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VARIATIONALLY COMPLETE REPRESENTATIONS ARE POLAR

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ABSTRACT. A recent result of C. Gorodski and G. Thorbergsson, involving classification, asserts that a variationally complete representation is polar. The aim of this paper is to give a conceptual and very short proof of this fact, which is the converse of a result of Conlon.

The concept of a variationally complete action was introduced by R. Bott [B] in 1956. Two years later, Bott and Samelson [BS] proved that s-representations (i.e. isotropy representation of semisimple symmetric spaces) are variationally complete. This class of representations contains examples of what L. Conlon [C] called polar representations, or more generally hyperpolar actions. He proved that a hyperpolar action of a compact Lie group on a complete Riemannian manifold is variationally complete. Polar representations were classified by J. Dadok [D] who proved that they are orbit equivalent to s-representations (see also [EH]). Recently, C. Gorodski and G. Thorbergsson classified variationally complete representations of compact Lie groups [GT]. From this classification they obtained that a variationally complete representation is also orbit equivalent to an s-representation (from this they obtained, with different methods, Dadok's list). So, they obtained the following equivalent theorem, some of whose history can be found in [TT, p. 196].

Theorem ([GT])**.** *A variationally complete orthogonal representation of a compact Lie group is polar.*

The object of this short note is to give a direct and geometric proof of the above theorem.

Recall that a compact connected Lie subgroup G of $SO(n)$ acts polarly on $\mathbb{R}^n$ if there exists an affine subspace which meets orthogonally all G-orbits. This is equivalent to the fact that the tangent space $T_v(G.v)$ of a principal orbit $G.v$ contains the tangent spaces of all orbits through points in the normal space $\nu_v(G.v)$. The G-action is called variationally complete if any G-transversal Jacobi field (i.e. produced by variations of transversal geodesics) that is tangent to orbits at two points is the restriction of some Killing field on $\mathbb{R}^n$ induced by the action. Recall that a geodesic $\gamma(t)$ in $\mathbb{R}^n$ is G-transversal if it is orthogonal to the G-orbit through

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$\gamma(t)$ for every t (or equivalently, for some t_0 since a Killing field projects constantly to any geodesic).

Proof. Let G be a compact connected Lie subgroup of $SO(n)$ such that the G-action is variationally complete, and let $v \in \mathbb{R}^n$ be a principal vector. Let ξ_v be a normal vector to $G.v$ at v whose shape operator A_{ξ_v} has all eigenvalues different from zero. (Such normal vectors define an open and dense subset of the normal space. This is because the determinant of the shape operator is a nonzero polynomial on the normal space at a given point v, since $A_v = -Id.$) Let $c(s)$ be a curve in $G.v$ with $c(0) = v$ and such that $w := c'(0) \neq 0$ is an eigenvector of A_{ξ_v} with associated eigenvalue λ. Extend ξ_v to a parallel normal field $\xi(s)$ to $G.v$ along $c(s)$. Let us consider the variation by G-transversal geodesics given by $\gamma_s(t) = c(s) + t\xi(s)$ and set $J(t) = \frac{\partial}{\partial s}_{|s=0}\gamma_s(t) = (1 - t\lambda)w$. Then $J(t)$ is a (G-transversal) Jacobi field along the geodesic $\gamma_0(t) = v + t\xi_v$ of $\mathbb{R}^n$. Observe that $J(0) = w \in T_v(G.v)$ and $J(1/\lambda) = 0 \in T_{\gamma_0(1/\lambda)}(G.\gamma_0(1/\lambda))$. By the variational completeness of the G-action, J is the restriction to γ_0 of a Killing field induced by G. Thus, for all t, $J(t) = (1 - t\lambda)w$, and so w, belongs to $T_{\gamma_0(t)}(G.\gamma_0(t))$. Since the eigenvectors of A_{ξ_v} generate $T_v(G.v)$, we obtain that $T_{\gamma_0(t)}(G.\gamma_0(t)) = T_v(G.v)$ for t small. This easily implies that G acts polarly. □

Remark. It is also true that a variationally complete action of a noncompact Lie subgroup G of $Iso(\mathbb{R}^n)$ is also polar. In fact, from [Di] (see also [O]) there always exists a principal orbit $G.p$ with a normal vector ξ_p whose shape operator A_ξ is positive definite (otherwise all G-orbits are parallel and totally geodesic), and so invertible. Then, the same proof applies.

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