INVERSE SCATTERING FOR THE NONLINEAR SCHRÖDINGER EQUATION II.
RECONSTRUCTION OF THE POTENTIAL AND THE NONLINEARITY
IN THE MULTIDIMENSIONAL CASE

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Abstract. We solve the inverse scattering problem for the nonlinear Schrödinger equation on $\mathbb{R}^n$, $n \geq 3$:

$$i\frac{\partial}{\partial t}u(t,x) = -\Delta u(t,x) + V_0(x)u(t,x) + \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+1)}u(t,x).$$

We prove that the small-amplitude limit of the scattering operator uniquely determines $V_j,j=0,1,\cdots$. Our proof gives a method for the reconstruction of the potentials $V_j,j=0,1,\cdots$. The results of this paper extend our previous results for the problem on the line.

1. Introduction

Let us consider the following nonlinear Schrödinger equation with a potential:

$$i\frac{\partial}{\partial t}u(t,x) = -\Delta u(t,x) + V_0(x)u(t,x) + F(x,u), u(0,x) = \phi(x),$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, $n \geq 3$. The potential, $V_0$, is a real-valued function, $F(x,u)$ is a complex-valued function, and $\Delta := \sum_{j=1}^{n} D_j^2$. We use the standard notation, $D_j := \frac{\partial}{\partial x_j}$ and for $\alpha := (\alpha_1,\cdots,\alpha_n)$, $D^\alpha := D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, with $|\alpha| := \sum_{j=1}^{n} \alpha_j$. We first construct the scattering operator for the nonlinear Schrödinger equation (1.1). For this purpose we introduce some assumptions and definitions.

Assumption A. Let $p$ satisfy $\rho < p < 1 + \frac{4}{n-1}$, where $\rho$ is the positive root of $\frac{n}{2p+1} = \frac{1}{\rho}$. Let $k$ be an integer such that $k > \frac{4}{p+1}$. Let $F = F_1 + iF_2$ with $F_1, F_2$ real-valued, and $u = r_1 + ir_2, r_1, r_2 \in \mathbb{R}$. We suppose that $F(0) = 0$ and that for all integers $\beta$ with $1 \leq \beta \leq k+1$ and all $\alpha$ with $|\beta + |\alpha|| \leq k+1$, we have that

$$\sum_{j=1}^{2} |\frac{\partial^\beta}{\partial r_1^{\beta_1} \partial r_2^{\beta_2}} D^\alpha F_j(x,u)| \leq C|u|^{|\max(0,p-\beta)|} \text{ for } |u| \leq \gamma,$$

for some $\gamma > 0$, and for all nonnegative integers, $\beta_1, \beta_2$, with $\beta = \beta_1 + \beta_2$. 

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We denote by $H_0$ the self-adjoint realization of $-\Delta$ in $L^2(\mathbb{R}^n)$ with domain the Sobolev space $W_{2,2}$. For the definition of the Sobolev spaces $W_{j,p}$, $j = 1, 2, \ldots, 1 \leq p \leq \infty$, see [1].

**Assumption B.** We assume that $V_0$ is real valued and that for some $\delta > (3n/2)+1$,

$$
\sup_{x \in \mathbb{R}^n} (1 + |x|)^{\delta} \left( \int_{|x-y| \leq 1} |D^{\alpha}V_0(y)|^{p_0} \, dy \right)^{1/p_0} < \infty,
$$

for all $|\alpha| \leq k + k_0$, with $k$ as in Assumption A. If $n = 3$, $p_0 = 2$ and $k_0 = 0$, and if $n \geq 4$, $p_0 > n/2$ and $k_0 := [(n-1)/2]$, where $[\sigma]$ denotes the integral part of $\sigma$. Moreover, assume that zero is neither an eigenvalue nor half-bound state (a resonance) of $H := H_0 + V_0$.

Zero is said to be a half-bound state of $H$ if the equation $H\phi = 0$ has a solution $\phi \notin L^2(\mathbb{R}^n)$, such that $(1 + |x|)^{1-\epsilon} \phi \in L^2(\mathbb{R}^n)$ for all $\epsilon > 0$.

