in h_\mu(\mathbf{A})$ is sufficiently close to f, there exists a free transverse triangulation $\mathcal{D}'$ of $\mathbf{A}\setminus\mathrm{Fix}(F')$, where $F'\in H_\mu(\mathbf{A})$ is lifted by f', such that $X\subset Q(\mathcal{D}')$. Moreover if X contains a nonnegative number and a nonpositive number, f' has a fixed point.*

Proof. There exists a finite subset $\mathcal{D}_1$ of $\mathcal{D}$ such that $X\subset Q(\mathcal{D}_1)$. If f' is sufficiently close to f, the closure of every disk of $\mathcal{D}_1$ is free (for F') and we have the same relation $X\subset Q(\mathcal{D}_1)$ for f'. We can complete this finite family to construct a free triangulation of $\mathbf{A}\setminus\mathrm{Fix}(F')$. If we perturb it slightly we will get a free transverse triangulation $\mathcal{D}'$ of $\mathbf{A}\setminus\mathrm{Fix}(F')$ such that $X\subset Q(\mathcal{D}')$.

For any integer $q\geq 1$, denote by $\mathrm{Fix}_{\geq q}F$ the set of fixed points of F whose rotation number has an absolute value larger than q. Consider $M>0$ such that the disks of $\mathcal{D}_1$ are contained in $\mathbf{T}^1\times]-M,M[$ and such that this annulus meets its image by F. Find an integer $q\geq 1$ which bounds strictly the rotation numbers of the fixed points of F which are in $\mathbf{T}^1\times[-M,M]$. If $f'\in h_\mu(\mathbf{A})$ is sufficiently close to f and if $F'\in H_\mu(\mathbf{A})$ is lifted by f', the set $\mathbf{T}^1\times]-M,M[$ meets its image by F' and q bounds strictly the rotation numbers of the fixed points of F' which are in $\mathbf{T}^1\times[-M,M]$. If f' has no fixed point, the disks of the family $\mathcal{D}_1^q$ obtained by lifting $\mathcal{D}_1$ to $\mathbf{A}_q$ are contained in the same connected component W' of $\mathbf{A}_q\setminus\pi_q^{-1}(\mathrm{Fix}(F'_{\geq q}))$ and this component is invariant by the lift of F'. Moreover the set $Q(\mathcal{D}_1^q)$ contains a nonnegative and a nonpositive number. Completing $\mathcal{D}_1^q$ and perturbing it we can find a free transverse triangulation $\mathcal{D}'$ of W' such that $Q(\mathcal{D}')$ has the same property. Since the triangulation $\mathcal{D}'$ is transitive by Proposition 2, the map f' must have a fixed point. $\square$

Corollary 1. *Fix $F\in H_\mu(\mathbf{A})$ and a lift $f\in h_\mu(\mathbf{A})$, where μ is a probability measure whose support is $\mathbf{A}$. If there exist two recurrent points z_1 and z_2 with rotation numbers $\rho(z_1)$ and $\rho(z_2)$ such that $\rho(z_1)<0<\rho(z_2)$, then f has a fixed point. Moreover every $f'\in h_\mu(\mathbf{A})$ sufficiently close to f has a fixed point.*

Proof. If these points are not fixed by F we can find a free transverse triangulation $\mathcal{D}$ of $\mathbf{A}\setminus\mathrm{Fix}(F)$ such that z_1 and z_2 belong to 2-cells. The hypothesis tells us that $Q(\mathcal{D})$ contains a strictly negative and a strictly positive number and we can apply Proposition 3. If at least one of them is fixed we can take an integer $q > \max(|\rho(z_1)|, |\rho(z_2)|)$ and consider a free transverse triangulation $\mathcal{D}$ of $\mathbf{A}_q \setminus \pi_q^{-1}(\mathrm{Fix}(F_{\geq q}))$ such that every preimage of z_1 or z_2 belongs to a disk of $\mathcal{D}$. We will get the same situation as above. □

