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A SIMPLE PROOF FOR SCHUR'S THEOREM

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ABSTRACT. In 1917 Schur gave a characterization of bounded analytic functions on the unit disc. We present a simple proof.

SCHUR'S THEOREM

Let $H(\Delta)$ be the set of analytic functions on the unit disc Δ in the complex plane, and let $B \subset H(\Delta)$ be the subset of functions f for which $f(\Delta) \subset \overline{\Delta}$. Schur [3] obtained the following result.

Theorem. *For functions $f : z \to \sum_{k=0}^{\infty} a_k z^k \in H(\Delta)$, the following conditions are equivalent.*

1) $f \in B$.

2) For all $N \in \mathbb{N}$ and for all $\lambda_0, \dots, \lambda_N \in \mathbb{C}$ we have

$$\sum_{k=0}^{N} \left| \sum_{n=k}^{N} a_{n-k} \lambda_n \right|^2 \leq \sum_{k=0}^{N} |\lambda_k|^2 .$$

There exist several proofs of this result; see e.g. [1], page 15, [2], page 40, page 180. We present a simple and elementary one. Our starting point is another characterization of B. As usual, we write H^2 for the Hardy space of functions $f : z \to \sum_{k=0}^{\infty} a_k z^k \in H(\Delta)$ for which

$$\|f\|_2^2 = \sum_{k=0}^{\infty} |a_k|^2 = \lim_{r \uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\vartheta})|^2 d\vartheta < \infty.$$

Lemma. *Let $f \in H(\Delta)$ and let $M_f : H(\Delta) \to H(\Delta)$ be the multiplication operator defined by $M_f(g) = f \cdot g$. Then the following conditions are equivalent.*

1) $f \in B$.

2) $M_f(H^2) \subset H^2$ and $\|M_f\| \leq 1$.

Proof. 1) $\Longrightarrow$ 2) is obvious.

2) $\Longrightarrow$ 1). We have $1 \in H^2$, hence $f \in H^2$, hence $f^2 \in H^2, \dots, f^n \in H^2$ and

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$\|f^n\|_2 \le 1$, i.e. for all n

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\vartheta})|^{2n} d\vartheta \le 1.$$

Thus $|f(z)| \le 1$ for all z. □

We are ready for the proof of Schur's theorem.

1) $\Longrightarrow$ 2). For every sequence $\mu_0, \mu_1, \ldots \in \mathbb{C}$ with $\mu_{N+1} = \mu_{N+2} = \ldots = 0$, we see that the polynomial $g : z \to \sum_{k=0}^{\infty} \mu_k z^k \in H^2$ and that

$$\|g\|_2^2 = \sum_{k=0}^{N} |\mu_k|^2.$$

We have

$$M_f(g)(z) = \sum_{m=0}^{\infty} \left(\sum_{l=0}^{m} a_{m-l}\mu_l \right) z^m,$$

so by the lemma we have

$$\sum_{m=0}^{N} \left| \sum_{l=0}^{m} a_{m-l}\mu_l \right|^2 \le \sum_{m=0}^{\infty} \left| \sum_{l=0}^{m} a_{m-l}\mu_l \right|^2 \le \sum_{k=0}^{N} |\mu_k|^2.$$

Take $\mu_k = \lambda_{N-k}$ $(k = 0, \ldots, N)$ and we obtain $\sum_{m=0}^{N} \left| \sum_{l=0}^{m} a_{m-l}\lambda_{N-l} \right|^2 \le \sum_{k=0}^{N} |\lambda_k|^2$. The change of variables $m = N - k$, $l = N - n$ leads to

$$\sum_{k=0}^{N} \left| \sum_{n=k}^{N} a_{n-k}\lambda_n \right|^2 \le \sum_{k=0}^{N} |\lambda_k|^2.$$

2) $\Longrightarrow$ 1). Let $z \in \Delta$; choose $\lambda_n = z^n$ and take the limit for $N \to \infty$. We obtain: $\frac{1}{1-|z|^2}|f(z)|^2 \le \frac{1}{1-|z|^2}$; i.e., $|f(z)| \le 1$.

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