Under Assumption B $H$ is self-adjoint with domain $W_{2,2}$ and it has no singular-continuous spectrum and no positive eigenvalues [5]. Moreover, the wave operators

$$
W_\pm := s - \lim_{t \to \pm \infty} e^{itH} e^{-itH_0}
$$

exist and $\text{Range} W_+ = \mathcal{H}_c$, the subspace of continuity of $H$. The scattering operator for the linear Schrödinger equation (equation (1.1) with $F = 0$) is given by

$$
S_L := W_+ W_-.
$$

Actually, these results are true under more general conditions. The crucial issue is that Yajima has proven that under Assumption B the wave operators and the adjoints, $W_\pm^*$, are bounded operators on $W_{l,p}, l = 0, 1, \cdots, k, 1 \leq p \leq \infty$. For this result see Theorem 1.2 of [23] and also [24]. This result and the intertwining relations for the wave operators, $e^{-itH} P_c = W_+ e^{-itH_0} W_+^*$, imply that the following $L^p - L^{\hat{p}}$ estimate follows from the corresponding result for $H_0$ (see [24]):

$$
\|e^{-itH} P_c\|_{B(L^p, L^{\hat{p}})} \leq \frac{C}{t^{n(p-1)/2}}, \quad t > 0,
$$

for $1 \leq p \leq 2$ and where $\frac{1}{p} + \frac{1}{\hat{p}} = 1$. By $P_c$ we denote the orthogonal projector onto $\mathcal{H}_c$. For any pair of Banach spaces $X, Y$, we denote by $\mathcal{B}(X, Y)$ the Banach space of all bounded operators from $X$ into $Y$. The $L^p - L^{\hat{p}}$ estimate in $\mathbb{R}^n, n \geq 3$, was first proven, under slightly different conditions, in [8].

The results of Yajima [24] allow us to extend to the case of $n \geq 3$ the method for the construction of the scattering operator for (1.1) and for the solution of the inverse scattering problem that we gave in [23] in the case of $n = 1$. The $L^p - L^{\hat{p}}$ estimate and the continuity of the wave operators on $W_{k,p}$ for the problem on the line was proven in [20] and [21] (see also [6]).

Let us denote [13],

$$
N_\delta(V_0) := \sup_{x \in \mathbb{R}^n} \left[ \int_{|x-y| < \delta} |V(y)|^{p_0} \, dy \right]^{1/p_0}.
$$

**Assumption C.** We assume that $N_\delta(D^{\alpha}V_0) < \infty$, $\delta > 0$, and that

$$
\lim_{\delta \to 0} N_\delta(D^{\alpha}V_0) = 0,
$$

where $|\alpha| \leq k - 1$, with $k$ and $p_0$ as in Assumption B.
We designate
\[ M := \{ u \in C(\mathbb{R}, W_{k,p+1}) : \sup_{t \in \mathbb{R}} (1 + |t|)^d \| u \|_{W_{k,p+1}} < \infty \} , \]
where \( d := \frac{p-1}{p+1} \). For functions \( u(t, x) \) defined in \( \mathbb{R}^{n+1} \) we denote \( u(t) \) for \( u(t, \cdot) \).

In the following theorem we construct the small-amplitude scattering operator.

**Theorem 1.1.** Suppose that Assumptions A, B and C are satisfied and that \( H \) has no eigenvalues. Then, there is a \( \delta > 0 \) such that for all \( \phi_- \in W_{k+1,2} \cap W_{k,1+\frac{1}{p}} \) with \( \| \phi_- \|_{W_{k+1,2}} + \| \phi_- \|_{W_{k,1+\frac{1}{p}}} \leq \delta \) there is a unique solution, \( u \), to (1.1) such that \( u \in C(\mathbb{R}, W_{k,2}) \cap M \) and
\[
\lim_{t \to -\infty} \| u(t) - e^{-itH} \phi_- \|_{W_{k,2}} = 0.
\]
Moreover, there is a unique \( \phi_+ \in W_{k,2} \) such that
\[
\lim_{t \to -\infty} \| u(t) - e^{-itH} \phi_+ \|_{W_{k,2}} = 0.
\]
Furthermore, \( e^{-itH} \phi_\pm \in M \) and
\[
\| u - e^{-itH} \phi_\pm \|_M \leq C \| e^{-itH} \phi_\pm \|_M^p .
\]

The scattering operator \( S_{V_0} : \phi_- \mapsto \phi_+ \) is injective on \( W_{k+1,2} \cap M \).

