COMMUTATOR CONDITIONS IMPLYING THE CONVERGENCE OF THE LIE–TROTTER PRODUCTS

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Abstract. In this paper we investigate commutator conditions for two strongly continuous semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) on a Banach space implying the convergence of the Lie–Trotter products \([T(\frac{t}{n})S(\frac{t}{n})]^n\). The results are then applied to various examples and, in particular, to the Ornstein–Uhlenbeck operator.

1. Introduction

In 1959 H.F. Trotter \[13\] obtained an explicit product formula for semigroups whose generator is the sum of two generators. Using this idea, more general product formulas were considered by P.R. Chernoff \[1\] Ch. 1, 2]. Following this approach one can prove the theorem below (cf. \[6\] Ch. III, Cor. 5.8).

**Theorem 1.** Let \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) be strongly continuous semigroups on a Banach space \(E\) with generators \((A, D(A))\) and \((B, D(B))\), respectively, satisfying the stability condition

\[
\| [T(\frac{t}{n})S(\frac{t}{n})]^n \| \leq Me^{\omega t}
\]

for all \(t \geq 0, n \in \mathbb{N}\), and some constants \(M \geq 1, \omega \in \mathbb{R}\). Consider the sum \(A + B\) on a subspace \(D \subseteq D(A) \cap D(B)\) and assume that \(D\) and \((\lambda_0 - A - B)D\) are dense in \(E\) for some \(\lambda_0 > \omega\).

Then the closure of \(A + B\) exists and generates a strongly continuous semigroup \((U(t))_{t \geq 0}\) given by the Lie–Trotter product formula

\[
U(t)f = \lim_{n \to \infty} [T(\frac{t}{n})S(\frac{t}{n})]^n f
\]

where the limit exists for all \(f \in E\) uniformly for \(t\) in compact intervals in \(\mathbb{R}_+\).

In \[8\] we gave an example of operators \(A\) and \(B\) such that the closure of the sum of \(A\) and \(B\) is a generator, but \[1\], and hence \[2\], is violated. Thus, to obtain the Lie–Trotter formula \(2\) a stability condition as in \(1\) is necessary.

On the other hand, in Theorem \[11\] one requires the range condition \((\lambda_0 - A - B)D = E\), which is sometimes hard to verify. The aim of this work is to replace the

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range condition by assumptions on the commutator of \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\), of \(A\) and \(B\), and their resolvents, respectively. At first glance, the commutator condition and introducing a new norm \(||| \cdot |||\) seems quite sophisticated. However, in the applications of our results in Section 4 and 5 we will see that we can use very simple norms. In particular, for the Ornstein–Uhlenbeck operator a natural Sobolev norm will work.

With our results we obtain a new approach for the Ornstein-Uhlenbeck semigroup in finite dimension as the limit of the Lie–Trotter products of a degenerate diffusion semigroup and a semigroup induced by a flow.

2. Commutator conditions for semigroups and generators

To use commutator conditions to establish the convergence of the Lie–Trotter products, we modify the stability estimate \((\mathbf{I})\).

**Definition 2.** Let \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) be semigroups on a normed vector space \((F, ||| \cdot |||)\). The semigroup \((T(t))_{t \geq 0}\) is called **exponentially bounded** if

\[|||T(t)||| \leq Me^{\omega t}\]

for all \(t \geq 0\) and some constants \(M \geq 1\), \(\omega \in \mathbb{R}\).

The semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) are called **locally Trotter-stable** if there exist constants \(t_0 > 0\) and \(M_{t_0} > 1\) such that

\[|||T(t_n)S(t_n)||| \leq M_{t_0}\]

for all \(t \in [0, t_0]\) and \(n \in \mathbb{N}\).

With this concept, we are able to state our main result.

**Theorem 3.** Let \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) be strongly continuous semigroups on a Banach space \(E\) and suppose that there exists a normed vector space \((F, ||| \cdot |||)\) which is densely embedded in \(E\) and invariant under both semigroups such that the following conditions hold:

(a) The two semigroups are exponentially bounded on \(F\) and locally Trotter-stable on \(E\) and \(F\).

(b) The commutator condition

\[|||T(t)S(t)f - S(t)T(t)f||| \leq t^\alpha M_1 |||f|||\]

is satisfied for all \(f \in F\), and \(t \in [0, \delta]\) with some \(\delta > 0\), and for some constants \(\alpha > 1\) and \(M_1 \geq 0\).

Then the Lie–Trotter product formula holds, i.e.,

\[U(t)f := \lim_{n \to \infty} \left[ T(t_n)S(t_n) \right]^n f\]

exists for all \(f \in E\) uniformly for \(t\) in compact intervals in \(\mathbb{R}^+\). Moreover, \((U(t))_{t \geq 0}\) is a strongly continuous semigroup with generator \((G, D(G))\) such that

\[G \geq \frac{\delta}{4 \alpha^2} [T(t)S(t)]_{t=0} \geq A + B ,\]

where each operator is defined on its natural domain.

