INVARIANT SUBSPACES AND REPRESENTATIONS OF CERTAIN VON NEUMANN ALGEBRAS

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Abstract. Let \((N, \alpha, G)\) be a covariant system and let \((\pi, U)\) be a covariant representation of \((N, \alpha, G)\) on a Hilbert space \(H\). In this note, we investigate the representation of the covariance algebra \(M\) and the \(\sigma\)-weakly closed subalgebra \(A\) generated by \(\pi(N)\) and \(\{U_g\}_{g \in G}\) in the case of \(G = \mathbb{Z}\) or \(\mathbb{R}\) when there exists a pure, full, \(A\)-invariant subspace of \(H\).

1. Introduction

If \(G\) is a locally compact group and \(\alpha : G \to \text{Aut}(N)\) is a continuous homomorphism of \(G\) into the group of *-automorphisms of a von Neumann algebra \(N\), then the triple \((N, \alpha, G)\) is called a covariant system. This notation was introduced by Doplicher, Kastler and Robinson in [2]. Covariant systems have turned out to be very interesting objects, both in theoretical physics and in mathematics. The covariant representation of \((N, \alpha, G)\) means a pair \((\pi, U)\) consisting of a unitary representation \(U\) of \(G\) and a \(\pi\)-representation of \(N\) with \(\pi\) and \(U\) operating over the same Hilbert space \(H\) such that

\[ \pi(\alpha_g(x)) = U_g \pi(x) U_g^* \quad (\forall x \in N, \forall g \in G). \]

The covariance algebra \(M\) of \((N, \alpha, G)\) is a von Neumann algebra generated by \(\pi(N)\) and \(\{U_g\}_{g \in G}\). When \(G\) has an order \(\geq\), we consider the \(\sigma\)-weakly closed subalgebra \(A\) of \(M\) which is generated by \(\pi(N)\) and \(\{U_g\}_{g \geq 0}\). The representation theory of \(M\) has been extensively studied by M. Takesaki in [16], M. Landstad in [5] and I. Raeburn in [13], among others. The covariance algebras provide us with a rich variety of examples of operator algebras. In this note, we consider the representation theory of \(M\) and \(A\) in the particular case of \(G = \mathbb{R}\) or \(\mathbb{Z}\). By inspiring the scattering theory of Lax and Phillips in [6], we study the representation of \(M\) and \(A\) to a crossed product and an analytic crossed product, respectively, using the theory of invariant subspace for \(A\). Therefore, our setting is as follows.

Let \(M\) be a von Neumann algebra acting on a Hilbert space \(H\) generated by a von Neumann algebra \(N\) and a unitary operator \(v\) satisfying \(vNv^* = N\), and let \(A\) be a \(\sigma\)-weakly closed subalgebra of \(M\) generated by \(N\) and the non-negative powers of \(v\). At first, we prove that if there is a pure, full, \(A\)-invariant subspace \(M\)
of $\mathcal{H}$, then $M$ is $*$-isomorphic to a (discrete) crossed product $N \rtimes_\alpha \mathbb{Z}$ of $N$ by a $*$-automorphism $\alpha = \text{adv}$, and that $\mathfrak{A}$ is simultaneously isomorphic to the analytic crossed product $N \rtimes_\alpha \mathbb{Z}_+$.

Similarly, we also consider the representation of a von Neumann algebra $M_0$ generated by a von Neumann algebra $N$ and a strongly continuous one-parameter unitary group $\{u_t\}_{t \in \mathbb{R}}$ satisfying $u_t N u_t^* = N$ for every $t$ in $\mathbb{R}$. Let $\mathfrak{B}$ be the $\sigma$-weakly closed subalgebra of $M_0$ generated by $N$ and $\{u_t\}_{t > 0}$. We prove that if there is a pure, full, $\mathfrak{B}$-invariant subspace of $\mathcal{H}$, then $M_0$ is $*$-isomorphic to a continuous crossed product, and that $\mathfrak{B}$ is simultaneously isomorphic to the related analytic crossed product.

Next, in §3, for a strongly continuous one-parameter unitary group $\{u_t\}_{t \in \mathbb{R}}$, we construct the unitary operator $v$ by the Cayley transform of the infinitesimal generator of $\{u_t\}_{t \in \mathbb{R}}$. From this unitary $v$ and $\{u_t\}_{t \in \mathbb{R}}$, we define the von Neumann algebras $M$, $M_0$ and the subalgebras $\mathfrak{A}$, $\mathfrak{B}$, respectively, as in §2, and we shall show that a closed subspace of $\mathcal{H}$ is pure, full, $\mathfrak{A}$-invariant if and only if it is pure, full, $\mathfrak{B}$-invariant (Proposition 3.2). Finally, we shall discuss the relation between a discrete crossed product and continuous crossed product.

2. Representation of certain von Neumann algebras to a crossed product

At first, we consider the representation of the covariance subalgebra in the case of $G = \mathbb{Z}$. Let $M$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ generated by a von Neumann algebra $N$ and a unitary operator $v$ satisfying $v N v^* = N$ and let $\mathfrak{A}$ be the $\sigma$-weakly closed subalgebra of $M$ generated by $N$ and non-negative powers of $v$. We now define the notion of invariant subspaces of $\mathcal{H}$ with respect to $\mathfrak{A}$ as in [8]-[10].