Note that in Theorem 1.1 we do not restrict \( F \) in such a way that energy is conserved. Moreover, for \( n = 3, \rho = 2 \) and \( \lim_{n \to \infty} \rho = 1 \). We prove Theorem 1.1 in Section 2 extending to this case the proof given in [23] in the case of \( n = 1 \). We construct the solution \( u(t, x) \) by means of the contraction mapping theorem in a ball, \( M_R \), of \( M \) with small enough radius, \( R \). It follows from Sobolev’s imbedding theorem [11] that \( |u(t, x)| < \gamma, t \in \mathbb{R}, x \in \mathbb{R}^n \), for all \( u(t, x) \in M_R \). This is the reason why we only have to assume that (1.2) holds for \( |u| \leq \gamma \). For results on scattering for the nonlinear Schrödinger equation in the case where \( V_0 = 0 \) see [16], [17], [18], [10], [9], [11], [3], [7], [2] and the references quoted there. In [8] the direct scattering for (1.1) with \( F = F(u) \) was studied for \( n \geq 3 \). The corresponding inverse problem was considered in [19]. For the case of the nonlinear Klein-Gordon equation on the line see [22].

Since we wish to reconstruct the potential, \( V_0 \), we consider the scattering operator that relates asymptotic states that are solutions to the linear Schrödinger equation with potential zero (1.1) with \( V_0 = F = 0 \):
\[
S := W_+^* S_{V_0} W_- .
\]

The first step is to reconstruct \( S_L \) from \( S \).

**Theorem 1.2.** Suppose that the assumptions of Theorem 1.1 are satisfied. Then for every \( \phi_- \in W_{k+1,2} \cap W_{k,1+\frac{1}{p}} \),
\[
\frac{d}{d\epsilon} S(\epsilon \phi) \bigg|_{\epsilon = 0} = S_L \phi ,
\]
where the derivative in the left-hand side of (1.13) exists in the strong convergence in $W_{k,2}$.

**Corollary 1.3.** Under the conditions of Theorem 1.1 the scattering operator, $S$, determines uniquely the potential $V_0$.

**Proof.** By Theorem 1.2 $S$ uniquely determines $S_L$. From $S_L$ we uniquely reconstruct the potential $V_0$ using the known results on the inverse scattering problem for the linear Schrödinger equation. See [4].

Let us now consider the case where $F(x, u) = \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)} u$. As we prove below we can also reconstruct the $V_j, j = 1, 2, \cdots$.

**Lemma 1.4.** Suppose that the conditions of Theorem 1.1 are satisfied, and moreover, that $F(x, u) = \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)} u$, for $|u| \leq \gamma$, for some $\gamma > 0$, where $j_0$ is an integer such that $j_0 \geq (p-3)/2$, and where $V_j \in W_{k,\infty}$ with $\|V_j\|_{W_{k,\infty}} \leq C^j, j = 1, 2, \cdots$, for some constant $C$. Then, for any $\phi \in W_{k+1,2} \cap W_{k,1+p}$ there is an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$:

(1.16)  
\[ i \langle (S_{\phi} - I)(\epsilon \phi), \phi \rangle_{L^2} = \sum_{j=1}^{\infty} e^{2(j_0+j)+1} \left[ \int \int dt dx V_j(x) |e^{-itH_0 \phi}|^{2(j_0+j+1)} + Q_j \right], \]

where $Q_1 = 0$ and $Q_j, j > 1$, depends only on $\phi$ and on $V_{l}$ with $l < j$. Moreover, for any $\hat{x} \in \mathbb{R}$, and any $\lambda \geq 1$, we denote $\phi_\lambda(x) := \phi(\lambda(x - \hat{x}))$. Then, if $\phi \neq 0$,