**Proof.** We denote the Lie–Trotter products by

\[P_n(t) := \left[ T(t_n)S(t_n) \right]^n\]
for $n \in \mathbb{N}$ and $t \geq 0$. Let $t_0 > 0$ such that the stability condition (3) is fulfilled on $(E, || \cdot ||)$ and $(F, || \cdot ||)$. Therefore, we obtain

$$\|P_n(t)\| = \|P_k(\frac{kt}{n})P_l(\frac{lt}{n})\| \leq M_t^2$$

for $t \in [0, \frac{3t_0}{2}]$ and $n = k + l$, $k, l \in \mathbb{N}$, such that $\frac{kt}{n}, \frac{lt}{n} \leq t_0$, $n \geq 2$. The analogous estimate also holds for $\| \cdot \|$. By induction we conclude that

(4) \quad $$\|P_n(t)\| \leq M_T \quad \text{and} \quad \|P_n(t)\| \leq M_T$$

for $(n, t) \in \mathbb{N} \times [0, T]$, $T > 0$, and a constant $M_T \geq 0$.

Now fix $T > 0$ and let $t \in [0, T]$ and $f \in F$. We choose $k, m, n \in \mathbb{N}$ with $0 < k \leq m$ such that $\frac{t}{m} \leq \delta$ and $m = kn$. Then, we obtain from the commutator condition (b) that

$$\|T(\frac{t}{m})S(\frac{k}{m})f - S(\frac{k}{m})T(\frac{t}{m})f\|$$

$$\leq \sum_{l=0}^{j-1} \|S(\frac{lt}{m}) (T(\frac{t}{m})S(\frac{k}{m}) - S(\frac{k}{m})T(\frac{t}{m})) S(\frac{k-1-lt}{m})f\|$$

$$\leq j \left( \frac{t}{m} \right)^\alpha M_2 ||f||$$

for $j \in \mathbb{N}$ and some constant $M_2 \geq 0$. Hence, by forming a telescope sum, we have

$$\| \left[ T(\frac{t}{m})S(\frac{k}{m}) \right] f - \left[ T(\frac{t}{m})S(\frac{k}{m}) \right] f \|$$

$$= \sum_{j=1}^{k-1} \left[ T(\frac{t}{m}) (T(\frac{t}{m})S(\frac{k}{m}) - S(\frac{k}{m})T(\frac{t}{m})) S(\frac{k}{m}) \right]^j \|f\|$$

$$\leq M_3 \sum_{j=1}^{k-1} j \left( \frac{t}{m} \right)^\alpha ||f||$$

$$= M_3 \frac{t^\alpha (k - 1) k}{2 m^\alpha} ||f||$$

for a suitable constant $M_3 \geq 0$. Note that

$$\| \left[ T(\frac{t}{m})S(\frac{k}{m}) \right]^j \| = \| \left[ T(\frac{t^j}{m^j}) \right] S(\frac{t^j}{m^j}) \|^j \| \leq M_T$$

for $j \in \mathbb{N}$ and $1 \leq j \leq m$. We now conclude that

(5) \quad $$\|P_n(t)f - P_m(t)f\| = \| \left[ T(\frac{t}{m})S(\frac{k}{m}) \right]^n f - \left[ T(\frac{t}{m})S(\frac{k}{m}) \right] f \|$$

$$\leq \sum_{l=0}^{n-1} \left[ T(\frac{t}{m})S(\frac{k}{m}) \right]^l \left[ T(\frac{t}{m})S(\frac{k}{m}) - \left[ T(\frac{t}{m})S(\frac{k}{m}) \right]^k \right] \left[ T(\frac{t}{m})S(\frac{k}{m}) \right]^{k(n-1-l)} f\|$$

$$\leq M_4 \frac{t^\alpha n(k - 1) k}{2 m^\alpha} ||f||$$

$$\leq M_4 \frac{t^\alpha k}{2 m^\alpha} ||f||$$

$$\leq M_5 \frac{k}{2 m^\alpha - 1} ||f||$$

for suitable constants $M_4, M_5 \geq 0$. 

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Thus, for arbitrary \(i, j \in \mathbb{N}\) and \(0 \leq t \leq T\), we have

\[
\|P_{2^i}(t)f - P_{2^j}(t)f\| \leq \sum_{l=i}^{j-1} \|P_{2^l}(t)f - P_{2^{l+1}}(t)f\| \\
\leq M_5 \sum_{l=i}^{j-1} \left( \frac{1}{2^{\alpha-1}} \right)^{l+1} \|f\|.
\]

The last expression converges to 0 as \(i, j \to \infty\), since \(\alpha > 1\). Hence,

\[
U(t)f := \lim_{i \to \infty} P_{2^i}(t)f
\]

exists for \(f \in F\) uniformly for \(t \in [0,T]\). Due to (4) we can extend \(U(t)\) to a bounded linear operator on \(E\).

