AN EXTENSION OF LUCAS’ THEOREM

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Abstract. Let $p$ be a prime. A famous theorem of Lucas states that \( \binom{mp+t}{n} \equiv \binom{m}{n} \binom{t}{n} \pmod{p} \) if $m, n, s, t$ are nonnegative integers with $s, t < p$. In this paper we aim to prove a similar result for generalized binomial coefficients defined in terms of second order recurrent sequences with initial values 0 and 1.

1. Introduction

Let $N = \{0, 1, 2, \cdots \}$, $Z^+ = \{1, 2, 3, \cdots \}$ and $Z^* = Z \setminus \{0\}$. Fix $A, B \in Z^*$. The Lucas sequence $\{u_n\}_{n \in \mathbb{N}}$ is defined as follows:

(1) $u_0 = 0, u_1 = 1$ and $u_{n+1} = Au_n - Bu_{n-1}$ for $n = 1, 2, 3, \cdots$.

Its companion sequence $\{v_n\}_{n \in \mathbb{N}}$ is given by

(2) $v_0 = 2, v_1 = A$ and $v_{n+1} = Av_n - Bv_{n-1}$ for $n = 1, 2, 3, \cdots$.

By induction, for $n = 0, 1, 2, \cdots$ we have

$$u_n = \sum_{0 \leq k < n} \alpha^k \beta^{n-1-k} \text{ and } v_n = \alpha^n + \beta^n$$

where

$$\alpha = \frac{A + \sqrt{\Delta}}{2}, \quad \beta = \frac{A - \sqrt{\Delta}}{2} \text{ and } \Delta = A^2 - 4B.$$ 

It follows that

$$v_n = 2u_{n+1} - Au_n, \quad u_{2n} = u_n v_n \text{ and } v_{2n} = v_n^2 - 2B^n \text{ for } n \in \mathbb{N}.$$ 

For $a, b \in \mathbb{Z}$ let $(a, b)$ denote the greatest common divisor of $a$ and $b$. A nice result of E. Lucas asserts that if $(A, B) = 1$, then $(u_m, u_n) = |u_{(m,n)}|$ for $m, n \in \mathbb{N}$ (cf. L. E. Dickson [1]).

In the case $A^2 = B = 1$, by induction on $n \in \mathbb{N}$ we find that $u_n = 0$ if $3 \mid n$, and

$$u_n = \begin{cases} 
1 & \text{if } A = -1 \& 3 \mid n - 1, \text{ or } A = 1 \& n \equiv 1, 2 \pmod{6}; \\
-1 & \text{if } A = -1 \& 3 \mid n + 1, \text{ or } A = 1 \& n \equiv -1, -2 \pmod{6}.
\end{cases}$$

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We set \([n] = \prod_{0 < k \leq n} u_k\) for \(n \in \mathbb{N}\), and regard an empty product as value 1. For \(n, k \in \mathbb{N}\) with \([n] \neq 0\), we define the Lucas \(u\)-nomial coefficient \([\frac{n}{k}]\) as follows:

\[
[n \atop k] = \begin{cases} 
\frac{[n]}{[k][n-k]} & \text{if } n \geq k, \\
0 & \text{otherwise.}
\end{cases}
\]

In the case \(A = 2\) and \(B = 1\), \([\frac{n}{k}]\) is exactly the binomial coefficient \(\binom{n}{k}\); when \(A = q + 1\) and \(B = q\) where \(q \in \mathbb{Z}\) and \([q] > 1\), \([\frac{n}{k}]\) coincides with Gaussian \(q\)-nomial coefficient \(\binom{n}{k}_q\) because \(u_j = (q^j - 1)/(q - 1)\) for \(j = 0, 1, 2, \cdots\). For generalized binomial coefficients formed from an arbitrary sequence of positive integers, the reader is referred to the elegant paper of D. E. Knuth and H. S. Wilf [5].