**Definition 2.1.** Let $\mathfrak{M}$ be a closed subspace of $\mathcal{H}$. We shall say that $\mathfrak{M}$ is: $\mathfrak{A}$-invariant, if $\mathfrak{A} \mathfrak{M} \subseteq \mathfrak{M}$; reducing, if $M \mathfrak{M} \subseteq \mathfrak{M}$; pure, if $\mathfrak{M}$ contains no non-trivial reducing subspace; and full, if the smallest reducing subspace containing $\mathfrak{M}$ is all of $\mathcal{H}$.

Since $v N v^* = N$, we put $\alpha(x) = v x v^* \ (\forall x \in N)$. We recall that the crossed product $N \rtimes_\alpha \mathbb{Z}$ of $N$ by the $*$-automorphism group $\{\alpha^n\}_{n \in \mathbb{Z}}$ is the von Neumann algebra acting on the Hilbert space $\ell^2(\mathbb{Z}, \mathcal{H})$ generated by the operators $\pi_\alpha(x) \ (\forall x \in N)$ and $S$ defined by the equations

$$\{ \pi_\alpha(x) \xi \}(n) = \alpha^{-n}(x) \xi(n) \ \ (\forall \xi \in \ell^2(\mathbb{Z}, \mathcal{H}), \ \forall n \in \mathbb{Z})$$

and

$$\{ S \xi \}(n) = \xi(n - 1) \ \ (\forall \xi \in \ell^2(\mathbb{Z}, \mathcal{H}), \ \forall n \in \mathbb{Z}).$$

We note that the analytic crossed product $N \rtimes_\alpha \mathbb{Z}_+$ determined by $N$ and $\alpha$ is defined to be the $\sigma$-weakly closed subalgebra of $N \rtimes_\alpha \mathbb{Z}$ generated by $\pi_\alpha(N)$ and the non-negative powers of $S$ (cf. [8]-[10]). Let $\{\alpha_t\}_{t \in \mathbb{T}}$ be the $*$-automorphism group of $N \rtimes_\alpha \mathbb{Z}$ which is dual to $\{\alpha^n\}_{n \in \mathbb{Z}}$ in the sense of Takesaki [10]. Then we have

**Theorem 2.2.** Let $M$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ generated by a von Neumann algebra $N$ and a unitary operator $v$ satisfying $v N v^* = N$ and let $\mathfrak{A}$ be the $\sigma$-weakly closed subalgebra of $M$ generated by $N$ and non-negative powers of $v$. Put $\alpha(x) = v x v^* \ (\forall x \in N)$. If there exists a pure, full, $\mathfrak{A}$-invariant
subspace $\mathfrak{M}$ of $\mathcal{H}$, then there exist a $*$-automorphism group $\{\gamma_t\}_{t \in \mathbb{T}}$ of $M$ and a $*$-isomorphism $\Phi$ from $M$ onto $N \rtimes_\alpha \mathbb{Z}$ such that

$$\Phi(x) = \pi_\alpha(x) \, (\forall x \in N), \quad \Phi(v) = S \quad \text{and} \quad \Phi \circ \gamma_t = \tilde{\alpha}_t \circ \Phi \, (\forall t \in \mathbb{T}).$$

**Proof.** Let $\mathfrak{M}$ be a pure, full, $\mathfrak{A}$-invariant subspace of $\mathcal{H}$. As in \cite[Proposition 3.1]{[8]}, the subspace $\mathfrak{M}$ has the following properties:

(i) $\mathfrak{M} \subset \mathfrak{M}$, 
(ii) $\bigcap_{k > 0} v^k \mathfrak{M} = \{0\}$, 
(iii) $\bigcup_{k < 0} v^k \mathfrak{M} = \mathcal{H}$.

Putting $\mathfrak{F} = \mathfrak{M} \oplus v\mathfrak{M}$, we have the decomposition of the Hilbert space $\mathcal{H}$:

$$\mathcal{H} = \sum_{n=-\infty}^{\infty} \oplus v^n \mathfrak{F}.$$ 

Let $P_n$ be the projection from $\mathcal{H}$ onto $v^n \mathfrak{F}$. Since $\mathfrak{F}$ is $N$-invariant and $vNv^* = N$, it follows that $P_n$ belongs to the commutant of $N$ and $\sum_{n=-\infty}^{\infty} P_n = I$. We define the one-parameter unitary group $\{V_t\}_{t \in \mathbb{T}}$ in the commutant of $N$ defined by

$$V_t = \sum_{n=-\infty}^{\infty} e^{int} P_n \quad (\forall t \in \mathbb{T}).$$

For each $t \in \mathbb{T}$, we see that

$$vV_t v^* = \sum_{n=-\infty}^{\infty} e^{int} vP_n v^* = \sum_{n=-\infty}^{\infty} e^{int} P_{n+1}$$

$$= \sum_{n=-\infty}^{\infty} e^{i(n-1)t} P_n = e^{-it} \sum_{n=-\infty}^{\infty} e^{int} P_n = e^{-it} V_t.$$

Setting $\gamma_t(x) = V_t^* x V_t$ for each $t \in \mathbb{T}$ and $x \in M$, we see that $\{\gamma_t\}_{t \in \mathbb{T}}$ is a $*$-automorphism group of $M$ such that $\gamma_t(v) = e^{-it} v \, (\forall t \in \mathbb{T})$. By \cite[19.9 Theorem]{[15]}, we have this proposition. \hfill $\square$

We now fix a pure, full, $\mathfrak{A}$-invariant subspace $\mathfrak{M}$ of $\mathcal{H}$. Then, by Theorem 2.2, there exist a $*$-automorphism group $\{\gamma_t\}_{t \in \mathbb{T}}$ of $M$ and a $*$-isomorphism $\Phi$ from $M$ onto $N \rtimes_\alpha \mathbb{Z}$ such that

$$\Phi(x) = \pi_\alpha(x) \, (\forall x \in N), \quad \Phi(v) = S \quad \text{and} \quad \Phi \circ \gamma_t = \tilde{\alpha}_t \circ \Phi \, (\forall t \in \mathbb{T}).$$