For arbitrary \(n \in \mathbb{N}\), we obtain from (5) that

\[
\|P_n(t)f - U(t)f\| \\
\leq \|P_n(t)f - P_{n2^k}(t)f\| + \|P_{n2^k}(t)f - P_{2^k}(t)f\| + \|P_{2^k}(t)f - U(t)f\| \\
\leq \sum_{k=0}^{l-1} \|P_{n2^k}(t)f - P_{n2^{k+1}}(t)f\| + \|P_{n2^k}(t)f - P_{2^k}(t)f\| + \|P_{2^k}(t)f - U(t)f\| \\
\leq M_5 \sum_{k=0}^{l-1} \left( \frac{1}{2^{\alpha-1}} \right)^{k+1} \|f\| + M_5 \frac{n}{(n2^l)^{\alpha-1}} \|\|f\| + \|P_{2^k}(t)f - U(t)f\| \\
\leq \left( \frac{2^{\alpha-1}M_5}{(2^{\alpha-1} - 1)n^{\alpha-1}} + \frac{M_5n^{2-\alpha}}{2^{l(\alpha-1)}} \right) \|f\| + \|P_{2^k}(t)f - U(t)f\|
\]

for \(l \in \mathbb{N}\).

Fix \(\epsilon > 0\). By the definition of the operators \(U(t)\), there exists \(l_0 \in \mathbb{N}\) such that

\[
\|P_{2^l}(t)f - U(t)f\| \leq \epsilon
\]

for \(f \in F\) and \(l \geq l_0\). Further, for \(n \in \mathbb{N}\) with \((\frac{1}{n})^{\alpha-1} \leq \frac{2^{\alpha-1}-1}{2^{\alpha-1}M_n}\) and \(l \geq \max\{l_0, l_1\}\) such that \(2^{l_1(\alpha-1)} \geq \left( \frac{Mn^{2-\alpha}}{\epsilon} \right)^{\alpha-1}\), we have

\[
\|P_n(t)f - U(t)f\| \leq \left( \frac{2^{\alpha-1}M_5}{(2^{\alpha-1} - 1)n^{\alpha-1}} + \frac{M_5n^{2-\alpha}}{2^{l(\alpha-1)}} \right) \|f\| + \|P_{2^l}(t)f - U(t)f\| \\
\leq 3\epsilon \|f\| + \|f\|)
\]

for \(f \in F\) uniformly for \(t \in [0,T]\). So the stability condition (4) implies that \(P_n(t)\) converges strongly in \(E\) uniformly for \(t \in [0,T]\).

To show the semigroup law for \((U(t))_{t \geq 0}\) and for rational numbers \(t\), we use an idea from the paper of Chernoff (cf. [3, Thm. 2.5.1]). To that purpose, let \(f \in E\)
and $\epsilon > 0$. Then

\[
\|U(t) f - U(\frac{t}{n}) U(\frac{t}{n}) f\| \\
\leq \|U(t) f - [T(\frac{t}{2n}) S(\frac{t}{2n})]^2 f\| + \| [T(\frac{t}{2n}) S(\frac{t}{2n})]^2 f - U(\frac{t}{n}) U(\frac{t}{n}) f\| \\
\leq \epsilon + \| [T(\frac{t}{2n}) S(\frac{t}{2n})]^n ( [T(\frac{t}{2n}) S(\frac{t}{2n})]^n f - U(\frac{t}{n}) f)\| \\
+ \| ( [T(\frac{t}{2n}) S(\frac{t}{2n})]^n - U(\frac{t}{n}) U(\frac{t}{n}) f)\| \\
\leq 3\epsilon
\]

for $n$ sufficiently large. This proves that $U(t) = U(\frac{t}{n}) U(\frac{t}{n})$. In an analogous way, one can show $U(nt) = U(t)^n$ for $n \in \mathbb{N}$, from which one obtains the semigroup law for rational numbers.

By the uniform convergence of the Lie–Trotter products on compact intervals, we obtain that $(U(t))_{t \geq 0}$ is a strongly continuous semigroup. The assertion concerning the generator $G$ follows from \[3 Prop. 4.1].

In the following, we show how commutator conditions on the generators $A$ and $B$ imply condition (b) of Theorem 3. These conditions can be verified in some interesting applications (see Section 4 and 5).

**Definition 4.** Let $(A, D(A))$ and $(B, D(B))$ be linear operators on a Banach space $E$. The operator

\[ C := AB - BA \quad \text{with domain} \]

\[ D(C) := D(AB) \cap D(BA) = \{ f \in D(A) \cap D(B) : Bf \in D(A) \text{ and } Af \in D(B) \} \]

is called the **commutator** of $A$ and $B$.

In the following we denote by $\| \cdot \|_C$ the **$C$–norm** defined by

\[ \|f\|_C := \|f\| + \|Cf\| \]

for all $f \in D(C)$.

The next lemma allows to relate the commutator of $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ to the commutator $C$.

**Lemma 5.** Let $(A, D(A))$ and $(B, D(B))$ be generators of strongly continuous semigroups $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ on a Banach space $E$, and let $(C, D(C))$ be the commutator of $A$ and $B$. Suppose that there exists a subspace $F \subseteq D(C)$ which is dense in $E$ and invariant under $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ such that for some $\delta > 0$ and $t \in [0, \delta]$ the functions

\[
\begin{cases}
    s \mapsto AS(t)T(s)f, \\ s \mapsto CS(s)f
\end{cases}
\]

are continuous and $s \mapsto AS(s)f$ is differentiable in $E$ for $s \in [0, \delta]$ and for all $f \in F$. Then the identity

\[ T(t)S(t)f - S(t)T(t)f = \int_0^t T(s) \left( \int_0^t S(t-r)CS(r)T(t-s)fds + \right) ds \\
\]

holds for all $f \in F$ and $t \in [0, \delta]$.