Let \(d > 1\) and \(q > 0\) be integers with \(d \mid q\). If \((A, B) = 1\) and \(d \mid u_k\) for \(k = 1, \cdots, q - 1\), then for any \(n \in \mathbb{N}\) we have

\[
d \mid u_n \iff d \text{ divides } (u_n, u_q) = |u_{(n, q)}| \iff q = (n, q) \iff q \mid n;
\]

this property is usually called the regular divisibility of \(\{u_n\}_{n \in \mathbb{N}}\). If \((d, u_k) = 1\) for all \(0 < k < q\), then we write \(q = d_s\) and call \(d\) a primitive divisor of \(u_q\) while \(q\) is called the rank of apparition of \(d\). When \((A, B) = 1\), \(q = d_s\), \(n \in \mathbb{N}\) and \(q \mid n\), we have

\[
(d, u_n) = ((d, u_q), u_n) = (d, (u_n, u_q)) = (d, u_{(n, q)}) = 1.
\]

When \(p\) is an odd prime not dividing \(B\), \(p_s\) exists because \(p \mid u_{p-(\frac{A}{B})}\) as is well known where \((-)\) denotes the Legendre symbol. On the other hand, drawing upon some ideas of A. Schinzel [6], C. L. Stewart [7] proved in 1977 that if \(A\) is prime to \(B\) and \(\alpha/\beta\) is not a root of unity, then \(u_n\) has a primitive prime divisor for each \(n > e^{452^{267}}\); P. M. Voutier [9] conjectured in 1995 that the lower bound \(e^{452^{267}}\) can be replaced by 30.

For \(m \in \mathbb{Z}\) we use \(\mathbb{Z}_m\) to denote the ring of rationals in the form \(a/b\) with \(a \in \mathbb{Z}\), \(b \in \mathbb{Z}^+\) and \((b, m) = 1\). When \(r \in \mathbb{Z}_m\), by \(x \equiv r \pmod{m}\) we mean that \(x\) can be written as \(r + my\) with \(y \in \mathbb{Z}_m\).

For convenience we set \(R(q) = \{x \in \mathbb{Z} : 0 \leq x < q\}\) for \(q \in \mathbb{Z}^+\).

Our main result is as follows.

**Theorem.** Suppose that \((A, B) = 1\), and \(A \neq \pm 1\) or \(B \neq 1\). Then \(u_k \neq 0\) for every \(k = 1, 2, 3, \cdots\). Let \(q \in \mathbb{Z}^+\), \(m, n \in \mathbb{N}\) and \(s, t \in R(q)\). Then

\[
\left[\frac{mq + s}{nq + t}\right] = \left(\frac{m}{n}\right) \left(\frac{s}{t}\right) u_{q+1}^{(nq+t)(m-n)+n(s-t)} \pmod{w_q}
\]

where \(w_q\) is the largest divisor of \(u_q\) prime to \(u_1, \cdots, u_{q-1}\). If \(q\) or \(m(n+t)+n(s+1)\) is even, then

\[
\left[\frac{mq + s}{nq + t}\right] = \left(\frac{m}{n}\right) \left(\frac{s}{t}\right) (-1)^{(mt-ns)(q-1)} B^{2\binom{nq+t}{(nq+t)(m-n)+n(s-t)}} \pmod{w_q}.
\]

**Remark 1.** Providing \((A, B) = 1\) and \(q \in \mathbb{Z}^+\), \((u_q, \prod_{0 < k < q} u_k) = 1\) if and only if \(u_d = \pm 1\) for all proper divisors \(d\) of \(q\) (this is because \((u_q, u_k) = |u_{(q,k)}|\)); therefore \(u_q\) is prime to \(u_1, \cdots, u_{q-1}\) if \(q\) is a prime.

When \(A = 2\) and \(B = 1\), we have \(u_k = k\) for all \(k \in \mathbb{N}\), hence the Theorem yields Lucas’ theorem which asserts that

\[
\left(\frac{mp + s}{np + t}\right) \equiv \left(\frac{m}{n}\right) \left(\frac{s}{t}\right) \pmod{p},
\]
where \( p \) is a prime and \( m, n, s, t \) are nonnegative integers with \( s, t < p \). In the case \( A = a + 1 \) and \( B = a \) where \( a \in \mathbb{Z} \) and \( |a| > 1 \), as \( u_{q+1} = (a^{q+1} - 1)/(a - 1) = a u_q + 1 \equiv 1 \pmod{u_q} \) for \( q \in \mathbb{Z}^+ \), our Theorem implies Theorem 3.11 of R. D. Fray [2].