On the other hand, we take another pure, full, $\mathfrak{A}$-invariant subspace $\mathfrak{N}$ of $\mathcal{H}$. As in the proof of Theorem 2.2, there exists a unitary group $\{W_t\}_{t \in \mathbb{T}}$ in the commutant of $N$ associated with $\mathfrak{N}$. Put $\rho_t(x) = W_t x W_t^*$ $(\forall x \in M)$. By Theorem 2.2, there exists a $*$-isomorphism $\Psi$ from $M$ onto $N \rtimes_\alpha \mathbb{Z}$ such that

$$\Psi(x) = \pi_\alpha(x) \, (\forall x \in N), \quad \Psi(v) = S \quad \text{and} \quad \Psi \circ \rho_t = \tilde{\alpha}_t \circ \Psi \, (\forall t \in \mathbb{T}).$$

Therefore, we have

$$\Phi \circ \gamma_t \circ \Phi^{-1} = \tilde{\alpha}_t = \Psi \circ \rho_t \circ \Psi^{-1} \quad (\forall t \in \mathbb{T}).$$

Since $\Phi^{-1} \circ \Psi$ is the identity map on $M$, we see that $\gamma_t = \rho_t \, (\forall t \in \mathbb{T})$ and so $V_t x V_t^* = W_t x W_t^* \, (\forall x \in M, \forall t \in \mathbb{T})$. Putting $A_t = W_t^* V_t \, (\forall t \in \mathbb{T})$, then $A_t$ is the unitary operator in the commutant of $M$ and, for all $s, t \in \mathbb{T}$, we have

$$A_t V_t^* A_s V_t = W_t^* V_t^* W_t^* V_s V_t = W_{t+s}^* V_{t+s} = A_{t+s}.$$
Thus, we have
\begin{equation}
A_{t+s} = A_t A_{\gamma^{-1}(A_s)} \quad (\forall s, t \in \mathbb{T}).
\end{equation}

The unitary family \( \{A_t\}_{t \in \mathbb{T}} \) in the commutant of \( M \) satisfying (2.1) is called a cocycle with respect to \( \mathfrak{M} \). Therefore we have

**Theorem 2.3.** Let \( M \) be a von Neumann algebra acting on a Hilbert space \( \mathcal{H} \) generated by a von Neumann algebra \( N \) and an unitary operator \( v \) satisfying \( vNv^* = N \) and let \( \mathfrak{A} \) be the \( \sigma \)-weakly closed subalgebra of \( M \) generated by \( N \) and non-negative powers of \( v \). Let \( \mathfrak{M} \) be a pure, full, \( \mathfrak{A} \)-invariant subspace of \( \mathcal{H} \). If \( \mathfrak{M} \) is another pure, full, \( \mathfrak{A} \)-invariant subspace of \( \mathcal{H} \), then there exists a cocycle \( \{A_t\}_{t \in \mathbb{T}} \) with respect to \( \mathfrak{M} \). Conversely, if \( \{A_t\}_{t \in \mathbb{T}} \) is a cocycle with respect to \( \mathfrak{M} \), then there exists a pure full \( \mathfrak{A} \)-invariant subspace of \( \mathcal{H} \) with the cocycle \( \{A_t\}_{t \in \mathbb{T}} \).

**Proof.** We only prove the converse. Assume that \( \{A_t\}_{t \in \mathbb{T}} \) is a cocycle with respect to \( \mathfrak{M} \). Put \( W_t = V_t^* A_t V_t \ (\forall t \in \mathbb{T}) \). Then we can easily check that \( \{W_t\}_{t \in \mathbb{T}} \) is a unitary group in the commutant of \( N \). Let \( W_t = \sum_{n=-\infty}^{\infty} e^{-i n t} Q_n \ (\forall t \in \mathbb{T}) \) be the spectral decomposition of \( W_t \). Putting \( \mathfrak{M} = \sum_{n=0}^{\infty} \mathfrak{B} Q_n \mathcal{H} \), then \( \mathfrak{M} \) is a pure, full, \( \mathfrak{A} \)-invariant subspace of \( \mathcal{H} \). In fact, for each \( x \in N \) and \( \xi \in \mathcal{H} \), we have \( xQ_n \xi \in \mathcal{N} \mathcal{H} \) for each \( n \) because \( W_t \) and \( A_t \) belong to \( \mathcal{N} \). Moreover, we see that
\begin{align*}
W_t vQ_n \xi & = V_t^* A_t vQ_n \xi = V_t^* v A_t Q_n \xi = e^{-it} v V_t^* A_t Q_n \xi \\
& = e^{-it} v W_t Q_n \xi = e^{-i(n+1)t} v Q_n \xi.
\end{align*}

It follows that \( v Q_n \xi \in \mathcal{Q}_{n+1} \mathcal{H} \ (\forall n \in \mathbb{N}) \). Therefore \( \mathfrak{M} \) is \( \mathfrak{A} \)-invariant. This completes the proof. \( \square \)

We next consider the case that \( G = \mathbb{R} \). Let \( \mathcal{H} \) be a von Neumann algebra acting on a Hilbert space \( \mathcal{H} \) and let \( \{u_t\}_{t \in \mathbb{R}} \) be a strongly continuous one-parameter unitary group on \( \mathcal{H} \) satisfying the condition \( u_t N u_t^* = N \ (\forall t \in \mathbb{R}) \). Let \( \mathcal{L}_0 \) be the von Neumann algebra generated by \( N \) and \( \{u_t\}_{t \in \mathbb{R}} \), and let \( \mathfrak{B} \) be the \( \sigma \)-weakly closed subalgebra of \( \mathcal{L}_0 \) generated by \( N \) and \( \{u_t\}_{t \geq 0} \).