(1.17)  
\[ V_j(\hat{x}) = \lim_{\lambda \to \infty} \lambda^{n+2} \frac{\int \int dt dx V_j(x) |e^{-itH_0 \phi}|^{2(j_0+j+1)}}{\int \int dt dx |e^{-itH_0 \phi}|^{2(j_0+j+1)}}. \]

**Corollary 1.5.** Under the conditions of Lemma 1.4 the scattering operator, $S$, determines uniquely the potentials $V_j, j = 0, 1, \cdots$.

**Proof.** By Corollary 1.3, $S$ uniquely determines $V_0$. Then the wave operators, $W_{\pm}$, are uniquely determined, and by (1.13), $S$ uniquely determines $S_{V_0}$. Finally by (1.16) and (1.17) $S_{V_0}$ uniquely determines $V_j, j = 1, 2, \cdots$.

We reconstruct the potentials $V_j, j = 0, 1, \cdots$, in the following way. First we obtain $S_L$ from $S$ using (1.13). By the method in [4] for inverse scattering for the linear Schrödinger equation we reconstruct $V_0$. We then reconstruct $S_{V_0}$ from $S$ using (1.13). Finally (1.16) and (1.17) give us, recursively, $V_j, j = 1, 2, \cdots$. Our formula (1.17) is an extension to our case of the reconstruction algorithm of [1].

In [15] Strauss proved that in the case $V_0 = 0$ and $F(x, u) = V(x)|u|^{p-1} u, x \in \mathbb{R}^n, p > 4$ if $n = 1, p > 3$ if $n = 2, p \geq 3$ if $n \geq 3$, and $V(x)$ a real-valued potential whose derivatives up to order $l$ are bounded, with $l > 3n/4$; then, the scattering operator uniquely determines $V$.

2. Scattering

By Theorem 3 on page 135 of [14],

(2.1)  
\[ \| \mathcal{F}^{-1}(1 + q^2)^{k/2}(\mathcal{F}f)(q) \|_{L^p} \]

is a norm that is equivalent to the norm of $W_{k,p}, 1 < p < \infty$. $\mathcal{F}$ denotes the Fourier transform. Then, by equation (1.2) of [24],

(2.2)  
\[ \| (I + H)^{l/2} f \|_{L^p} \]
defines a norm that is equivalent to the norm of $W_{l,p}, l = 0, 1, \cdots, k, 1 < p < \infty$. We will use this equivalence below without further comments. This implies that estimate (1.6) holds in the norm on $\mathcal{B}(W_{l,p},W_{l,0}), l = 0, 1, \cdots, k$.

The following inequality is proven in Theorem 9.2 on page 141 of [13]:

$$
\| (D^a V_0) \|_{L^2} \leq C_1 N_\delta (D^a V_0) \| \phi \|_{W_{2,2}} + C_2 N_1 (D^a V_0) \| \phi \|_{L^2},
$$

where $C_1$ is independent of $\delta$. Let us denote $R(\rho) := (H + \rho)^{-1}$ and $R_0(\rho) := (H_0 + \rho)^{-1}$. Equation (2.5) implies that if Assumption C holds, given $a < 1$, there is $\rho_0 > 0$ such that

$$
\| V_0 R_0(\rho) \|_{B(L^2)} \leq a < 1
$$

for all $\rho \geq \rho_0 > 0$. Moreover, $\rho_0$ depends on $V_0$ only through $N_\delta(V_0)$. It follows that

$$
R(\rho) = R_0(\rho) (I + V_0 R_0(\rho))^{-1} = R_0(\rho) \sum_{t=0}^{\infty} (-1)^t (V_0 R_0(\rho))^t
$$

for all $\rho \geq \rho_0$. Taking derivatives in (2.5) term by term we prove that

$$
\| R(\rho) \|_{B(W_{j,2},W_{j+2,2})} \leq C, j = 0, 1, 2, \cdots, k - 1.
$$

It follows that if $k$ is odd,

$$
\| (H_0 + \rho)^{(k+1)/2} (R(\rho))^{(k+1)/2} \|_{B(L^2)} \leq C.
$$

Then, for some constants $C_1, C_2$,

$$
C_1 \| \phi \|_{W_{k+1,2}} \leq \left\| (I + H)^{(k+1)/2} \phi \right\|_{L^2} \leq C_2 \| \phi \|_{W_{k+1,2}}.
$$