Theorem 3 of B. Wilson [10] follows from our Theorem in the special case \( A = 1 \), \( B = -1 \) and \( s \geq t \). Wilson used a result of Kummer concerning the highest power of a prime dividing a binomial coefficient; see Knuth and Wilf [5] for various generalizations of Kummer’s theorem. Our proof of the Theorem is more direct; we don’t use Kummer’s theorem in any form.

**Example.** (i) Set \( A = 4 \) and \( B = 1 \). Then
\[

d_0 = 0, \quad d_1 = 1, \quad d_2 = 4, \quad d_3 = 15, \quad d_4 = 56, \quad d_5 = 209, \quad d_6 = 780.
\]
Clearly \( p = 13 \) is the largest primitive divisor of \( d_6 = 780 \). By the Theorem,
\[
\begin{align*}
\left( \frac{71}{25} \right) &= \left( \frac{11 \times 6 + 5}{4 \times 6 + 1} \right) \\
&= \left( \frac{11}{4} \right) \left( -1 \right)^{11 \times 1 - 4 \times 5} \\
&= -330 \times 209 \equiv -5 \times 1 \equiv 8 \pmod{13}.
\end{align*}
\]

(ii) Take \( A = 1 \) and \( B = -7 \). Then \( d_3 = 3 = 8 \) and \( d_4 = 15 \). By the Theorem,
\[
\begin{align*}
\left( \frac{35}{10} \right) &= \left( \frac{11 \times 3 + 2}{3 \times 3 + 1} \right) \\
&= \left( \frac{11}{3} \right) \left[ 15^{10(11-3)+3(2-1)} \right] \\
&= 3 \pmod{8}.
\end{align*}
\]

2. **Several Lemmas**

**Lemma 1.** Let \( n \) and \( k \) be positive integers with \( n > k \) and \( |n| \neq 0 \). Then
\[
\left( \frac{n}{k} \right) = u_{k+1} \left( \frac{n-1}{k} \right) - Bu_{n-k-1} \left( \frac{n-1}{k-1} \right).
\]

If \( 2 \mid A \) and \( 2 \nmid B \), then \( \left( \frac{n}{k} \right) \equiv \left( \frac{n}{k} \right) \pmod{2} \).

**Proof.** Clearly the right hand side of (6) coincides with
\[
\begin{align*}
&\frac{u_{k+1}}{k} \frac{1}{k} \frac{1}{n-1-k} - Bu_{n-k-1} \frac{1}{k-1} \frac{1}{n-k} \\
&= \frac{1}{k} \frac{1}{n-k} (u_{k+1} u_{n-k} - Bu_k u_{n-k-1}) = \frac{n-1}{k}.
\end{align*}
\]

where in the last step we use the identity \( u_{k+1} u_{n-k} - Bu_k u_{n-k-1} = u_{k+1} \) which can be easily proved by induction on \( k \in \mathbb{Z}^+ \).

Now suppose that \( 2 \nmid (A-1)B \). Then \( u_1, u_3, u_5, \ldots \) are odd and \( u_2, u_4, u_6, \ldots \) are even. If
\[
\left( \frac{n-1}{k} \right) \equiv \left( \frac{n-1}{k} \right) \pmod{2} \quad \text{and} \quad \left( \frac{n-1}{k-1} \right) \equiv \left( \frac{n-1}{k-1} \right) \pmod{2},
\]
then (6) yields that
\[
\begin{align*}
\left( \frac{n}{k} \right) &\equiv (k+1) \left( \frac{n-1}{k} \right) - (n-k-1) \left( \frac{n-1}{k-1} \right) \\
&\equiv (k+1) \left( \frac{n}{k} \right) - n \left( \frac{n-1}{k-1} \right) \equiv \left( \frac{n}{k} \right) \pmod{2}.
\end{align*}
\]

So \( \left( \frac{n}{k} \right) \equiv \left( \frac{n}{k} \right) \pmod{2} \) by induction. \( \square \)
Remark 2. In light of Lemma 1, by induction, if \( n \in \mathbb{N} \) and \([n] \neq 0\), then \([n^k] \in \mathbb{Z}\) for all \( k \in \mathbb{N} \). This was also realized by W. A. Kimball and W. A. Webb \([4]\). In 1989 Knuth and Wilf \([5]\) proved that generalized binomial coefficients, formed from a regularly divisible sequence of positive integers, are always integral.