**Proof.** Let $t \in [0, \delta]$ with $\delta > 0$. We apply the fundamental theorem of calculus to the continuously differentiable functions

\[ s \mapsto T(s)S(t)T(t-s)f \quad \text{and} \quad r \mapsto S(t-r)AS(r)f \]
for $0 \leq r, s \leq \delta$ and $f \in F$. Since the operator $A$ is closed, we obtain by the differentiability of the second function that

$$\frac{d}{dr} S(t-r)AS(r)f = S(t-r)CS(r)f$$

for $0 \leq r \leq \delta$ and $f \in F$. Thus, we have

$$T(t)S(t)f - S(t)T(t)f = \int_0^t \frac{d}{ds} \{T(s)S(t)T(t-s)f\} ds$$

$$= \int_0^t T(s) (AS(t) - S(t)A) T(t-s)f ds$$

$$= \int_0^t T(s) \left( \int_0^s \frac{d}{dr} \{S(t-r)AS(r)T(t-s)f\} dr \right) ds$$

$$= \int_0^t T(s) \left( \int_0^t S(t-r)CS(r)T(t-s)f dr \right) ds$$

for $f \in F$ and $t \in [0, \delta]$. 

As an easy consequence of Lemma 5 we obtain the following result.

**Corollary 6.** Let $(A, D(A))$ and $(B, D(B))$ be generators of strongly continuous semigroups $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ on a Banach space $E$ and let $(C, D(C))$ be the commutator of $A$ and $B$. Suppose that:

(a) there exists a normed vector space $(F, ||\cdot||)$, $F \subseteq D(C)$, which is dense in $E$ and invariant under both semigroups, and the norm $||\cdot||$ is finer than the $C$-norm $||\cdot||_C$;

(b) there exists $\delta > 0$ such that the functions

\[
\begin{cases}
  s \mapsto AS(t)S(s)f, & s \mapsto CS(s)f \\
  s \mapsto AS(s)f
\end{cases}
\]

are continuous and $\frac{d}{ds} AS(s)f$ is differentiable in $E$ for $s, t \in [0, \delta]$ and $f \in F$, and

(c) the two semigroups are exponentially bounded on $F$ and locally Trotter-stable on $E$ and $F$.

Then the conclusion of Theorem 3 holds.

**Proof.** By the exponential boundedness of the semigroups and Lemma 5 we obtain the estimate

$$\|T(t)S(t)f - S(t)T(t)f\| = \| \int_0^t T(s) \left( \int_0^s S(t-r)CS(r)T(t-s)f dr \right) ds \|$$

$$\leq Mt^2 \sup_{0 \leq r, s \leq t} \|S(r)T(t-s)f\|_C$$

$$\leq \tilde{M}t^2 \|f\|$$

for $f \in F$, $t \in [0, \delta]$ and some constants $M, \tilde{M} \geq 0$. Thus, assumption (b) of Theorem 3 is fulfilled. Since condition (a) of Theorem 3 was assumed explicitly, the assertion follows. 

\[\square\]
3. Commutator conditions for resolvent operators

In the following we express the commutator condition on the semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) in Theorem 3 by a corresponding condition on the powers of the resolvent operators \(R(\lambda, A)\) and \(R(\mu, B)\), respectively. In special cases, only a commutator condition on \(R(\lambda, A)\) and \(R(\mu, B)\) will be necessary.

First, we state the following result on the asymptotic behaviour of the gamma function.

**Lemma 7.** The gamma function defined by
\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad \text{for } \Re(z) > 0
\]
satisfies
\[
\lim_{n \to \infty} n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = 1 \quad \text{for } a, b \in \mathbb{R}.
\]

**Proof.** By Stirling’s Formula we can conclude that
\[
\Gamma(x + 1) = \sqrt{2\pi} x^{x+1/2} e^{-x} (1 + \phi(x))
\]
for \(x \in \mathbb{R}\), where \(\phi\) is a function converging to zero as \(\frac{1}{x}\) (see [1, p. 652]). Therefore, we have
\[
n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = \left(\frac{n+a-1}{n+b-1}\right)^{n+a-1/2} \left(\frac{n}{n+b-1}\right)^{b-a} e^{b-a} \frac{1 + \phi(n+a-1)}{1 + \phi(n+b-1)}
\]
for \(a, b \in \mathbb{R}\) which converges to 1 as \(n \to \infty\).

In the sequel, we assume without loss of generality that \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) are bounded semigroups. Note that we now use a commutator condition on the semigroups which is slightly different from the one in Theorem 3.

**Theorem 8.** Let \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) be bounded strongly continuous semigroups on a Banach space \(E\) with generators \((A, D(A))\) and \((B, D(B))\), respectively, and suppose that there exists a normed vector space \((F; \| \cdot \|)\) which is embedded in \(E\). Furthermore, let \(\alpha, \beta \geq 0\) and \(M \geq 0\). Then the following statements are equivalent:

(i) The semigroups satisfy the commutator condition
\[
\|T(t)S(s)f - S(s)T(t)f\| \leq \ell^\alpha s^\beta M\|f\|\]
for all \(f \in F\) and \(s, t \geq 0\).