**Definition 2.4.** Let \( \mathfrak{M} \) be a closed subspace of \( \mathcal{H} \). We shall say that \( \mathfrak{M} \) is: \( \mathfrak{B} \)-invariant, if \( \mathfrak{B} \mathfrak{M} \subset \mathfrak{M} \); reducing, if \( \mathcal{L}_0 \mathfrak{M} \subset \mathfrak{M} \); pure, if \( \mathfrak{M} \) contains no non-trivial reducing subspace; and full, if the smallest reducing subspace containing \( \mathfrak{M} \) is all of \( \mathcal{H} \).

Since \( u_t \) and \( N \) satisfy the condition \( u_t N u_t^* = N \ (\forall t \in \mathbb{R}) \), we can define the \( \sigma \)-weakly continuous \( * \)-automorphism \( \beta_t \) of \( N \) implemented by the unitary operator \( u_t \ (\forall t \in \mathbb{R}) \). Recall that the continuous crossed product \( N \rtimes_{\beta} \mathbb{R} \) of \( N \) by \( \{\beta_t\}_{t \in \mathbb{R}} \) is the von Neumann algebra acting on a Hilbert space \( L^2(\mathbb{R}, \mathcal{H}) \) generated by the operators \( \pi_{\beta}(x) \) and \( \lambda(t) \) defined by the equations, for \( \forall x \in N \),
\begin{align*}
\{\pi_{\beta}(x)\xi\}(t) & = \beta_{-t}(x)\xi(t) \quad (\forall \xi \in L^2(\mathbb{R}, \mathcal{H}), \ \forall t \in \mathbb{R}) \\
\{\lambda(t)\xi\}(s) & = \xi(s-t) \quad (\forall \xi \in L^2(\mathbb{R}, \mathcal{H}), \ \forall s, t \in \mathbb{R}).
\end{align*}

The analytic crossed product \( N \rtimes_{\beta} \mathbb{R} \) determined by \( N \) and \( \{\beta_t\}_{t \in \mathbb{R}} \) is defined to be the \( \sigma \)-weakly closed subalgebra of \( N \rtimes_{\beta} \mathbb{R} \) generated by \( \pi_{\beta}(N) \) and \( \{\lambda(t)\}_{t \geq 0} \).
Theorem 2.5. Let $M_0$ be the von Neumann algebra generated by $N$ and a unitary group $\{u_t\}_{t \in \mathbb{R}}$ satisfying $u_t N u_t^* = N (\forall t \in \mathbb{R})$ and let $\mathfrak{B}$ be the $\sigma$-weakly closed subalgebra of $M_0$ generated by $N$ and $\{u_t\}_{t \geq 0}$. If there exists a pure, full, $\mathfrak{B}$-invariant subspace $\mathfrak{M}$ of $\mathcal{H}$, then there exist a one-parameter group $\{\theta_t\}_{t \in \mathbb{R}}$ of $*$-automorphisms on $M_0$ and a $*$-isomorphism $\Theta$ from $M_0$ onto $N \rtimes_{\beta} \mathbb{R}$ such that

$$\Theta(x) = \pi_\beta(x) (\forall x \in N), \quad \Theta(u_t) = \lambda(t) \quad \text{and} \quad \Theta \circ \theta_t = \hat{\beta}_t \circ \Theta (\forall t \in \mathbb{R}),$$

where $\{\hat{\beta}_t\}_{t \in \mathbb{R}}$ is the $*$-automorphism of $N \rtimes_{\beta} \mathbb{R}$ which is dual to $\{\beta_t\}_{t \in \mathbb{R}}$. Further $\Theta$ maps $\mathfrak{A}$ onto $N \rtimes_{\beta} \mathbb{R}_+$. 

Proof. Let $\mathfrak{M}$ be a pure, full, $\mathfrak{B}$-invariant subspace of $\mathcal{H}$. Then it is clear that $\mathfrak{M}$ has the following properties:

(i) $\mathfrak{B} \mathfrak{M} \subset \mathfrak{M}$, (ii) $\bigcap_{t>0} u_t \mathfrak{M} = \{0\}$, (iii) $\bigcup_{t<0} u_t \mathfrak{M} = \mathcal{H}$.

Let $P_t$ be the projection from $\mathcal{H}$ onto $u_t \mathfrak{M}$ ($\forall t \in \mathbb{R}$). Since $N \mathfrak{M} \subset \mathfrak{M}$ and $N = u_t N u_t^*$, for each $t \in \mathbb{R}$, $u_t \mathfrak{M}$ is $N$-invariant, and so $P_t$ belongs to $N'$. Since $\mathfrak{M}$ is pure and full, we can easily check that the projections $\{P_t\}_{t \in \mathbb{R}}$ are a spectral family. Thus, we obtain the strongly continuous unitary group $\{U_t\}_{t \in \mathbb{R}}$ of $N'$ defined by

$$U_t = \int_{\mathbb{R}} e^{-i\lambda t} dP_\lambda \quad (\forall t \in \mathbb{R}).$$