Lemma 2. Let \( q \) be a positive integer. Then \( u_{q+1}^2 \equiv B^q \pmod{u_q} \). If \( 2 \mid q \), then \( u_{q+1} \equiv -Bq/2 \pmod{d} \) for any primitive divisor \( d \) of \( u_q \).

Proof. As
\[
\begin{pmatrix}
  u_q & u_{q-1} \\
  u_{q+1} & u_q
\end{pmatrix} = \begin{pmatrix}
  u_{q-1} & u_{q-2} \\
  u_q & u_{q-1}
\end{pmatrix} \begin{pmatrix}
  A & 1 \\
  -B & 0
\end{pmatrix}
\]
\[
= \cdots = \begin{pmatrix}
  u_1 & u_0 \\
  u_2 & u_1
\end{pmatrix} \begin{pmatrix}
  A & 1 \\
  -B & 0
\end{pmatrix}^{q-1},
\]
we have \( u_{q+2} - u_{q-1}u_{q+1} \equiv B^{q-1} \) and hence \( u_{q+1}^2 \equiv -Bu_{q-1}u_{q+1} \equiv B^q \pmod{u_q} \).

Now assume that \( q = 2n \) where \( n \in \mathbb{Z}_+ \). Let \( d \) be a primitive divisor of \( u_q \). Since \( u_n \equiv 0 \pmod{d} \) and \( (d, u_n) = 1 \), we have \( d \mid u_n \) and hence
\[
u_{q+1} = \frac{Aq + u_q}{2} = \frac{Au_n + u_n^2 - 2B^n}{2} = u_{n+1} - Bn \equiv -B^n \pmod{d}.
\]
This ends the proof. \( \square \)

Lemma 3. Let \( k, q \in \mathbb{Z}_+ \). Then
\[
u_{kq + l} \equiv u_{q+1}^{kq}u_l \pmod{u_q} \quad \text{for } l = 0, 1, 2, \ldots.
\]

If \( u_q \neq 0 \), then
\[
u_{kq} \equiv u_{q+1}^{kq-1} + (k - 1)Aq/2 \pmod{u_q}.
\]

Proof. Let \( l \in \mathbb{N} \). By Lemma 2 of Z.-W. Sun \([8]\),
\[
u_{kq + l} = \sum_{r=0}^{k} \binom{k}{r} c^{k-r} u_q^r u_{l+r}
\]
where \( c = -Bu_{q-1} = u_{q+1} - Au_q \). Clearly \( u_{kq + l} \equiv u_{q+1}^{kq}u_l \pmod{u_q} \). In the case \( u_q \neq 0 \),
\[
u_{kq} \equiv \sum_{r=1}^{k} \frac{1}{k} \binom{k}{r} c^{k-r} u_q^{r-1} u_r = \sum_{r=1}^{k} \frac{(k - 1)}{r - 1} \frac{u_q^{r-1} u_r}{r} c^{k-r} u_r.
\]
For any prime \( p \) and integer \( r > 3 \) we have
\[
p^{r-2} \geq (1 + 1)^{r-2} \geq 1 + (r - 2) + 1 = r,
\]
therefore \( u_q^{-2}/r \in \mathbb{Z}_{u_q} \) for \( r = 3, 4, \ldots \). If \( 2 \mid u_q \) and \( 2 \nmid A \), then \( 2 \mid B \) (otherwise \( u_q \equiv u_{q-1} \equiv \cdots \equiv u_1 \neq 0 \pmod{2} \)), as \( u_{q+1} \equiv B^q \pmod{u_q} \) we have \( c \equiv u_{q+1} \equiv 1 \pmod{2} \). Thus (8) holds providing \( u_q \neq 0 \). \( \square \)