(ii) The resolvent operators satisfy the commutator condition
\[
\|R(\lambda, A)^n R(\mu, B)^n f - R(\mu, B)^n R(\lambda, A)^n f\| \leq \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)!^2 \lambda^{\alpha+n} \mu^{\beta+n}} M\|f\|\]
for all \(f \in F\), \(\lambda, \mu > 0\) and \(n \in \mathbb{N}\).
Proof. (i) ⇒ (ii). Let $f \in F$ and $\lambda, \mu > 0$. We use the Laplace representation of the resolvent (see [6, Ch. II, Thm. 1.10]) and obtain

$$
\| R(\lambda, A)^n R(\mu, B)^n f - R(\mu, B)^n R(\lambda, A)^n f \|
\leq \frac{1}{(n-1)^2} \int_0^\infty \int_0^\infty r^{n-1} s^{n-1} e^{-\lambda r} e^{-\mu s} \| T(r)S(s)f - S(s)T(r)f \| ds dr
\leq \frac{M}{(n-1)^2} \int_0^\infty \int_0^\infty r^{n-1+\alpha} s^{n-1+\beta} e^{-\mu s} dr ds \| f \|
= \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)^2 \lambda^{1+n} \mu^{1+n}} \int_0^\infty r^{n-1+\alpha} e^{-r} dr \int_0^\infty s^{n-1+\beta} e^{-s} ds \| f \|
= \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)^2 \lambda^{1+n} \mu^{1+n}} M \| f \|
$$

for all $n \in \mathbb{N}$.

(ii) ⇒ (i). Recall that by the Post–Widder Inversion Formula

$$
T(t)f = \lim_{n \to \infty} \left[ \frac{\tau}{\pi} R(\frac{\tau}{\pi}, A) \right]^n t
$$

for $f \in E$ uniformly for $t$ in compact intervals in $\mathbb{R}_+$ ([6, Ch. III, Cor. 5.5]). Let $f \in F$ and $s, t \geq 0$. Applying Lemma 7 we conclude that

$$
\frac{n^{2n}}{t^n s^n} \| R(\frac{\tau}{\pi}, A)^n R(\frac{\tau}{\pi}, B)^n f - R(\frac{\tau}{\pi}, B)^n R(\frac{\tau}{\pi}, A)^n f \|
\leq \frac{n^{2n}}{t^n s^n} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)^2 \lambda^{1+n} \mu^{1+n}} M \| f \|
= n^{-\alpha+\beta} \lambda^{\alpha} \mu^{\beta} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n+1)^2} M \| f \|
\longrightarrow t^{\alpha} s^{\beta} M \| f \|,
$$

as $n \to \infty$ which implies (i).

We now discuss some cases where it is enough to impose a commutator condition on the resolvent operators and not on all their powers.

**Proposition 9.** Let $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ be bounded strongly continuous semigroups on a Banach space $E$ with generators $(A, D(A))$ and $(B, D(B))$, respectively. Let $\alpha, \beta \geq 0$ such that $\alpha + \beta \geq 2$. Then the following statements are equivalent:

(i) The semigroups satisfy the commutator condition

$$
\| T(t)S(s)f - S(s)T(t)f \| \leq t^\alpha s^\beta M_1 \| f \|
$$

for all $f \in E$, $s, t \geq 0$, and a constant $M_1 \geq 0$.

(ii) The resolvent operators satisfy the commutator condition

$$
\| R(\lambda, A)R(\mu, B)f - R(\mu, B)R(\lambda, A)f \| \leq \frac{M_2}{\lambda^{\alpha+1} \mu^{\beta+1}} \| f \|
$$

for all $f \in E$, $\lambda, \mu > 0$, $n \in \mathbb{N}$, and a constant $M_2 \geq 0$.

Moreover, if $\alpha + \beta > 2$, then the semigroups $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ commute.

**Proof.** The implication (i) ⇒ (ii) is proved analogously as in Theorem [6].

(ii) ⇒ (i). For simplicity, we define

$$
\mathcal{R}_A(\frac{\tau}{\pi}) := \frac{\tau}{\pi} R(\frac{\tau}{\pi}, A), \mathcal{R}_B(\frac{\tau}{\pi}) := \frac{\tau}{\pi} R(\frac{\tau}{\pi}, B)
$$
and the commutator

$$\left[ R_A(\frac{t}{n}), R_B(\frac{t}{n}) \right] := \frac{t}{n} R(\frac{t}{n}, A) \frac{t}{n} R(\frac{t}{n}, B) - \frac{t}{n} R(\frac{t}{n}, B) \frac{t}{n} R(\frac{t}{n}, A)$$

for $n \in \mathbb{N}$ and $t \geq 0$. Since the semigroups are bounded there exists a constant $M \geq 0$ such that

$$\| (R_A(\frac{t}{n}))^j \|, \| (R_B(\frac{t}{n}))^j \| \leq M$$

for all $j, n \in \mathbb{N}$ such that $1 \leq j \leq n$, and for all $t, s \geq 0$. By assumption, we have

$$\| [R_A(\frac{t}{n}), R_B(\frac{t}{n})] \| \leq M_2 \left( \frac{t}{n} \right)^{\alpha + \beta}$$