Corollary 2. *Let $(F_n)_{n\geq 0}$ be a sequence in $H_\mu(\mathbf{A})$ which converges to $F \in H_\mu(\mathbf{A})$ and $(f_n)_{n\geq 0}$ a sequence of lifts in $h_\mu(\mathbf{A})$ which converges to $f \in h_\mu(\mathbf{A})$, where μ is a probability measure whose support is $\mathbf{A}$. If the integers $|R(f_n)|$ are uniformly bounded, then $R(f)$ is finite and there exists a uniform bound to the rotation numbers of the recurrent points of the F_n.*

Proof. Corollary 1, applied to a translation of f or f_n, tells us that $R(f)$ and $R(f_n)$ are intervals of $\mathbf{Z}$. Moreover, if $\rho^- \in R(f)$ and $\rho^+ \in R(f)$ satisfy $\rho^+ - \rho^- \geq 2$, then every integer $\rho \in [\rho^- + 1, \rho^+ - 1]$ belongs to $R(f_n)$ for n large enough. We deduce that $R(f)$ is finite and that the set of rotation numbers of recurrent points of F is bounded. For the same reasons the set of rotation numbers of the recurrent points of each F_n is bounded. We want to find a uniform bound; it is sufficient to find it for n large enough.

Define $M = \max_n |R(f_n)| \geq 0$ and consider $q = M + 2$. We can find an interval $I \in \mathbf{Z}$ of cardinality $M+1$ such that if $k \in I$, the map F' of $\mathbf{A}_q$ lifted by $f' = T^{-k} \circ f$ is different from the identity. If F is different from the identity, any choice is good; if F is the identity, there exists p such that $T^{-p} \circ f$ is the identity and we choose I disjoint from $p + q\mathbf{Z}$. For every $n \geq 0$, there exists $k \in I$ such that $R(f_n)$ doesn't meet I. That means that the homeomorphism F'_n of $\mathbf{A}_q$ lifted by $f'_n = T^{-k} \circ f_n$ is fixed point free. To prove the corollary it is sufficient to fix $k \in I$ and to show that the rotation numbers of the recurrent points of F_n are uniformly bounded if F'_n is fixed point free. Consider a free transverse triangulation $\mathcal{D}$ of $\mathbf{A}_q \setminus \mathrm{Fix}(F')$ and the set $\mathcal{Q}(\mathcal{D})$ defined for the lift f' or F', fix a number $\rho \in Q(\mathcal{D})$ and take a finite sub-family $\mathcal{D}_1$ such that $\rho \in Q(\mathcal{D}_1)$. If n is large enough, then $\mathcal{D}_1$ is a family of disks whose closure is free for F'_n. If z is a recurrent point of F_n of rotation number ρ', then by extending the family $\mathcal{D}_1$ and perturbing it we can construct a free transverse triangulation $\mathcal{D}$ of $\mathbf{A}_q$ for F'_n such that $\rho \in Q(\mathcal{D})$ and such that every preimage of z in $\mathbf{A}_q$ belongs to a disk of $\mathcal{D}$. We deduce that we can find in $Q(\mathcal{D})$ rational numbers arbitrarily close to ρ'/q. This implies that $\rho'/q \in [\rho - 1, \rho + 1]$ because F'_n is fixed point free. □

3. Proofs of the theorems

Proof of Theorem 1. We consider a probability measure μ on $\mathbf{A}$, a homeomorphism $F \in H_\mu(\mathbf{A})$ and a lift $f \in h_\mu(\mathbf{A})$. We suppose that the set $R(f)$ is bounded. Every fixed point of F has a rotation number and the set of nonfixed points of F can be covered by free disks. So it is sufficient to prove that if U is a free disk of F such that $\mu(U) > 0$, then μ-a.e. recurrent point of F in U has a rotation number.