Since $u_s P_\lambda u_s = u_{\lambda-s}$, we have for every $s, t \in \mathbb{R}$,

$$u_s U_t u_s = \int_{\mathbb{R}} e^{-i\lambda t} dP_{\lambda-s} = \int_{\mathbb{R}} e^{-i\lambda t} dP_{\lambda-s}$$

$$= e^{-ist} \int_{\mathbb{R}} e^{-i\lambda t} dP_\lambda = e^{-ist} U_t.$$ 

Therefore $U_t x U_t^*$ $= x$ ($\forall x \in N \forall t \in \mathbb{R}$) and $U_t u_s U_t^* = e^{-ist} u_s$ ($\forall s, t \in \mathbb{R}$). Thus the $*$-automorphism group $\{\theta_t\}_{t \in \mathbb{R}}$ of $M_0$ defined by $\theta_t(x) = U_t x U_t^*$ ($\forall x \in M_0$, $\forall t \in \mathbb{R}$) satisfies $\theta_t(u_s) = e^{-ist} u_s$ ($\forall s, t \in \mathbb{R}$). Therefore we have the proposition from [15, 19.9 Theorem].

We now fix a pure, full, $\mathfrak{B}$-invariant subspace $\mathfrak{M}$ of $\mathcal{H}$. As in the case of $G = \mathbb{Z}$, we can consider the notion of cocycle with respect to $\mathfrak{M}$. By Theorem 2.5, there exist a one-parameter group $\{\theta_t\}_{t \in \mathbb{R}}$ of $*$-automorphisms on $M_0$ which is implemented by the unitary group $\{U_t\}_{t \in \mathbb{R}}$ and a $*$-isomorphism $\Theta$ from $M_0$ onto $N \rtimes_{\beta} \mathbb{R}$ such that

$$\Theta(x) = \pi_\beta(x) (\forall x \in N), \quad \Theta(u_t) = \lambda(t) \quad \text{and} \quad \Theta \circ \theta_t = \hat{\beta}_t \circ \Theta (\forall t \in \mathbb{R}).$$

We take another pure, full, $\mathfrak{B}$-invariant subspace $\mathfrak{N}$ of $\mathcal{H}$. As in the proof of Theorem 2.5, there exists a one-parameter unitary group $\{W_t\}_{t \in \mathbb{R}}$ associated with $\mathfrak{N}$. Put $\sigma_t(x) = W_t x W_t^*$ for any $x \in M_0$. Then, by Theorem 2.5, there exists a $*$-isomorphism $\Pi$ from $M_0$ onto $N \rtimes_{\beta} \mathbb{R}$ such that

$$\Pi(x) = \pi_\beta(x) (\forall x \in N), \quad \Pi(u_t) = \lambda(t) \quad \text{and} \quad \Pi \circ \sigma_t = \hat{\beta}_t \circ \Pi (\forall t \in \mathbb{R}).$$

Put $B_t = W_t^* U_t$ ($\forall t \in \mathbb{R}$). Then $B_t$ is a unitary operator in the commutant of $M_0$ satisfying $B_{t+s} = B_t \theta_{-t}(B_s)$ ($\forall s, t \in \mathbb{R}$). We shall say that the unitary family $\{B_t\}_{t \in \mathbb{R}}$ is a cocycle with respect to $\mathfrak{M}$. As in Theorem 2.3, we have the following:
Theorem 2.6. Let $M_0$ be the von Neumann algebra generated by $N$ and a unitary group $\{u_t\}_{t \in \mathbb{R}}$ satisfying $u_t N u_t^* = N (\forall t \in \mathbb{R})$ and let $\mathfrak{B}$ be the $\sigma$-weakly closed subalgebra of $M_0$ generated by $N$ and $\{u_t\}_{t \geq 0}$. Let $\mathfrak{M}$ be a pure, full, $\mathfrak{B}$-invariant subspace of $\mathcal{H}$. If we take another pure, full, $\mathfrak{B}$-invariant subspace $\mathfrak{M}$ of $\mathcal{H}$, then there exists a cocycle $\{B_t\}_{t \in \mathbb{R}}$ with respect to $\mathfrak{M}$. Conversely, if $\{B_t\}_{t \in \mathbb{R}}$ is a cocycle with respect to $\mathfrak{M}$, then there exists a pure, full, $\mathfrak{B}$-invariant subspace of $\mathcal{H}$ with the cocycle $\{B_t\}_{t \in \mathbb{R}}$.

3. Representation of the continuous case $G = \mathbb{R}$

Let $N$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\{u_t\}_{t \in \mathbb{R}}$ be a strongly continuous one-parameter unitary group on $\mathcal{H}$. We consider the von Neumann algebra $M_0$ generated by $N$ and $\{u_t\}_{t \in \mathbb{R}}$, and the $\sigma$-weakly closed subalgebra $\mathfrak{B}$ of $M_0$ generated by $N$ and $\{u_t\}_{t \geq 0}$. Let $A$ be the infinitesimal generator of $\{u_t\}_{t \in \mathbb{R}}$ defined by

$$A \xi = \lim_{t \to 0^+} \frac{u_t \xi - \xi}{t} \quad (\forall \xi \in D(A)),$$

where $D(A)$ is the set of all elements for which the limit exists. It is well-known that the Cayley transform $v$ of $A$, that is, $v = (I + A)(I - A)^{-1}$, is a unitary operator on $\mathcal{H}$. For the unitary operator $v$, let $M$ be the von Neumann algebra generated by $N$ and $v$, and let $\mathfrak{A}$ be the $\sigma$-weakly closed subalgebra generated by $N$ and the non-negative powers of $v$ as in §2.