Lemma 4. Assume that \((A, B) = 1\), \( q \in \mathbb{Z}_+ \) and \( u_k \neq 0 \) for all \( k \in \mathbb{Z}_+ \). Then for any \( m, n \in \mathbb{N} \) and \( s, t \in R(q) \) we have
\[
u_{q+1} \left[ \frac{mq + s}{nq + t} \right] \equiv \nu_{q+1} \left[ \frac{mq}{nq} \right] \left[ \frac{s}{t} \right] \nu_{u_{q+1}^{k(m-n)+n(s-t)}} \pmod{w_q}
\]
where \( w_q \) is the largest divisor of \( u_q \) prime to \( u_1, \ldots, u_{q-1} \).
Proof. Let \( m, n \in \mathbb{N} \) and \( s, t \in R(q) \). If \( m < n \), then \( mq + s < (m + 1)q \leq mq + t \) and hence \( \left[ \frac{mq + s}{mq + t} \right] = 0 = \left[ \frac{mq}{mq + t} \right] \). If \( m = n \) and \( s < t \), then \( \left[ \frac{mq + s}{mq + t} \right] = 0 = \left[ \frac{mq}{mq + t} \right] \). Below we assume that \( m \geq n \) and \( mq + s \geq mq + t \).

As \( (A, B) = 1 \), \( (u_q, u_{q+1}) = |u_{q(q+1)}| = 1 \). Observe that \( w_q \) is prime to \( u_{q+1} \prod_{0 < r < q} u_r \), and

\[
\frac{[mq + s]}{[mq + t]} = \frac{\prod_{0 < m - n q < j \leq mq} u_j}{\prod_{0 < j \leq nq} u_j} \times \frac{\prod_{0 < r \leq s} u_{mq + r}}{\prod_{0 < r \leq t} u_{mq + r}} \times \begin{cases} \prod_{0 < r \leq s-t} u_{(m-n)q + r}^{-1} & \text{if } s \geq t, \\ \prod_{0 < r \leq t-s} u_{(m-n)q - r} & \text{if } s < t. \end{cases}
\]

By Lemma 3, \( u_{kq + r} \equiv u_{q+1}^k u_r \mod w_q \) for any \( k, r \in \mathbb{N} \). So

\[
\frac{[mq + s]}{[mq + t]} \equiv \frac{[mq]}{[mq]} \times \frac{\prod_{0 < r \leq s} u_{q+1}^s u_r}{\prod_{0 < r \leq t} u_{q+1}^t u_r} \times \begin{cases} \prod_{0 < r \leq s-t} u_{q+1}^{-1} u_{(m-n)q + r} & \text{if } s \geq t, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
\equiv \frac{[mq]}{[mq]} \left[ \frac{s}{t} \right] u_{q+1}^{m-nt} \times \begin{cases} u_{q+1}^{(n-m)(s-t)}/[s-t] & \text{if } s \geq t, \\ 0 & \text{if } s < t, \end{cases}
\]

\[
\equiv \frac{[mq]}{[mq]} \left[ \frac{s}{t} \right] u_{q+1}^{(m-n)q + s} \mod w_q.
\]

This concludes the proof. \( \square \)

3. Proof of the Theorem

Let us first show that \( u_1, u_2, u_3, \ldots \) are all nonzero.

If \( \Delta = 0 \), then \( \alpha = \beta = A/2 \) and hence

\[
u_k = \sum_{0 \leq r < k} \alpha^r \beta^{k-1-r} = k \left( \frac{A}{2} \right)^{k-1} \neq 0 \text{ for } k = 1, 2, 3, \ldots .
\]

Suppose that \( u_k = 0 \) for some \( k \in \mathbb{Z}^+ \). Then \( \Delta \neq 0 \), \( \alpha \neq \beta \) and \( \alpha^k = \beta^k \). Since the field \( \mathbb{Q}(\sqrt{\Delta}) \) contains the root \( \alpha/\beta \neq \pm 1 \) of unity, by Propositions 13.1.5 and 13.1.6 of K. Ireland and M. Rosen \([3]\) there exists a positive integer \( D \) such that \( \Delta = -D^2 \) and \( \alpha/\beta \in \{ \pm i \} \), or \( \Delta = -3D^2 \) and \( \alpha/\beta \in \{ \pm \omega, \pm \omega^2 \} \) where \( \omega = (-1 + \sqrt{-3})/2 \). In the former case, \( (A + Di)/(A - Di) \in \{ \pm i \} \); hence \( A^2 = D^2 \) and \( 2B = (A^2 - \Delta)/2 = D^2 \). This is impossible since \( A \) or \( B \) is odd. Thus the latter case happens. Now that