In the case when $k$ is even we have that

$$
R(\rho)^{(k+1)/2} = R(\rho)^{k/2} (H + \rho)^{-1/2}.
$$

Again using Theorem 9.2 on page 141 of [13] we obtain that

$$
\| V_0^{1/2} \phi \| \leq C_1 [N_\delta(V_0)]^{1/2} \| \phi \|_{W_{1,2}} + C_2 [N_1(V_0)]^{1/2} \| \phi \|_{L^2}.
$$

Then, if $\rho$ is large enough,

$$
\| (H + \rho)^{1/2} \phi \|^2 = ((H_0 + V_0 + \rho) \phi, \phi) \geq C \| \phi \|^2_{W_{1,2}},
$$

and we have that

$$
\| (H + \rho)^{-1/2} \|_{B(L^2,W_{1,2})} \leq C.
$$

Hence, by (2.3), (2.9) and (2.12), equation (2.8) also holds for $k$ even.

The proofs of Theorem 1.1, Theorem 1.2, and Lemma 1.4 follow as in [23]. We give details below for the convenience of the reader.

Proof of Theorem 1.1. By Sobolev’s imbedding theorem [11] $L^\infty$ is continuously imbedded in $W_{k,1+p}$ and it follows by standard arguments (see [9] and (2.10) below) that $u \in C(\mathbf{R}, W_{k,2}) \cap M$ is a solution to (1.1) with $\lim_{t \to -\infty} \| u(t) - e^{-itH} \phi \|_{W_{1,2}} = 0$ for some $\phi \in W_{k,2}$, if and only if $u$ is a solution to the following integral equation:

$$
u(t) = e^{-itH} \phi + \frac{1}{t} \int_{-\infty}^{t} e^{-i(t-\tau)H} F(x,u(\tau)) \, d\tau.$$
Let us designate
\begin{equation}
Q(u) := \frac{1}{t} \int_{-\infty}^{t} e^{-i(t-\tau)H} F(x, u(\tau)) \, d\tau.
\end{equation}

For \( R > 0 \) let us denote \( M_R := \{ u \in M : \|u\|_M \leq R \} \). By Assumption A, (1.14) and since \( L_\infty \subset W_{k,p+1} \), there is an \( R_0 > 0 \) such that if \( u \in M_{R_0} \),
\begin{equation}
\|Q(u) - Q(v)\|_{W_{k,p+1}} \leq C (1 + |t|)^{-d} (\|u\|_M + \|v\|_M)^{p-1} \|u - v\|_M,
\end{equation}
where we used that \( d > 1 \) and that \( p d > 1 \). Moreover, by (2.15) with \( v(t) = 0 \),
\begin{equation}
\|Q(u(t))\|_{W_{k,2}}^2 \leq CR \int_{-\infty}^{t} d\tau \left( (I + H)^{k/2} F(x, u(\tau)), (I + H)^{k/2} Q(u) \right)_{L^2} \leq C \int_{-\infty}^{t} d\tau \|F(x, u(\tau))\|_{W_{k,1+1/p}} \times (1 + |\tau|)^{-d} \|u\|_M^p \leq C \int_{-\infty}^{t} d\tau (1 + |\tau|)^{-d(p+1)} \|u\|_M^{2p} \leq C(1 + \max\{0, -t\})^{-(d+p-1)} \|u\|_M^{2p}.
\end{equation}