The next proposition embodies an important idea of [6] and is the key result of our approach. For completeness, we give the proof.

Proposition 3.1. Keep the notations as above. Let $\mathfrak{M}$ be a closed subspace of $\mathcal{H}$. Then $\mathfrak{M}$ is $\mathfrak{A}$-invariant if and only if $\mathfrak{M}$ is $\mathfrak{B}$-invariant.

Proof. We only need to prove that a closed subspace $\mathfrak{M}$ of $\mathcal{H}$ is $v$-invariant if and only if $\mathfrak{M}$ is $u_t$-invariant for all $t > 0$. Let $A$ be a infinitesimal generator of $\{u_t\}_{t \in \mathbb{R}}$. Setting $R(\lambda, A) = (\lambda I - A)^{-1}$, we see that

$$v = R(1, A) + AR(1, A) = 2R(1, A) - I,$$

and making use of the Laplace transform representation of $R(1, A)$, we have

$$v \xi = 2 \int_0^\infty e^{-t} u_t \xi \ dt - \xi \quad (\forall \xi \in \mathcal{H}).$$

Let $\xi \in \mathfrak{M}$. Since $u_t \xi \in \mathfrak{M}$ for all $t > 0$, we have $v \xi \in \mathfrak{M}$.

To prove the converse, we first show that $R(\lambda, A)\mathfrak{M} \subset \mathfrak{M}$ for all $\lambda > 0$. Now the resolvent is analytic on the resolvent set and can be expanded in a power series as follows:

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n \{R(\lambda_0, A)\}^{n+1},$$

valid for $|\lambda_0 - \lambda| |R(\lambda_0, A)| < 1$. For $\lambda_0 > 0$, we have $|R(\lambda_0, A)| \leq \frac{1}{\lambda_0}$ so that the above series holds for $|\lambda - \lambda_0| < \lambda_0$. It follows from this expansion that $R(\lambda_0, A)\mathfrak{M} \subset \mathfrak{M}$ implies $R(\lambda, A)\mathfrak{M} \subset \mathfrak{M}$ for all $|\lambda - \lambda_0| < \lambda_0$. Assuming $v\mathfrak{M} \subset \mathfrak{M}$, one infers from (3.1) that $R(1, A)\mathfrak{M} \subset \mathfrak{M}$ and hence by a stepwise process using (3.3) that
\( R(\lambda, A)M \subset \mathcal{M} \) for all \( \lambda > 0 \). Hence for \( \xi \) in \( \mathcal{M} \) and \( \eta \) in the orthogonal complement of \( \mathcal{M} \)

\[
0 = \langle R(\lambda, A)\xi, \eta \rangle = \int_0^\infty e^{-\lambda t} \langle u_t \xi, \eta \rangle \, dt \quad (\forall \lambda > 0).
\]

By the Laplace transform uniqueness theorem, we have \( \langle u_t \xi, \eta \rangle = 0 \) and hence \( u_t \mathcal{M} \subset \mathcal{M} \) for all \( t > 0 \). This completes the proof. \( \square \)

From Proposition 3.1, we have the following:

**Proposition 3.2.** Keep the notation as above. Then

(i) \( \mathcal{M} = \mathcal{M}_0 \). Moreover, if \( \mathcal{M}_0 \) has a separating vector, then \( \mathfrak{A} = \mathfrak{B} \).

(ii) A closed subspace \( \mathcal{M} \) of \( \mathcal{H} \) is pure, full, \( \mathfrak{A} \)-invariant if and only if \( \mathcal{M} \) is pure, full, \( \mathfrak{B} \)-invariant.

**Proof.** We only prove (i). By Proposition 3.1, a closed subspace \( \mathcal{M} \) is reducing for \( \mathcal{M} \) if and only if \( \mathcal{M} \) is reducing for \( \mathcal{M}_0 \). Hence the commutant of \( \mathcal{M} \) is equal to the commutant of \( \mathcal{M}_0 \), and so \( \mathcal{M} = \mathcal{M}_0 \).

We next prove that \( \mathfrak{A} = \mathfrak{B} \). To do this, we need the following notations. If \( \mathfrak{C} \) is an algebra of \( \mathcal{M}_0 \) and \( \mathfrak{L} \) is a lattice of projections in \( B(\mathcal{H}) \), then we write

\[
\text{Lat}\mathfrak{C} = \{ P \in B(\mathcal{H})_p \mid (I - P)TP = 0, \forall T \in \mathfrak{C} \}
\]

and

\[
\text{AlgLat}\mathfrak{L} = \{ T \in B(\mathcal{H}) \mid (I - P)TP = 0, \forall P \in \mathfrak{L} \},
\]

where \( B(\mathcal{H})_p \) is the set of all projections in \( B(\mathcal{H}) \).