We denote by Φ the first return map defined a.e. on U and by ν the time of first return. The measurable function ν is integrable on U and we have

$$\int_U \nu \, d\mu = \mu\left(\bigcup_{k\geq 0} F^k(U)\right).$$

Indeed, we have the following measurable partitions (modulo sets of measure zero):

$$U = \bigsqcup_{i\geq 1} U_i \quad \text{and} \quad \bigcup_{k\geq 0} F^k(U) = \bigsqcup_{i\geq 1} \bigsqcup_{0\leq j\leq i-1} F^j(U_i) \ ,$$

where $U_i = \nu^{-1}(\{i\})$; therefore

$$\mu\left(\bigcup_{k\geq 0} F^k(U)\right) = \sum_{i\geq 1} \sum_{0\leq j\leq i-1} \mu(U_i) = \sum_{i\geq 1} i\,\mu(U_i) = \int_U \nu\, d\mu.$$

Fix a connected component $\widetilde{U}$ of $\pi^{-1}(U)$ and consider $\tau : U \to \mathbf{Z}$ such that

$$\tau(z) = k \quad \text{iff} \quad f^{\nu(z)}(\widetilde{z}) \in T^k(\widetilde{U}) \ ,$$

where $\widetilde{z}$ is the preimage of z contained in $\widetilde{U}$.

Using the fact that the disk U, taken alone, defines a transitive family of free disks of F and using the properties of the sets $Q(\alpha)$ stated in Section 2, we deduce that the function τ/ν is bounded. More precisely we recall that there exists a fixed point of $f \circ T^{-k}$ for every integer k surrounded by two points in the image of τ/ν.

Since the function ν is integrable, it is the same for τ and we know by the Poincaré Recurrence Theorem and the Birkhoff Ergodic Theorem that almost every point of U satisfies the following properties:

i) z is a recurrent point of Φ;

ii) $\displaystyle\lim_{n\to\pm\infty} \frac{\tau(z) + \cdots + \tau(\Phi^{n-1}(z))}{n} = \tau^*(z)$ exists;

ii) $\displaystyle\lim_{n\to\pm\infty} \frac{\nu(z) + \cdots + \nu(\Phi^{n-1}(z))}{n} = \nu^*(z)$ exists.

The recurrent points of F are exactly the recurrent points of Φ because U is open. We deduce that almost every recurrent point of F in U has a rotation number which is

$$\lim_{n\to\pm\infty} \frac{\tau(z) + \cdots + \tau(\Phi^{n-1}(z))}{\nu(z) + \cdots + \nu(\Phi^{n-1}(z))} = \frac{\tau^*(z)}{\nu^*(z)}.$$

If the support of μ is $\mathbf{A}$ we know using the corollaries in Section 2 that the rotation numbers are uniformly bounded and that the measure has a rotation number. □

Remark. If the support of μ is not $\mathbf{A}$ the second part is obviously false. Suppose that $f(x, k) = (x + k + 1/2, k)$ if $k \in \mathbf{Z}$ and that F is fixed point free. One can construct easily a discrete invariant measure without rotation number.

Proof of Theorem 2. We consider a probability measure μ on $\mathbf{A}$, a sequence $(F_n)_{n\geq 0}$ in $H_\mu(\mathbf{A})$ which converges to F and a sequence of lifts $(f_n)_{n\geq 0}$ which converges to a lift f of F. We suppose that the integers $|R(f_n)|$ are uniformly bounded. Corollary 2 tells us that $R(f)$ is bounded and that the rotation numbers of recurrent points of the F_n are bounded independently of n. By Theorem 1 we can define the rotation number ρ_n of μ for each f_n, $n \geq 0$, and the rotation number ρ of μ for f. We know that the sequence $(\rho_n)_{n\geq 0}$ is bounded and want to prove that it converges to ρ. Consider an integer $k \geq 1$ such that the rotation numbers of recurrent points of each F_n or F are contained in $[-k, k]$. Taking $f'_n = T^{k+1} \circ f_n$ instead of f_n and $\mathbf{A}_{2k+2}$ instead of $\mathbf{A}$ we can suppose that every F_n (and F) is fixed point free and that every ρ_n (and ρ) belongs to $]0, 1[$. Using the results of Section 2 we deduce that for any free transverse triangulation $\mathcal{D}$ of $\mathbf{A}$ for F_n or F, we have $Q(\mathcal{D}) \subset]0, 1[$.