for all $n \in \mathbb{N}$, $t \geq 0$ and some constant $M_2 \geq 0$. Let $f \in E$ and $t, s \geq 0$. Then

$$\| T(t)S(s)f - S(s)T(t)f \| \leq \lim_{n \to \infty} \| (R_A(\frac{t}{n}))^n (R_B(\frac{t}{n}))^n f - (R_B(\frac{t}{n}))^n (R_A(\frac{t}{n}))^n f \|$$

for all $n \in \mathbb{N}$. Moreover, we can estimate

$$\| [R_A(\frac{t}{n}), R_B(\frac{t}{n})] f \| \leq \sum_{j=0}^{n-1} \| (R_A(\frac{t}{n}))^{n-1-j} [R_A(\frac{t}{n}), R_B(\frac{t}{n})] (R_A(\frac{t}{n}))^j f \|$$

$$\leq nM^2M_2 \left( \frac{t}{n} \right)^{\alpha} \left( \frac{t}{n} \right)^{\beta} \| f \|,$$

$$\| [R_A(\frac{t}{n}), R_B(\frac{t}{n})]^{n-1} f \| \leq (n - 1)M^2M_2 \left( \frac{t}{n} \right)^{\alpha} \left( \frac{t}{n} \right)^{\beta} \| f \|,$$

and by induction

$$\| (R_A(\frac{t}{n}))^j , (R_B(\frac{t}{n}))^j \| \leq jM^2M_2 \left( \frac{t}{n} \right)^{\alpha} \left( \frac{t}{n} \right)^{\beta} \| f \|$$

for each $j, n \in \mathbb{N}$ such that $1 \leq j \leq n$. Therefore, equation (11) yields

$$\| T(t)S(s)f - S(s)T(t)f \| \leq \lim_{n \to \infty} \| (R_A(\frac{t}{n}))^n (R_B(\frac{t}{n}))^n f - (R_B(\frac{t}{n}))^n (R_A(\frac{t}{n}))^n f \|$$

$$\leq \lim_{n \to \infty} \frac{n^2 t^{\alpha + \beta}}{n^{\alpha + \beta}} M^2M_2 \| f \|$$

$$\leq t^\alpha s^\beta M_3 \| f \|$$

for some constant $M_3 \geq 0$.

Using the estimate (11), it follows that the semigroups commute if $\alpha + \beta > 2$. \(\square\)

A similar commutator condition for the resolvent operators in the context of nonautonomous evolution equations was studied in [9].
4. Applications to semigroups from quantum mechanics

First, we apply our results to groups arising in quantum mechanics. Let \((T(t))_{t \in \mathbb{R}}\) and \((S(t))_{t \in \mathbb{R}}\) be a pair of unitary groups on a complex Hilbert space \(H\) satisfying the so-called Weyl relation (see \([10\text{ p. } 274]\))
\[
T(t)S(s) = e^{i\lambda t}S(s)T(t) \quad \text{for all } s, t \in \mathbb{R}.
\]
(12)

These semigroups are evidently exponentially bounded and locally Trotter–stable on \(H\). Moreover, condition (b) of Theorem \([3]\) follows from the estimate
\[
\|T(t)S(t)f - S(t)T(t)f\| = \| \left( e^{it^2} - 1 \right) S(t)T(t)f \| \leq t^2 \sum_{k=1}^{\infty} \frac{(t^2)^{k-1}}{k!} \|f\| \leq t^2 e^{t^2} \|f\|
\]
for \(f \in H\) and \(t \geq 0\). Therefore, the Lie–Trotter product formula holds.

Based on this relation, P.E.T. Jorgensen, R.T. Moore [7, Ch. 11] and H. Suzuki [12] treated generalized Weyl relations leading to the following result.

**Corollary 10.** Let \((T(t))_{t \geq 0}, (S(t))_{t \geq 0}\) and \((V(t))_{t \geq 0}\) be strongly continuous semigroups on a Banach space \(E\) with generators \((A, D(A)), (B, D(B))\) and \((C, D(C))\), respectively. Suppose that there exists a subspace \(F \subseteq D(C)\) which is dense in \(E\) and invariant under the semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) such that the following conditions hold:
(a) The semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) are exponentially bounded on \((F, \| \cdot \|_C)\) and locally Trotter–stable on both \(E\) and \((F, \| \cdot \|_C)\).
(b) The generalized Weyl relation holds, i.e.
\[
T(t)S(t)f = V(t^2)S(t)T(t)f
\]
for all \(f \in E\) and \(t \geq 0\).

Then the conclusion of Theorem \([3]\) holds.