By Proposition 3.1, it is clear that \( \text{Lat}\mathfrak{A} = \text{Lat}\mathfrak{B} \). Since \( \text{AlgLat}\mathfrak{B} \) contains \( \mathfrak{B} \), we have the following inclusions:

\[
\mathfrak{A} \subset \mathfrak{B} \subset \text{AlgLat}\mathfrak{A}.
\]

If \( \mathfrak{A} \subset \text{AlgLat}\mathfrak{A} \), then there exists a non-zero element \( x \in \text{AlgLat}\mathfrak{A} \) such that \( x \notin \mathfrak{A} \). Hence there is a normal linear functional \( \phi \) in the predual \( (\mathcal{M}_0)^* \) of \( \mathcal{M}_0 \) such that \( \phi(x) = 1 \) and \( \phi|\mathfrak{A} = 0 \). Since \( \mathcal{M}_0 \) has a separating vector, by [7, Corollary 1.13.7], there are non-zero vectors \( \xi \) and \( \eta \) in \( \mathcal{H} \) such that \( \phi(\xi) = \langle y\xi, \eta \rangle \) \( (\forall y \in \mathcal{M}) \). Hence, for each \( y \in \mathfrak{A} \), we have

\[
\langle y\xi, \eta \rangle = \phi(y) = 0.
\]

This implies that \( [\mathfrak{A}\xi] \perp \eta \), where \( [\mathfrak{A}\xi] \) denotes the closed subspace of \( \mathcal{H} \) spanned by \( \mathfrak{A}\xi \). Since \( [\mathfrak{A}\xi] \subset \text{Lat}\mathfrak{A} \) and \( x \in \text{AlgLat}\mathfrak{A} \), \( x\xi \) belongs to \( [\mathfrak{A}\xi] \). This implies that

\[
1 = \phi(x) = \langle x\xi, \eta \rangle = 0.
\]

This is a contradiction and so \( \mathfrak{A} = \mathfrak{B} \). This completes the proof. \( \square \)

We remark that if the commutant of a von Neumann algebra \( \mathcal{M}_0 \) is properly infinite, then \( \mathcal{M}_0 \) always has a separating vector (cf. [1, Corollary 11]).

We now consider the case that \( \mathfrak{N} \) and \( u_t \) satisfy the condition \( u_t \mathfrak{N}u_t^* = \mathfrak{N} \) \( (\forall t \in \mathbb{R}) \). If there exists a pure, full, \( \mathfrak{B} \)-invariant subspace \( \mathcal{M} \) of \( \mathcal{H} \), then, by Theorem 2.5, \( \mathcal{M}_0 \) is \( * \)-isomorphic to a continuous crossed product. Moreover, from
onto

However, Katayama showed in [4, Theorem 3.5] that if there exists a \( t \in \mathbb{R} \) such that \((N_{t}, \varphi_{N})\) does not satisfy the condition \( \varphi_{N}^{\ast} = N \), then \( M \) is \( * \)-isomorphic to a discrete crossed product. So it is natural to ask when \( N \) and \( \varphi \) satisfy the condition \( \varphi_{N}^{\ast} = N \). Recall that the unitary operator \( \varphi \) has the following form:

\[
\varphi_{t} = 2 \int_{0}^{\infty} e^{-t} u_{t} \xi dt - \xi \quad (\forall \xi \in \mathcal{H}).
\]

It is clear that if, for every \( t \in \mathbb{R} \), \( u_{t} \) belongs to the commutant of \( N \), then \( \varphi \) is also in \( N' \). In this case, \( \varphi \) and \( N \) satisfy the condition \( \varphi_{N}^{\ast} = N \). But, in general, \( \varphi \) and \( N \) do not satisfy the condition \( \varphi_{N}^{\ast} = N \). In fact, we can give the following:

**Example 3.3.** Let \( N \) be a von Neumann algebra acting on a Hilbert space \( \mathcal{H} \) and let \( \{ \beta_{t} \}_{t \in \mathbb{R}} \) be a \(*\)-automorphism group of \( N \) such that there exists a \( t_{0} \in \mathbb{R} \) such that \( \beta_{t_{0}} \) is outer. Recall that a continuous crossed product \( N \rtimes_{\beta_{t}} \mathbb{R} \) is the von Neumann algebra generated by \( \pi_{\beta}(N) \) and \( \{ \lambda(t) \}_{t \in \mathbb{R}} \). For each \( x \in N \) and \( t \in \mathbb{R} \), it is clear that

\[
\pi_{\beta}(\beta_{t}(x)) = \lambda(t) \pi_{\beta}(x) \lambda(t)^{\ast}
\]

and

\[
\pi_{\beta}(N) = \lambda(t) \pi_{\beta}(N) \lambda(t)^{\ast} \quad (\forall t \in \mathbb{R}).
\]

For the unitary group \( \{ \lambda(t) \}_{t \in \mathbb{R}} \), we obtain the unitary operator \( \varphi \) defined by the form

\[
\varphi_{t} = 2 \int_{0}^{\infty} e^{-t} \lambda(t) \xi dt - \xi \quad (\forall \xi \in L^{2}(\mathbb{R}, \mathcal{H})).
\]

By choosing an appropriate representation for \( N \rtimes_{\beta} \mathbb{R} \), we shall assume that \( N \rtimes_{\beta} \mathbb{R} \) has a separating vector. In this case, by Proposition 3.2, the von Neumann algebra generated by \( \pi_{\beta}(N) \) and \( \varphi \) coincides with \( N \rtimes_{\beta} \mathbb{R} \), and the \( \sigma \)-weakly closed subalgebra generated by \( \pi_{\beta}(N) \) and the non-negative powers of \( \varphi \) also coincides with \( N \rtimes_{\beta} \mathbb{R}_{+} \). Hence, by Theorem 2.2, if \( \pi_{\beta}(N) \) and \( \varphi \) satisfy the condition \( \varphi \pi_{\beta}(N) \varphi^{\ast} = \pi_{\beta}(N) \), then there is a \(*\)-isomorphism \( \Phi \) from \( N \rtimes_{\beta} \mathbb{R} \) onto \( N \rtimes_{\alpha} \mathbb{Z} \) such that \( \Phi(N \rtimes_{\beta} \mathbb{R}_{+}) = N \rtimes_{\alpha} \mathbb{Z}_{+} \) for some \(*\)-automorphism \( \alpha \) of \( N \). Since there exists a faithful normal canonical conditional expectation of \( N \rtimes_{\alpha} \mathbb{Z} \) onto \( \pi_{\alpha}(N) \) (cf. [8–10]), there is a faithful normal conditional expectation of \( N \rtimes_{\beta} \mathbb{R} \) onto \( \pi_{\beta}(N) \). However, Katayama showed in [4] Theorem 3.5 that if there exists a \( t_{0} \in \mathbb{R} \) such that \( \beta_{t_{0}} \) is outer, then there does not exist any normal conditional expectation of \( N \rtimes_{\beta} \mathbb{R} \) onto \( \pi_{\beta}(N) \), which is a contradiction. Hence, \( \pi_{\beta}(N) \) and \( \varphi \) do not satisfy the condition \( \varphi \pi_{\beta}(N) \varphi^{\ast} = \pi_{\beta}(N) \).

