A SIMPLE PROOF FOR SCHUR’S THEOREM

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ABSTRACT. In 1917 Schur gave a characterization of bounded analytic functions on the unit disc. We present a simple proof.

**Schur’s theorem**

Let \( H(\Delta) \) be the set of analytic functions on the unit disc \( \Delta \) in the complex plane, and let \( B \subset H(\Delta) \) be the subset of functions \( f \) for which \( f(\Delta) \subset \overline{\Delta} \). Schur [3] obtained the following result.

**Theorem.** For functions \( f: z \to \sum_{k=0}^\infty a_k z^k \in H(\Delta) \), the following conditions are equivalent.

1) \( f \in B \).

2) For all \( N \in \mathbb{N} \) and for all \( \lambda_0, \ldots, \lambda_N \in \mathbb{C} \) we have

\[
\left| \sum_{k=0}^N \sum_{n=k}^\infty a_{n-k} \lambda_n \right|^2 \leq \sum_{k=0}^N |\lambda_k|^2.
\]

There exist several proofs of this result; see e.g. [1], page 15, [2], page 40, page 180. We present a simple and elementary one. Our starting point is another characterization of \( B \). As usual, we write \( H^2 \) for the Hardy space of functions \( f: z \to \sum_{k=0}^\infty a_k z^k \in H(\Delta) \) for which

\[
\|f\|_2^2 = \sum_{k=0}^\infty |a_k|^2 = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.
\]

**Lemma.** Let \( f \in H(\Delta) \) and let \( M_f: H(\Delta) \to H(\Delta) \) be the multiplication operator defined by \( M_f(g) = f \cdot g \). Then the following conditions are equivalent.

1) \( f \in B \).

2) \( M_f(H^2) \subset H^2 \) and \( \|M_f\| \leq 1 \).

**Proof.** 1) \( \implies \) 2) is obvious.

2) \( \implies \) 1). We have \( 1 \in H^2 \), hence \( f \in H^2 \), hence \( f^2 \in H^2, \ldots, f^n \in H^2 \) and
\[ \|f^n\|_2 \leq 1, \text{ i.e. for all } n \]
\[ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{2n} \, d\theta \leq 1. \]

Thus \( |f(z)| \leq 1 \) for all \( z \).

We are ready for the proof of Schur’s theorem.

1) \( \implies \) 2). For every sequence \( \mu_0, \mu_1, \ldots \in \mathbb{C} \) with \( \mu_{N+1} = \mu_{N+2} = \ldots = 0 \), we see that the polynomial \( g : z \to \sum_{k=0}^{N} \mu_k z^k \in H^2 \) and that

\[ \|g\|_2^2 = \sum_{k=0}^{N} |\mu_k|^2. \]

We have

\[ M_f(g)(z) = \sum_{m=0}^{\infty} \left( \sum_{l=0}^{m} a_{m-l} \mu_l \right) z^m, \]

so by the lemma we have

\[ \sum_{m=0}^{N} \left( \sum_{l=0}^{m} a_{m-l} \mu_l \right)^2 \leq \sum_{m=0}^{\infty} \left( \sum_{l=0}^{m} a_{m-l} \mu_l \right)^2 \leq \sum_{k=0}^{N} |\mu_k|^2. \]

Take \( \mu_k = \lambda_{N-k} \) \( (k = 0, \ldots, N) \) and we obtain

\[ \sum_{m=0}^{\infty} \left( \sum_{l=0}^{m} a_{m-l} \lambda_{N-l} \right)^2 \leq \sum_{k=0}^{N} |\lambda_k|^2. \]

The change of variables \( m = N - k, \ l = N - n \) leads to

\[ \sum_{k=0}^{N} \left( \sum_{n=k}^{N} a_{n-k} \lambda_n \right)^2 \leq \sum_{k=0}^{N} |\lambda_k|^2. \]

2) \( \implies \) 1). Let \( z \in \Delta \); choose \( \lambda_n = z^n \) and take the limit for \( N \to \infty \). We obtain:

\[ \frac{1}{1-|z|^2} |f(z)|^2 \leq \frac{1}{1-|z|^2}; \text{ i.e., } |f(z)| \leq 1. \]

References


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