**Proof.** Fix \(\delta > 0\). We can estimate the commutator of \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) as
\[
\|T(t)S(t)f - S(t)T(t)f\| = \| (V(t^2) - Id) [S(t)T(t)] f \|
\]
\[
= \| \int_0^{t^2} V(s)C [S(t)T(t)]fds \|
\]
\[
\leq t^2 M \|f\|_C
\]
for \(f \in F\), \(t \in [0, \delta]\) and some constant \(M \geq 0\). So Theorem \([3]\) implies the assertion. \(\square\)

5. Applications to Ornstein–Uhlenbeck operators

In this section we consider the (finite–dimensional) Ornstein–Uhlenbeck operator which has been studied e.g. in \([5]\). Let \(E := C_0(\mathbb{R}^d)\) or \(L^p(\mathbb{R}^d), 1 \leq p < \infty\). For any symmetric, positive semi–definite matrix \(A := (a_{ij})\) and a matrix \(B := (b_{ij}) \in \mathcal{L}(\mathbb{R}^d)\), the Ornstein–Uhlenbeck operator is defined by
\[
[O]f(x) := \sum_{i,j=1}^{d} a_{ij} D_{ij} f(x) + \sum_{i,j=1}^{d} b_{ij} x_i D_i f(x) = \langle \nabla, A \nabla f(x) \rangle + \langle B x, \nabla f(x) \rangle
\]
(13)
for each \(f \in S(\mathbb{R}^d), x \in \mathbb{R}^d\), the Schwartz space of rapidly decreasing functions, \(x \in \mathbb{R}^d\), and \(\nabla := \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right)\). Without loss of generality, we write \(A\) as
diag(a₁, ..., a_d) where a_k > 0 for 1 ≤ k ≤ j, a_{j+1} = ... = a_d = 0 and define the operators A and B as the closure of

\[ A f := \langle \nabla, A \nabla f \rangle \quad \text{and} \quad [B f](x) := \langle B x, \nabla f(x) \rangle \]

for \( f \in \mathcal{S}(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \), respectively (cf. [14, Ch. II.6]). The operator \((A, D(A))\) generates a strongly continuous semigroup \((T(t))_{t \geq 0}\) given by

\[ T(t)f(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\sum_{i=1}^{d} \frac{|x_i-s_i|^4}{4a_i t}} f(s_1, \ldots, s_j, x_{j+1}, \ldots, x_d) ds_1 \cdots ds_j \]

for \( t > 0, x \in \mathbb{R}^d \), and \( f \in E \) (see [3]). Furthermore, the operator \((B, D(B))\) generates the strongly continuous semigroup \((S(t))_{t \geq 0}\) given by

\[ S(t)f(x) = f(e^{tB}x) \]

for all \( f \in E \) and \( x \in \mathbb{R}^d \) (see [3, Ch. II, Sec. 3.28]).

These semigroups and their generators have the following useful properties.

**Lemma 11.** Let \((T(t))_{t \geq 0}, (S(t))_{t \geq 0}\) and their generators \((A, D(A))\) and \((B, D(B))\) be as above. Then the following properties hold:

(a) The semigroups \((T(t))_{t \geq 0}\) and \((S(t))_{t \geq 0}\) are locally Trotter–stable on \( E \).

(b) The commutator \(C\) of \((A, D(A))\) and \((B, D(B))\) is given by

\[ Cf := ABf - BAf = 2\langle B A \nabla, \nabla f \rangle \]

for all \( f \in \mathcal{S}(\mathbb{R}^d) \).

(c) For \( t \geq 0\) the functions

\[ s \mapsto A S(t)T(s)f, s \mapsto C S(s)f \]

are continuous and

\[ s \mapsto A S(s)f \]

is differentiable

in \( E \) for \( s \geq 0 \) and for all \( f \in \mathcal{S}(\mathbb{R}^d) \).

(d) The operator \( \langle B A \nabla, \nabla \rangle \) commutes with \((T(t))_{t \geq 0}\) on \( \mathcal{S}(\mathbb{R}^d) \) and

\[ \langle B A \nabla, \nabla S(t)f \rangle = S(t)\langle e^{tB} B A e^{tB} \nabla, \nabla f \rangle \]

for \( f \in \mathcal{S}(\mathbb{R}^d) \), where \( B^* \) denotes the transpose of the matrix \( B \).

**Proof.** Clearly, the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \) is invariant under both semigroups and both generators \( A \) and \( B \). Moreover, \((T(t))_{t \geq 0}\) is a semigroup of contractions on \( E \) and

\[ \| S(t)f \|_{C_0} \leq \| f \|_{C_0}, \quad \| S(t)f \|_{L^p} \leq e^{t\omega} \| f \|_{L^p} \]

for \( f \in E \) and \( \omega := \frac{\text{tr}(B)}{p} \). It follows that the semigroups are locally Trotter–stable on \( E \).

To determine the commutator of \((A, D(A))\) and \((B, D(B))\) on \( \mathcal{S}(\mathbb{R}^d) \), we note that

\[ \frac{\partial}{\partial x_k} \langle B x, \nabla f(x) \rangle = \langle B e_k, \nabla f(x) \rangle + \langle B x, \frac{\partial}{\partial x_k} \nabla f(x) \rangle \]

for \( 1 \leq k \leq d, f \in \mathcal{S}(\mathbb{R}^d), x \in \mathbb{R}^d \), and the \( k \)-th unit vector \( e_k \). Hence, we have

\[ \frac{\partial^2}{\partial x_k^2} \langle B x, \nabla f(x) \rangle = 2\langle B e_k, \frac{\partial}{\partial x_k} \nabla f(x) \rangle + \langle B x, \frac{\partial^2}{\partial x_k^2} f(x) \rangle \]
for $f \in S(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Therefore, we obtain from identity (10) that

