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METACOMPACT SUBSPACES OF PRODUCTS OF ORDINALS

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Abstract. Let X be a subspace of the product of finitely many ordinals. X is countably metacompact, and X is metacompact iff X has no closed subset homeomorphic to a stationary subset of a regular uncountable cardinal. A theorem generalizing these two results is: X is λ-metacompact iff X has no closed subset homeomorphic to a $(\kappa_1, \dots, \kappa_n)$-stationary set where $\kappa_1 < \lambda$.

1. Sources

This paper combines two lines of research. The first is the investigation of countably metacompact subspaces of the product of ordinals by Kemoto and Smith in [7] and [8]. A synthesis of the main theorems from these papers is

Theorem 1.1. *If X is a subspace of the product of two ordinals, or a subspace of ω_1^n, then X is countably metacompact. However, not every subspace of ω_1^ω is countably metacompact.*

The second line is an investigation of D-spaces. Van Douwen and Lutzer [1] proved that a subspace of a linearly ordered space is a D-space iff it is metacompact iff it has no closed subset homeomorphic to a stationary subset of a regular, uncountable cardinal. Stanley ([11] and [3]) proved the same equivalence for subspaces of the product of finitely many ordinals.

Theorem 1.2. *Let X be a subspace of the product of finitely many ordinals. The following are equivalent:*

1. *X is a D-space.*
2. *X is metacompact.*
3. *X is metalindelöf.*
4. *X has no closed subset homeomorphic to a stationary subset of a regular uncountable cardinal.*

Kemoto, Tamano, and Yajima ([9]) proved that metacompactness, screenability, and weak submetalindelöfness are equivalent for subspaces of the product of two ordinals.

Considering these results, it is natural to conjecture first, that every subspace of a finite product of ordinals is countably metacompact, and second, there is a

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unified proof which yields Theorem 1.1, Theorem 1.2, and an analogous statement about λ-metacompactness. We confirm these conjectures by proving the following theorem.

Theorem 1.3. *Let X be a subspace of the product of finitely many ordinals. X is λ-metacompact iff X has no closed subset homeomorphic to a $(\kappa_1, \ldots, \kappa_n)$-stationary set where $\kappa_1 < \lambda$. In particular, $(\lambda = \omega_1)$, X is countably metacompact, and $(\lambda = \infty)$, X is metacompact iff X has no closed subset homeomorphic to a stationary subset of a regular uncountable cardinal.*

2. Metacompactness

Definition 2.1. Let κ be an infinite cardinal, X be a set, and $\mathcal{V}$ a family of subsets of X. We say that $\mathcal{V}$ is *point-$< \kappa$* iff for all $x \in X$, $|\{V \in \mathcal{V} : x \in V\}| < \kappa$. It is usual to say *point-finite* when $\kappa = \omega$, and *point-countable* when $\kappa = \omega_1$.

It is convenient and harmless (in the context of metacompactness) to require that refinements be precise. Also, we frequently use the original family of sets to index itself and subsequent refinements.

Definition 2.2. We say that $\mathcal{V}$ is a *partial refinement* of $\mathcal{U}$ if $\mathcal{V}$ has the form $\{V(U) : U \in \mathcal{U}\}$ and $V(U) \subset U$ for all U. If additionally, $\bigcup \mathcal{V} = \bigcup \mathcal{U}$, we say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$.

Definition 2.3. We say that a space X is *metacompact* if every open cover $\mathcal{U}$ of X has a point finite open refinement $\mathcal{V}$. We say that a space X is *λ-metacompact* if every open cover $\mathcal{U}$ of X, with $|\mathcal{U}| < \lambda$, has a point finite open refinement $\mathcal{V}$. *Countably metacompact* is a synonym of ω_1-metacompact.

To include Theorem 1.2 as a particular case of Theorem 1.3, we wish metacompact to be a particular case of λ-metacompact. Let ∞ be a symbol such that $\lambda < \infty$ for all cardinals λ. Then ∞-metacompact is a synonym for metacompact.

A space X is called *λ-metalindelöf* if every open cover $\mathcal{U}$ of X, with $|\mathcal{U}| < \lambda$, has a point countable open refinement $\mathcal{V}$. The methods of this paper show that a subspace of a finite product of ordinals is λ-metalindelöf iff it is λ-metacompact.

The union of two metacompact spaces need not be metacompact. Consider, for example, the Moore plane (aka Niemytzki plane or the tangent disk space). The upper half-plane and the discrete x-axis are metacompact, but their union, the whole space, is not. In clause 3 of Corollary 2.6, we show that for subspaces of finite products of ordinals, the union of finitely many metacompact spaces is metacompact. (When we finish, Theorem 1.3 with Theorem 3.9 will show that the union of countably many metacompact spaces is metacompact.)