Let us first prove the uniqueness. For \( u, v \) any pair of solutions to (1.1) that satisfy (1.10) we have that
\begin{equation}
u(t) - v(t) = Q(u(t)) - Q(v(t)).\end{equation}
Let us denote \( u_T := \chi(-\infty, T)(t) u(t) \), where \( \chi(-\infty, T)(t) \) is the characteristic function of \((-\infty, T), T \in \mathbb{R} \). \( v_T \) is similarly defined. It follows from (2.17) that
\begin{equation}
\|u_T(t) - v_T(t)\|_{\tilde{M}}^2 < 1/2 \|u_T(t) - v_T(t)\|_{\tilde{M}}^2 \text{ for some } T \text{ negative enough},
\end{equation}
where \( \tilde{M} \) is defined as \( M \), but with a slightly smaller \( p \). Then, \( u(t) = v(t) \) for \( t \leq T \), and the standard uniqueness result implies that \( u = v \). The uniqueness of \( \phi_+ \) is obvious by the unitarity of \( e^{-itH} \) in \( L^2 \).

By Sobolev’s imbedding theorem,
\begin{equation}
\|e^{-itH} \phi_-\|_{W_{k,p+1}} \leq C \|e^{-itH} \phi_-\|_{W_{k,1+2}} \leq C \| (H + I)^{(k+1)/2} e^{-itH} \phi_-\|_{L^2} \leq C \| (H + I)^{(k+1)/2} \phi_-\|_{L^2} \leq C \| \phi_-\|_{W_{k,1+2}}.
\end{equation}

By (1.6) and (2.19):
\begin{equation}
\|e^{-itH} \phi_-\|_M \leq C \left[ \| \phi_-\|_{W_{k+1,2}} + \| \phi_-\|_{W_{k,1+1/2}} \right].
\end{equation}

Let us take \( R \leq R_0 \) so small that \( C(2R)^{p-1} \leq 1/2 \), with \( C \) as in (2.15), and \( \delta > 0 \) such that \( C \delta \leq R/4 \), with \( C \) as in (2.20). Then, the map \( u \mapsto e^{-itH} \phi_- + Q u \) is a contraction from \( M_R \) into \( M_R \) for all \( \phi_- \in W_{k+1,2} \cap W_{k,1+1/2} \) with \( \| \phi_-\|_{W_{k+1,2}} + \)
The contraction mapping theorem implies that this map has a unique fixed point that is a solution to (2.13) in $M_R$. Moreover,

$$
\|u\|_M \leq \|e^{-itH}\phi_-\|_M + \frac{1}{2}\|u\|_M,
$$

and then

$$
\|u\|_M \leq C\|e^{-itH}\phi_-\|_M.
$$

Equation (1.12) for $\phi_-$ follows from (2.13), (2.15) with $v = 0$ and (2.22). By (2.13) and (2.16) $u \in C(R, W_{k,2})$ and (1.10) holds.

We now define

$$
\phi_+ = \phi_- + \frac{1}{i} \int_{-\infty}^{\infty} e^{itH}F(x,u(t))\,dt.
$$

Estimating as in (2.16) we prove that $\phi_+ \in W_{k,2}$ and that

$$
\|\phi_+ - \phi_-\|_{W_{k,2}} \leq C\|u\|_M^p.
$$

Equation (1.13) follows from (2.21), (2.22) and (2.24). By (2.23) and (2.24)

$$
u(t) = e^{-itH}\phi_+ - \frac{1}{i} \int_{t}^{\infty} e^{-i(t-\tau)H}F(x,u(\tau))\,d\tau.
$$

We prove (1.11) estimating as in (2.16). In a similar way we prove that

$$
\left\|\int_{t}^{\infty} e^{-i(t-\tau)H}F(x,u(\tau))\,d\tau\right\|_M \leq C\|u\|^\frac{p}{2}_M,
$$

and it follows that

$$
\|u\|_M \leq C\|e^{-itH}\phi_+\|_M.
$$

Equation (1.12) for $\phi_+$ follows from (2.25), (2.26) and (2.27). We prove that $S_{V_0}$ is injective as in the proof of uniqueness above.