$$[ABf - BAf](x) = 2 \sum_{k=1}^{d} \langle B e_k a_k \frac{\partial}{\partial x_k}, \nabla f(x) \rangle + \sum_{k=1}^{d} \langle B x, a_k \frac{\partial^2}{\partial x_k^2} f(x) \rangle$$

$$- \langle B x, \nabla \sum_{k=1}^{d} a_k \frac{\partial^2}{\partial x_k^2} f(x) \rangle$$

$$= 2 \langle BA \nabla, \nabla f(x) \rangle$$

for $f \in S(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

Clearly, the operators $A$ and $B$ are continuous for the usual topology on the space $S(\mathbb{R}^d)$ induced by the family of seminorms $p_{\alpha, n} = \sup_{x \in \mathbb{R}^d} |x^n D^\alpha f(x)|$.

By a straightforward computation, using [11, Thm. 7.2 and Thm. 7.4], we obtain that the statements in (c) are fulfilled for the usual topology on $S(\mathbb{R}^d)$. Since this topology is finer than the topology on $E$, assertion (c) is proved.

Observe that $(T(t))_{t \geq 0}$ is a convolution semigroup and that convolution commutes with the differential operators $\frac{\partial^2}{\partial x_k^2}$ ($1 \leq k \leq d$). Therefore, the commutator of $(A, D(A))$ and $(B, D(B))$ commutes with $(T(t))_{t \geq 0}$. For the semigroup $(S(t))_{t \geq 0}$ we have the following commutator relation

$$\nabla S(t)f = \sum_{k=1}^{d} S(t)(e^{tb} e_k, \nabla f) e_k,$$

and therefore

$$\langle BA \nabla, \nabla S(t)f \rangle = S(t)\langle e^{tb} B A e^{tb} \nabla, \nabla f \rangle$$

for $f \in S(\mathbb{R}^d)$. $\square$

We can now apply Corollary 3 The fact that the closure of $A + B$ is a generator seems to be known (cf. [5]).

**Proposition 12.** Let $(T(t))_{t \geq 0}$, $(S(t))_{t \geq 0}$ be the strongly continuous semigroups on $E$ given by (14) and (15) generated by $(A, D(A))$ and $(B, D(B))$, respectively. Then the conclusion of Theorem 3 holds.

**Proof.** On $S(\mathbb{R}^d)$ we define the norm

$$\|||f||| := \|f\| + \sum_{1 \leq i, j \leq d} \left\| \frac{\partial^2}{\partial_i \partial_j} f \right\|$$

for $f \in S(\mathbb{R}^d)$. This norm is finer than the $C^\infty$-norm $\| \cdot \|_C$.

In the next step, we show the exponential boundedness of the semigroups $(T(t))_{t \geq 0}$ and $(S(t))_{t \geq 0}$ on $(S(\mathbb{R}^d), ||| \cdot |||)$. This is obvious for $(T(t))_{t \geq 0}$ which is $||| \cdot |||\cdot$-contractive on $S(\mathbb{R}^d)$. On the other hand, we have

$$\left\| \frac{\partial^2}{\partial_i \partial_j} S(t)f \right\| = \|S(t)(e^{tb} e_{ij} e^{tb} \nabla, \nabla f) \| \leq e^{2t\omega} \|S(t)\| \cdot |||f|||$$
where
\[
e_{ij} := \begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{pmatrix}
\]
and \( \tilde{\omega} := \max_{1 \leq i, j \leq d} |b_{ij}| \). Therefore, we can estimate
\[
\||S(t)f||| \leq d^2 e^{2t\tilde{\omega}} \|S(t)\| \cdot ||f||
\]
for \( f \in S(\mathbb{R}^d) \).

Finally, this implies the local Trotter–stability on \((S(\mathbb{R}^d), ||\cdot||||)\)
\[
|||T(\frac{t}{n})S(\frac{t}{n})|||^n \leq e^{2t\tilde{\omega}}||f||
\]
if \( E = C_0(\mathbb{R}^d) \), and
\[
|||T(\frac{t}{n})S(\frac{t}{n})|||^n \leq e^{t(\omega+2\tilde{\omega})}||f||
\]
if \( E = L^p(\mathbb{R}^d) \). We can now apply Corollary 6 which concludes the proof.

As a concrete example we mention a semigroup appearing in mathematical finance (2).

**Examples 13.** On the space \( E := C_0(\mathbb{R}^2) \) or \( L^p(\mathbb{R}^2) \) \((1 \leq p < \infty)\) the Cauchy problem
\[
\begin{align*}
\frac{d}{dt} u(t, x, y) &= \frac{\partial^2}{\partial x^2} u(t, x, y) + x \frac{\partial}{\partial y} u(t, x, y), \quad t \geq 0, \\
u(0, x, y) &= f(x, y),
\end{align*}
\]
is investigated in [2]. Taking
\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
the Cauchy problem (17) can be reformulated as an Ornstein–Uhlenbeck operator \( O \) defined in (13). Therefore, all assumptions of Proposition 12 are satisfied and \( O \) generates a semigroup given as the limit of the Lie–Trotter products.

**References**


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