\[
\frac{A + D\sqrt{-3}}{A - D\sqrt{-3}} = \frac{A^2 - 3D^2 + 2AD\sqrt{-3}}{A^2 + 3D^2} \in \left\{ \frac{-1 \pm \sqrt{-3}}{2}, \frac{1 \pm \sqrt{-3}}{2} \right\},
\]

we have \( A^2 - 3D^2 = \pm 2AD \) and hence \( A^2 \in \{ D^2, 9D^2 \} \). If \( A^2 = D^2 \), then \( B = (A^2 - \Delta)/4 = D^2 \), hence \( (A, B) > 1 \) or \( A^2 = B = 1 \); if \( A^2 = 9D^2 \), then \( B = (A^2 - \Delta)/4 = 3D^2 \) and hence \( 3 | (A, B) \). This leads to a contradiction.
Next we show (4).
Let \( u_0' = 0 \), \( u_1' = 1 \) and \( u_j' = v_qu_j - B^ju_j' = B^ju_j' \) for \( j = 1, 2, 3, \ldots \). Note that \( \alpha^q + \beta^q = vq \) and \( \alpha^q/\beta^q = Bq \). Fix \( k \in \mathbb{Z}^+ \). If \( \Delta = A^2 - 4B \neq 0 \), then

\[
\frac{u_{kq}}{u_q} = \frac{(\alpha^q - \beta^q)/(\alpha - \beta)}{(\alpha^q - \beta^q)/(\alpha - \beta)} = \frac{(\alpha^q)^k - (\beta^q)^k}{\alpha^q - \beta^q} = u_k',
\]

if \( \Delta = 0 \), then \( \alpha = \beta = A/2 \), \( u_q = q(A/2)^{-1} \), \( u_{kq} = kq(A/2)^{kq-1} \) and

\[
u_k' = \sum_{0 \leq r < k} (\alpha^q)^r(\beta^q)^{k-1-r} = k \left( \frac{A}{2} \right)^{q(k-1)} = \frac{u_{kq}}{u_q}.
\]

So we always have \( u_{kq}/u_q = u_k' \). By (8),

\[
\frac{u_{kq}}{ku_q} \equiv r_k \pmod{u_q} \quad \text{where} \quad r_k = u_{k-1}q \left\{ \begin{array}{ll}
(k - 1)Au_q/2 & \text{if } 2 \mid u_q, \\
0 & \text{otherwise}.
\end{array} \right.
\]

Notice that \( (r_k, u_q) = 1 \) if \( 2 \nmid u_q \), and \( (r_k, u_q/A) = 1 \) if \( 2 \mid u_q \).

Suppose \( m > n > 0 \). We assert that

\[
(10) \quad \prod_{0 \leq k < n} \frac{u_{(m-k)q}}{u_{(n-k)q}} \equiv \left( \frac{m}{n} \right)^{u_{n(m-n)}} (\pmod{u_q}).
\]

If \( 2 \nmid u_q \) or \( 4 \mid u_q \), then \((r_k, u_q) = 1 \) for all \( k = 1, 2, 3, \ldots \), hence

\[
\prod_{0 \leq k < n} \frac{u_{(m-k)q}}{u_{(n-k)q}} \equiv \prod_{0 \leq k < n} \frac{m-k}{n-k} \times \prod_{0 \leq k < n} \frac{u_{(m-k)q}/((m-k)u_q)}{u_{(n-k)q}/((n-k)u_q)}
\]

\[
\equiv \left( \frac{m}{n} \right) \prod_{0 \leq k < n} \frac{u_{q+1}^{m-k} + (m-k-1)Au_q/2}{u_{q+1}^{n-k} + (n-k-1)Au_q/2}
\]

\[
\equiv \left( \frac{m}{n} \right) \prod_{0 \leq k < n} \left( u_{q+1}^{m-n} + (m-n,Au_q/2) \right) \equiv \left( \frac{m}{n} \right) \left( u_{q+1}^{n(m-n)} + n(m-n),Au_q/2 \right)
\]

\[
\equiv \left( \frac{m}{n} \right) u_{q+1}^{n(m-n)} + \frac{m(m-1)}{2}(m-2)\left(n-1 \right)Au_q \equiv \left( \frac{m}{n} \right) u_{q+1}^{n(m-n)} \pmod{u_q}.
\]