Definition 2.4. Let α be an ordinal number, and let n be a natural number. If $y \in \alpha^n$ and $z \in (\{-1\} \cup \alpha)^n$ we define

$$z < y \quad \text{iff} \quad z_i < y_i \text{ for all } i \leq n,$$

$$z \leq y \quad \text{iff} \quad z_i \leq y_i \text{ for all } i \leq n,$$

$$(z, y] \quad = \quad \{a \in \alpha^n : z < a \leq y\}.$$

Such "intervals" form a basis for the product topology on α^n.

Lemma 2.5. *Let κ be an infinite cardinal and let $(y^\mu : \mu < \kappa)$ be a κ-sequence of n-tuples of ordinals. There is $H \in [\kappa]^\kappa$ such that if $\mu < \nu$ are both in H, then $y_i^\mu \leq y_i^\nu$ for all $i \leq n$.*

Proof. Define $c : [\kappa]^2 \to \{0, 1\}$ as follows: $c(\mu, \nu) = 0$ if $y_i^\mu \leq y_i^\nu$ for all $i \leq n$, and $c(\mu, \nu) = 1$ otherwise. By the Dushnik-Miller-Erdös Theorem [2], either there is $H \in [\kappa]^\kappa$ as desired, or there is $H' \in [\kappa]^\omega$ such that $c(\mu, \nu) = 1$ whenever both μ and ν are in H'. Towards a contradiction, assume the latter. Define $c : [H']^2 \to \{1, 2, \ldots, n\}$ so that if $c(\mu, \nu) = i$, then $y_i^\mu > y_i^\nu$. By Ramsey's Theorem, there is an i and $H'' \in [H']^\omega$ such that $c'(\mu, \nu) = i$ whenever both μ and ν are in H''. Then y_i^μ, $\mu \in H''$, is an infinite strictly decreasing sequence of ordinals. Contradiction! □

Corollary 2.6. *Let α be an ordinal number, let κ be an infinite cardinal, and let n be a natural number.*

1. *Let $\mathcal{V}$ be a point-$< \kappa$ open cover of $Y \subset \alpha^n$. There is $\mathcal{V}'$, a point-$< \kappa$ open in α^n family such that $V = V' \cap Y$ for all $V \in \mathcal{V}$.*
2. *If $Y \subset X \subset \alpha^n$, $\mathcal{U}$ is an open cover of X with $|\mathcal{U}| < \lambda$, and Y is λ-metacompact, then there is $\mathcal{V}''$, a point-$< \kappa$ open in X partial refinement of $\mathcal{U}$ satisfying $Y \subset \bigcup \mathcal{V}''$.*
3. *If $\mathcal{Y}$ is a finite family of λ-metacompact subspaces of α^n, then $\bigcup \mathcal{Y}$ is λ-metacompact.*

Proof. For each $y \in Y$ and $V \in \mathcal{V}$ such that $y \in V$, choose $z(y, V) \in (\{-1\} \cup \alpha)^n$ so that $(z(y, V), y] \cap Y \subset V$. For each $V \in \mathcal{V}$, set $V' = \bigcup_{y \in V} (z(y, V), y]$. Clearly V' is open in α^n and $V' \cap Y = V$. We claim that $\mathcal{V}' = \{V' : V \in \mathcal{V}\}$ is point-$< \kappa$. Towards a contradiction, assume that there are κ-sequences $(V^\mu : \mu < \kappa)$ and $(y^\mu : \mu < \kappa)$ and an $a \in \alpha^n$ satisfying

$$z(y^\mu, V^\mu)_i < a_i \leq y_i^\mu \text{ for all } \mu < \kappa \text{ and all } i \leq n.$$

Apply Lemma 2.5 to $(y^\mu : \mu < \kappa)$ to get H. Let $\nu \in H$. Then y^ν witnesses that $\mathcal{V}$ is not point-$< \kappa$. Contradiction!

For clause 2, let Y, X, and $\mathcal{U}$ be as hypothesized. For each $U \in \mathcal{U}$, set $U' = U \cap Y$. Because Y is λ-metacompact, there is a point-finite open (in Y) refinement of $\{U' : U \in \mathcal{U}\}$. Apply clause 1 to get $\mathcal{V}'$. For each $U \in \mathcal{U}$, set $V''(U) = V'(U) \cap U$. Then $\{V''(U) : U \in \mathcal{U}\}$ is the desired partial refinement.