Proof of Theorem 1.2. Since $S(0) = 0$ and $W_{\pm}$ are bounded on $W_{k,2}$, it is enough to prove that

$$s - \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} (S_V(\epsilon \phi) - \epsilon \phi) = 0.
$$

By (2.20) and (2.22) with $\phi_-$ replaced by $\epsilon \phi$ we have

$$
\|u\|_M \leq C\epsilon \left[\|\phi_-\|_{W_{k+1,2}} + \|\phi_-\|_{W_{k,1+\frac{1}{p}}}\right].
$$

To prove (2.28) we estimate the integral on the right-hand side of (2.23) as in (2.16), with the aid of (2.29).

Proof of Lemma 1.4. By the contraction mapping theorem,

$$
u(t) = e^{-itH}\epsilon \phi + \sum_{n=1}^{\infty} Q^n e^{-itH}\epsilon \phi.
$$

Equation (1.16) follows from (2.23) and (2.30). By Sobolev’s imbedding theorem [1], $W_{k+1,2} \subset L^q$, $2 \leq q \leq \infty$. Then, estimating as in (2.19) we prove that
\[ \|e^{-itH}\phi\|_{L^q} \leq C_q \|e^{-itH}\phi\|_{W_{k+1,2}} \leq C_q \|\phi\|_{W_{k+1,2}}, \quad 2 \leq q \leq \infty, \] and it follows from (1.6) that

\[ \int \int dt \, dx \left| e^{-itH} \phi \right|^2 (j_0 + j + 1) < \infty, \quad j = 1, 2, \ldots. \]  

(2.31)

For \( \lambda \geq 1 \) and \( \hat{x} \in \mathbb{R}^n \) we denote by \( H_\lambda \) the following self-adjoint operator in \( L^2 \):

\[ H_\lambda := H_0 + V_\lambda(x), \quad \text{where} \quad V_\lambda(x) = \frac{1}{\lambda^2} V_0(\frac{x}{\lambda} + \hat{x}). \]

(2.32)

Since \( H \) has no eigenvalues, we have that \( H_\lambda \) has no eigenvalues, i.e., \( H_\lambda > 0 \). Moreover, as \( N_\delta(D^\alpha V_\lambda) \leq N_\delta(D^\alpha V_0) \) for \( \lambda \geq 1 \), equation (2.8) holds with \( H_\lambda \) instead of \( H \) with the same \( C_1, C_2 \) for all \( \lambda \geq 1 \).

Let us denote \( \tilde{t} := \lambda^2 t \) and \( \tilde{x} := \lambda (x - \hat{x}) \). We have that

\[ (e^{-itH_\lambda} \phi)(\tilde{x}) = (e^{-itH} \phi)(x). \]

(2.33)

Equation (2.33) implies that

\[ I_j := \lambda^{n+2} \int \int dt \, dx \, V_j(x, \tilde{x}) \left| e^{-itH} \phi \right|^2 (j_0 + j + 1) \]

(2.34)

By Theorem VIII.20 on page 286 of [12] and (2.32)

\[ s - \lim_{\lambda \to \infty} e^{-itH_\lambda} \phi = e^{-itH_0} \phi, \]

(2.35)

where the limit exists in the strong topology on \( W_{k+1,2} \). By Sobolev’s imbedding theorem, the limit in (2.35) also exists in the strong topology on \( L^q \), \( 2 \leq q \leq \infty \). Moreover,

\[ \left\| e^{-itH_\lambda} \phi \right\|_{L^\infty} \leq C \left\| \phi \right\|_{W_{k+1,2}}. \]

(2.36)

By (1.6) and (2.33),

\[ \left\| e^{-itH_\lambda} \phi \right\|_{L^{p+1}} = \lambda^n \left\| e^{-itH} \phi \right\|_{L^{p+1}} \leq C \frac{1}{\lambda^d} \left\| \phi \right\|_{L^{p+1}} \]

(2.37)

with \( d := \frac{n+1}{p+1} \). Equation (1.17) follows from (2.34), (2.35), (2.36), (2.37) and the dominated convergence theorem. Note that \( 2(j_0 + j + 1) \geq p + 1 \), that \( d(p+1) > 1 \), and that \( V_j \) is continuous.

References


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