A SIMPLE PROOF FOR SCHUR’S THEOREM

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(Communicated by Juha M. Heinonen)

Abstract. In 1917 Schur gave a characterization of bounded analytic functions on the unit disc. We present a simple proof.

SCHUR’S THEOREM

Let $H(\Delta)$ be the set of analytic functions on the unit disc $\Delta$ in the complex plane, and let $B \subset H(\Delta)$ be the subset of functions $f$ for which $f(\Delta) \subset \overline{\Delta}$. Schur [3] obtained the following result.

Theorem. For functions $f : z \to \sum_{k=0}^{\infty} a_k z^k \in H(\Delta)$, the following conditions are equivalent.

1) $f \in B$.

2) For all $N \in \mathbb{N}$ and for all $\lambda_0, \ldots, \lambda_N \in \mathbb{C}$ we have

$$\sum_{k=0}^{N} \left| \sum_{n=k}^{N} a_{n-k} \lambda_n \right|^2 \leq \sum_{k=0}^{N} |\lambda_k|^2.$$ 

There exist several proofs of this result; see e.g. [1], page 15, [2], page 40, page 180. We present a simple and elementary one. Our starting point is another characterization of $B$. As usual, we write $H^2$ for the Hardy space of functions $f : z \to \sum_{k=0}^{\infty} a_k z^k \in H(\Delta)$ for which

$$\|f\|^2_2 = \sum_{k=0}^{\infty} |a_k|^2 = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$ 

Lemma. Let $f \in H(\Delta)$ and let $M_f : H(\Delta) \to H(\Delta)$ be the multiplication operator defined by $M_f(g) = f \cdot g$. Then the following conditions are equivalent.

1) $f \in B$.

2) $M_f(H^2) \subset H^2$ and $\|M_f\| \leq 1$.

Proof. 1) $\implies$ 2) is obvious.

2) $\implies$ 1). We have $1 \in H^2$, hence $f \in H^2$, hence $f^2 \in H^2$, $f^n \in H^2$ and
\[ \|f^n\|_2 \leq 1, \text{ i.e. for all } n \]
\[ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 n d\theta \leq 1. \]

Thus \(|f(z)| \leq 1\) for all \(z\).

We are ready for the proof of Schur’s theorem.

1) \(\implies\) 2). For every sequence \(\mu_0, \mu_1, \ldots, \in \mathbb{C}\) with \(\mu_{N+1} = \mu_{N+2} = \ldots = 0\), we see that the polynomial \(g : z \rightarrow \sum_{k=0}^{N} \mu_k z^k \in H^2\) and that
\[ \|g\|^2 = \sum_{k=0}^{N} |\mu_k|^2. \]

We have
\[ M_f(g)(z) = \sum_{m=0}^{\infty} \left( \sum_{l=0}^{m} a_{m-l} \mu_l \right) z^m, \]
so by the lemma we have
\[ \sum_{m=0}^{N} \left| \sum_{l=0}^{m} a_{m-l} \mu_l \right|^2 \leq \sum_{m=0}^{\infty} \left| \sum_{l=0}^{m} a_{m-l} \mu_l \right|^2 \leq \sum_{k=0}^{N} |\mu_k|^2. \]

Take \(\mu_k = \lambda_{N-k} (k = 0, \ldots, N)\) and we obtain
\[ \sum_{m=0}^{N} \left| \sum_{l=0}^{m} a_{m-l} \lambda_{N-l} \right|^2 \leq \sum_{k=0}^{N} |\lambda_k|^2. \]

The change of variables \(m = N - k, l = N - n\) leads to
\[ \sum_{k=0}^{N} \left| \sum_{n=k}^{N} a_{n-k} \lambda_n \right|^2 \leq \sum_{k=0}^{N} |\lambda_k|^2. \]

2) \(\implies\) 1). Let \(z \in \Delta\); choose \(\lambda_n = z^n\) and take the limit for \(N \rightarrow \infty\). We obtain:
\[ \frac{1}{1-|z|^2} |f(z)|^2 \leq \frac{1}{1-|z|^2}; \text{ i.e., } |f(z)| \leq 1. \]

REFERENCES


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