Finally, we discuss the relation between a continuous crossed product and a discrete crossed product.

**Theorem 3.4.** If a crossed product \( N \rtimes_{\beta} \mathbb{R} \) admits a separating vector (for example, \( N \rtimes_{\beta} \mathbb{R} \) is properly infinite), then the following two conditions are equivalent:

(i) There exists a \(*\)-isomorphism \( \Phi \) from \( N \rtimes_{\beta} \mathbb{R} \) onto \( N \rtimes_{\alpha} \mathbb{Z} \) such that \( \Phi(N \rtimes_{\beta} \mathbb{R}_{+}) = N \rtimes_{\alpha} \mathbb{Z}_{+} \) for some \(*\)-automorphism \( \alpha \) of \( N \).

(ii) \( \beta_{t} \) is inner for all \( t \in \mathbb{R} \).

**Proof.** (i) \( \Rightarrow \) (ii) Since \( \Phi \) is the \(*\)-isomorphism satisfying \( \Phi(N \rtimes_{\beta} \mathbb{R}_{+}) = N \rtimes_{\alpha} \mathbb{Z}_{+} \), we have \( \Phi(\pi_{\beta}(N)) = \pi_{\alpha}(N) \). Since there exists a normal conditional expectation of \( N \rtimes_{\alpha} \mathbb{Z} \) onto \( \pi_{\alpha}(N) \), there also exists a normal conditional expectation of \( N \rtimes_{\beta} \mathbb{R} \) onto \( \pi_{\beta}(N) \). Hence, by [4] Theorem 3.6, we have that \( \beta_{t} \) is inner for each \( t \in \mathbb{R} \).
(ii) ⇒ (i) Since $\beta t$ is inner for all $t \in \mathbb{R}$, there exists a unitary operator $v_t$ in $N$ such that $\beta_t$ is implemented by the unitary operator $v_t$. Putting $u_t = \lambda(t)\pi_\beta(v_t)^*$ for all $t \in \mathbb{R}$, we can show that $\{u_t\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group in $\pi_\beta(N)'$. Moreover, we see that the von Neumann algebra generated by $\pi_\beta(N)$ and $\{u_t\}_{t \in \mathbb{R}}$ is equal to $N \rtimes_{\beta} \mathbb{R}$. Similarly, the $\sigma$-weakly closed subalgebra of $N \rtimes_{\beta} \mathbb{R}$ generated by $\pi_\beta(N)$ and $\{u_t\}_{t > 0}$ is $N \rtimes_{\beta} \mathbb{R}^+$. For the unitary group $\{u_t\}_{t \in \mathbb{R}}$, we can construct the unitary operator $v$ on $L^2(\mathbb{R}, \mathcal{H})$ as follows:

$$v\xi = 2\int_0^\infty e^{-t}u_t\xi dt - \xi \quad (\forall \xi \in L^2(\mathbb{R}, \mathcal{H}))$$

Since $N \rtimes_{\beta} \mathbb{R}$ admits a separating vector, by Proposition 3.2, $N \rtimes_{\beta} \mathbb{R}$ is also generated by $\pi_\beta(N)$ and $v$, and $N \rtimes_{\beta} \mathbb{R}^+$ is generated by $\pi_\beta(N)$ and the non-negative powers of $v$. Since, for every $t \in \mathbb{R}$, $u_t$ belongs to the commutant of $\pi_\beta(N)$, $v$ is also in $\pi_\beta(N)'$. Thus $v$ and $\pi_\beta(N)$ satisfy the condition $v\pi_\beta(N)v^* = \pi_\beta(N)$. Putting $\mathcal{M} = L^2(\mathbb{R}^+, \mathcal{H})$, it is clear that $\mathcal{M}$ is a pure, full, $N \rtimes_{\beta} \mathbb{R}^+$-invariant subspace of $L^2(\mathbb{R}, \mathcal{H})$. For each $t \in \mathbb{R}$, we have

$$u_t\mathcal{M} = \lambda(t)\pi_\beta(v_t)^*\mathcal{M} \subset \lambda(t)\mathcal{M} = \pi_\beta(v_t)^*\pi_\beta(v_t)\mathcal{M} \subset u_t\mathcal{M}.$$ 

It follows that $u_t\mathcal{M} = (\lambda(t)\mathcal{M})$ ($\forall t \in \mathbb{R}$), and so $\mathcal{M}$ is pure and full for $\{u_t\}_{t \in \mathbb{R}}$. Therefore, by Proposition 3.2, $\mathcal{M}$ is also pure and full for $v$. Thus, by Theorem 2.2, we have (i). This completes the proof. 

\[\square\]

References


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