Clause 3 follows from clause 2. □

Example 2.7. Let Y be a subset of a regular, uncountable cardinal κ. Let $\mathcal{U}$ be an open cover of Y by bounded open sets (for example, $\{[0, y] \cap Y : y \in Y\}$). It is easy to verify that if $\mathcal{U}$ has a point-$< \kappa$ open refinement, then Y is not stationary in κ. Contrapositively, if Y is stationary, then Y is not metacompact. If a space X has a closed subset homeomorphic to a stationary subset Y of κ, then X is not.

If we assume that X is a subspace of an ordinal, the converse of the preceding sentence is true (it is a special case of Theorem 1.3), and seems to be folklore. Theorem 1.1 and Theorem 1.2 suggest that this converse extends to subspaces of the product of finitely many ordinals. The following example shows that for $\omega_1 < \lambda < \infty$, the one dimensional notion of stationary is not sufficient.

Example 2.8. Let $\kappa_1 < \kappa_2$ be regular, uncountable cardinals. Let $\{A_\zeta : \zeta \in \kappa_1\}$ be a family of stationary subsets of κ_2. Set $X = \{(\zeta, \nu) \in \kappa_1 \times \kappa_2 : \nu \in A_\zeta \wedge \zeta \in \kappa_1\}$.

If we choose $\{A_\zeta : \zeta \in \kappa_1\}$ to be pairwise disjoint sets of ordinals of cofinality at least κ_1, then X has no subspace homeomorphic to a stationary subset of κ_1.

For $\theta < \kappa_1$, set $U_\theta = \{(\zeta, \nu) \in X : \zeta \leq \theta\}$. Then $\mathcal{U} = \{U_\theta : \theta \in \kappa_1\}$ is an open cover of X with $|\mathcal{U}| = \kappa_1$. Let $\mathcal{V} = \{V_\theta : \theta \in \kappa_1\}$ be an open refinement of $\mathcal{U}$. We will show that $\mathcal{V}$ is not point-$< \kappa_1$. For each $(\zeta, \nu) \in X$, choose $\theta(\zeta, \nu) < \kappa_1$, $\eta_{\zeta,\nu} < \kappa_1$, and $\mu_{\zeta,\nu} < \kappa_2$ so that

$$\big((\eta_{\zeta,\nu}, \zeta] \times (\mu_{\zeta,\nu}, \nu]\big) \cap X \subset V_{\theta(\zeta,\nu)}.$$

Via the Pressing Down Lemma ([10], Chapter II, Lemma 6.15), for each $\zeta < \kappa_1$ find $\theta(\zeta) < \kappa_1$, $\eta_\zeta < \kappa_1$, $\mu_\zeta < \kappa_2$ and A'_ζ, a stationary subset of A_ζ, so that

$$\big((\eta_\zeta, \zeta] \times (\mu_\zeta, \nu]\big) \cap X \subset V_{\theta(\zeta)}$$

for all $\nu \in A'_\zeta$. Apply the Pressing Down Lemma again to find $\eta < \kappa_1$ and B stationary in κ_1 so that

$$\big((\eta, \zeta] \times (\mu_\zeta, \nu]\big) \cap X \subset V_{\theta(\zeta)}$$

for all $\nu \in A'_\zeta$ and all $\zeta \in B$. Set $\eta^* = \eta + 1$ and $\mu^* = \sup\{\mu_\zeta : \zeta \in B\} + 1$. Then $(\eta^*, \mu^*) \in V_{\theta(\zeta)}$ for all $\zeta \in B$. Recall that $\zeta \leq \theta(\zeta)$, so $|\{\theta(\zeta) : \zeta \in B\}| = \kappa_1$.

3. $(\kappa_1, \ldots, \kappa_n)$-stationary sets

First, we set notation for concatenating n-tuples. If $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_n)$, then $a^\frown b$ is the $(m+n)$-tuple $c = (c_1, \ldots, c_{m+n})$, where $c_i = a_i$ for $i \leq m$ and $c_{m+j} = b_j$ for $j \leq n$.

Definition 3.1. For $A \subset \kappa_1 \times \ldots \times \kappa_n$, we define a tree $T(A)$. The ith level, denoted $l_i(A)$, is defined to be $\{b \in \kappa_1 \times \ldots \times \kappa_i : (\exists c \in \kappa_{i+1} \times \ldots \times \kappa_n)(b^\frown c \in A)\}$. In particular, $l_0(A) = \emptyset$ and $l_n(A) = A$. For $b \in l_i(A)$, let $A_b = \{c \in \kappa_{i+1} \times \ldots \times \kappa_n : b^\frown c \in A)\}$ and set $N(A, b) = \{\xi \in \kappa_{i+1} : b^\frown \xi \in l_{i+1}(A)\}$.