Indeed the set $Q(\mathcal{D})$ meets $]0,1[$ because the rotation number of any recurrent point belongs to this interval and cannot meet $]-\infty,0]$ or $[1,\infty[$), otherwise they would be a fixed point of F (or F_n).

Consider a full free transverse triangulation $\mathcal{D}=(V_\alpha)_{\alpha\in A}$ of F such that every disk has a diameter <1. Then a finite subfamily $\mathcal{D}_1=(V_\alpha)_{\alpha\in A_1}$ of $\mathcal{D}$ such that the measure of $W=\mathrm{int}(\overline{\bigcup_{\alpha\in A_1}V_\alpha})$ is larger than $1-\varepsilon$. By the transitivity of $\mathcal{D}$ we know that there exists a larger finite subfamily $\mathcal{D}_2=(V_\alpha)_{\alpha\in A_2}$ and $M_1>0$ such that for every α and α' in A_1 there exists a $\mathcal{D}_2$-chain from V_α to $V_{\alpha'}$ of length $\leq M_1$.

Let $\widetilde{\mathcal{D}}=(V_\alpha)_{\alpha\in\widetilde{A}}$, $\widetilde{\mathcal{D}}_1=(V_\alpha)_{\alpha\in\widetilde{A}_1}$ and $\widetilde{\mathcal{D}_2}=(V_\alpha)_{\alpha\in\widetilde{A}_2}$ be the lifted families. There exists $M_2>0$ such that for every $\alpha\in\widetilde{A}_1$ and $\alpha'\in\widetilde{A}_1$ there is an integer $k\in\mathbf{Z}$ and a $\widetilde{\mathcal{D}_2}$-chain from V_α to $V_{\alpha'+k}$ of length $\leq M_1$ and such that V_α and $V_{\alpha'+k}$ can be projected by p_1 on an interval of length $\leq M_2$.

Consider the first return map Φ defined μ-a.e. on W, the time of first return ν, and the function

$$\lambda : z\mapsto \sum_{i=0}^{\nu(z)-1} r(F^i(z)) = p_1\circ f^{\nu(z)}(\widetilde{z})-p_1(\widetilde{z}) \ , \ \ \widetilde{z}\in\pi^{-1}(\{z\}).$$

If $\Phi(z)$ is defined, using Baire's theorem, we can find z' close to z such that

- every point $F^i(z')$, $0\leq i\leq\nu(z)$, belongs to V_{α_i}, $\alpha_i\in A$;
- α_0 and $\alpha_{\nu(z)}$ are in A_1;
- $z\in\overline{V}_{\alpha_0}$ and $F(z)\in\overline{V}_{\alpha_{\nu(z)}}$.

If $\widetilde{z}\in\pi^{-1}(\{z\})$, we can lift the chain $(V_{\alpha_i})_{0\leq i\leq\nu(z)}$ to get a $\widetilde{\mathcal{D}}$-chain from V_α to $V_{\alpha'}$ where

- α and α' are in $\widetilde{A_1}$;
- $\widetilde{z}\in\overline{V}_\alpha$ and $f(\widetilde{z})\in\overline{V}_{\alpha'}$.

We can add to this chain a $\widetilde{\mathcal{D}}_2$-chain from $V_{\alpha'}$ to a translated $V_{\alpha+k}$ to get a $\widetilde{\mathcal{D}}$-chain from V_α to $V_{\alpha+k}$ of length between $\nu(z)$ and $\nu(z)+M_1$. The fact that each disk has a diameter <1 and that V_α and $V_{\alpha+k}$ are projected in an interval of length $\leq M_2$ implies that $|k-\lambda(z)|\leq M_2+2$. The fact that $Q(\mathcal{D})\subset]0,1[$ implies that $0<k<\nu(z)+M_1$. Therefore

$$\begin{aligned}-\nu(z)(M_1+M_2+3)<-M_2-2<\lambda(z)\\ <\nu(z)+M_1+M_2+2\leq\nu(z)(M_1+M_2+3)\end{aligned}$$

and

$$\left|\frac{\lambda(z)}{\nu(z)}\right|\leq M_1+M_2+3=M_3.$$

As in the proof of Theorem 1, ν and λ are integrable. From Theorem 1 we know that for μ-a.e. point $z\in W$:

- **i)** z is a recurrent point of Φ;
- **ii)** $\displaystyle\lim_{n\to\pm\infty}\frac{\nu(z)+\cdots+\nu(\Phi^{n-1}(z))}{n}=\nu^*(z)$ exists;
- **iii)** $\displaystyle\lim_{n\to\pm\infty}\frac{\lambda(z)+\cdots+\lambda(\Phi^{n-1}(z))}{n}=\lambda^*(z)$ exists;
- **iv)** the rotation number $\rho(z)$ exists and we have $\rho(z)=\dfrac{\lambda^*(z)}{\nu^*(z)}$.

Using the fact that ρ is F-invariant, that μ is Φ-invariant and the Birkhoff Ergodic Theorem we obtain

$$\int_{\bigcup_{k\geq 0} F^k(W)} \rho\, d\mu = \int_W \nu\, \rho\, d\mu = \int_W \nu^*\, \rho\, d\mu = \int_W \lambda^*\, d\mu = \int_W \lambda\, d\mu.$$

The first equality is obtained by arguing as in the proof of Theorem 1.

For n large enough $\mathcal{D}_2$ is a family of free disks of f_n and we have the same situation: we define Φ_n, ν_n and λ_n on W with the same inequalities, we define ρ_n on $\mathbf{A}$, ν_n^* and λ_n^* on W with the same equality.

We write

$$\begin{aligned}\left|\int_{\bigcup_{k\geq 0} F^k(W)} \rho\, d\mu - \int_{\bigcup_{k\geq 0} F_n^k(W)} \rho_n\, d\mu\right| &= \int_W (\lambda - \lambda_n)\, d\mu \\ &= \int_W \left(\frac{\lambda}{\nu} - \frac{\lambda_n}{\nu_n}\right) \nu\, d\mu + \int_W \frac{\lambda_n}{\nu_n}(\nu - \nu_n)\, d\mu.\end{aligned}$$

First we obtain

$$\left|\int_W \frac{\lambda_n}{\nu_n}(\nu - \nu_n)\, d\mu\right| \leq \int_W M_3|\nu' - \nu_n'|\, d\mu \leq M_3 \int_W (\nu' + \nu_n')\, d\mu \leq 2M_3\varepsilon\ ,$$

where $\nu' = \nu - 1$ and $\nu_n' = \nu_n' - 1$. Then we write

$$\left|\frac{\lambda}{\nu} - \frac{\lambda_n}{\nu_n}\right| \nu \leq 2M_3\nu.$$

Using the fact that the boundary of W has measure 0 (because the triangulation is full)) we deduce that the function $\dfrac{\lambda}{\nu} - \dfrac{\lambda_n}{\nu_n}$ tends to zero a.e. when $n \to +\infty$. So by the Lebesgue Dominated Convergence Theorem, for n large enough we have

$$\left|\int_W \left(\frac{\lambda}{\nu} - \frac{\lambda_n}{\nu_n}\right) \nu\ , d\mu\right| \leq \varepsilon.$$

Using the fact that all the rotation numbers belong to $[0,1]$ we deduce that

$$\left|\int_{\mathbf{A}} \rho\, d\mu - \int_{\bigcup_{k\geq 0} F^k(W)} \rho\, d\mu\right| \leq \varepsilon$$

and

$$\left|\int_{\mathbf{A}} \rho_n\, d\mu - \int_{\bigcup_{k\geq 0} F_n^k(W)} \rho_n\, d\mu\right| \leq \varepsilon.$$

We have obtained

$$|\rho_n - \rho| \leq (2M_3 + 3)\varepsilon$$

for n large enough, and the sequence $(\rho_n)_{n\geq 0}$ converges to ρ. □

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