In the case \( u_q \equiv 0 \pmod{2} \), by the above method

\[
\prod_{0 \leq k < n} \frac{u_{(m-k)q}}{u_{(n-k)q}} \equiv \left( \frac{m}{n} \right)^{u_{n(m-n)}} \pmod{u_{q+1}^2};
\]

as \( v_q = 2u_{q+1} - Au_q \equiv 0 \pmod{2} \) and \( B \equiv 1 \pmod{2} \) (otherwise \( A, u_1, u_2, u_3, \ldots \) are all odd), we also have

\[
\prod_{0 \leq k < n} \frac{u_{(m-k)q}}{u_{(n-k)q}} \equiv \prod_{0 \leq k < n} \frac{u_{(m-k)q}}{u_{(n-k)q}'} \equiv \left( \frac{m}{n} \right)^{u_{q+1}^{n(m-n)}} \pmod{u_q}.
\]

by Lemma 1. This proves (10).

Now we claim that

\[
(11) \quad \left( \frac{mq}{nq} \right) \equiv \left( \frac{m}{n} \right) u_{q+1}^{n(m-n)q} \pmod{u_q}.
\]
This is obvious if \( m \leq n \) or \( n = 0 \). In the case \( m > n > 0 \), if \( 0 < j < nq \) and \( q \not| j \), then \( (u_{nq-j}, w_q) = 1 \) and
\[
\frac{u_{mq-j}}{u_{nq-j}} = \frac{u_{(m-n)q+nq-j}}{u_{nq-j}} \equiv \frac{u_{m-n}}{u_{nq-j}} \mod w_q
\]
by Lemma 3; thus
\[
\left[ \begin{array}{c} m \\ n \end{array} \right] = \prod_{0 \leq j < nq} \frac{u_{mq-j}}{u_{nq-j}} = \prod_{0 \leq k < n} \frac{u_{(m-k)q}}{u_{nq-j}} \times \prod_{0 < j < nq} \frac{u_{mq-j}}{u_{nq-j}} \\
\equiv \left( \begin{array}{c} m \\ n \end{array} \right) u_{q+1}^{n(m-n)} \times u_{q+1}^{(m-n)(nq-n)} = \left( \begin{array}{c} m \\ n \end{array} \right) u_{q+1}^{(m-n)nq} \mod w_q.
\]

In view of (9) and (11),
\[
\left[ \begin{array}{c} mq + s \\ nq + t \end{array} \right] \equiv \left( \begin{array}{c} m \\ n \end{array} \right) u_{q+1}^{(m-n)nq} \times \left[ \begin{array}{c} s \\ t \end{array} \right] t^{(m-n)+n(s-t)} u_{q+1}^{t(m-n)+n(s-t)} \\
\equiv \left( \begin{array}{c} m \\ n \end{array} \right) \left[ \begin{array}{c} s \\ t \end{array} \right] u_{q+1}^{(nq+t)(m-n)+n(s-t)} \mod w_q.
\]

Finally we say something about (5). If \( 2 \mid q \), then
\[
(nq + t)(m - n) + n(s - t) \equiv t(m - n) + n(s - t) \equiv mt - ns \mod 2,
\]
and \( u_{q+1} \equiv -B^{t/2} \mod w_q \) by Lemma 2. When \( q \) is odd and \( l = m(n+t)+n(s+1) \) is even,
\[
(nq + t)(m - n) + n(s - t) \equiv (n + t)(m - n) + n(s - t) \equiv l \equiv 0 \mod 2,
\]
and \( u_{q+1}^2 \equiv B^q \mod w_q \) by Lemma 2. Thus (5) follows from (4) if \( 2 \mid ql \). We are done.

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References


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