Definition 3.2. Let $(\kappa_1, \ldots, \kappa_n)$ be a nondecreasing sequence of uncountable regular cardinals. We say that $A \subset \kappa_1 \times \ldots \times \kappa_n$ is $(\kappa_1, \ldots, \kappa_n)$-*stationary* iff $N(A, b)$ is stationary in κ_{i+1} for all $i < n$ and $b \in l_i(A)$. If $\kappa_i = \kappa$ for all i, we write κ^n-stationary in place of $(\kappa_1, \ldots, \kappa_n)$ -stationary. Let $\overline{\kappa}^n$ denote $\{a \in \kappa^n : a_1 < \ldots < a_n\}$. It is easy to verify that $A \subset \kappa^n$ is κ^n-stationary iff $A \cap \overline{\kappa}^n$ is κ^n-stationary. Notions equivalent to κ^n-stationary were called stafull in [4] and [5]. Kemoto remarks that even for $A \subset \overline{\omega}_1^2$, this notion differs from inductively stationary of [8]. Sometimes we will say stationary in place of $(\kappa_1, \ldots, \kappa_n)$-stationary.

We will need the following observations.

Lemma 3.3. *Let $X \subset \kappa_1 \times \ldots \times \kappa_n$, and let $T = l_m(X)$ be $(\kappa_1, \ldots, \kappa_m)$-stationary. If for all $t \in T$, X_t is $(\kappa_{m+1}, \ldots, \kappa_n)$-stationary, then X is $(\kappa_1, \ldots, \kappa_n)$-stationary.*

Lemma 3.4. *If $Y \subset \kappa$, then Y is stationary in κ iff Y is (κ)-stationary. If X is $(\kappa_1, \ldots, \kappa_n)$-stationary, then for all $j \leq n$, for all $b \in l_j(X)$, X_b is closed (in X) and homeomorphic to a $(\kappa_j, \ldots, \kappa_n)$-stationary set. However, X need not contain any subset homeomorphic to a stationary subset of κ_1.*

Lemma 3.5. *If Y is $(\kappa_1, \ldots, \kappa_n)$-stationary and C_j is club in κ_j for each j, then $Y \cap (C_1 \times \ldots \times C_n)$ is $(\kappa_1, \ldots, \kappa_n)$-stationary.*

Definition 3.6. For regular cardinals $\rho_1 \leq \rho_2$, an index set $I \subset \rho_1$, and a family $(C_i)_{i\in I}$ of club subsets of ρ_2, set

$$\bigwedge_{i\in I} C_i = \{\gamma \in \rho_2 : (\forall i \in \gamma \cap I)(\gamma \in C_i)\}.$$

We mention two special cases. First, if $\rho_1 < \rho_2$ and $\rho_1 \leq \min C_i$ for all $i \in I$, then $\bigwedge_{i\in I} C_i$ is $\bigcap_{i\in I} C_i$, which is club in ρ_2. Second, if $I = \rho_1 = \rho_2$, then $\bigwedge_{i\in I} C_i$ is the diagonal intersection, which is club in ρ_2. In the general case, $\bigwedge_{i\in I} C_i$ is club in ρ_2 by the "diagonal intersection proof" (see, for example, [10] Ch. II, Lemma 6.14).

In contrast to the familiar $n = 1$ case, if $n > 1$, a superset of a $(\kappa_1, \ldots, \kappa_n)$-stationary set is not necessarily $(\kappa_1, \ldots, \kappa_n)$-stationary. The next lemma explains this situation.

Lemma 3.7. *Let $Y \subset \{a \in \kappa_1 \times \ldots \times \kappa_n : (\forall i < n)(a_i < a_{i+1})\}$. There are $C_1, \ldots, C_n$, with each C_j club in κ_j, such that*

1. *$Y \cap (C_1 \times \ldots \times C_n)$ is $(\kappa_1, \ldots, \kappa_n)$-stationary and closed in Y if Y contains a $(\kappa_1, \ldots, \kappa_n)$-stationary set.*
2. *$Y \cap (C_1 \times \ldots \times C_n) = \emptyset$ if Y contains no $(\kappa_1, \ldots, \kappa_n)$-stationary set.*

Proof. We define $e_Y : T(Y) \to \{0, 1\}$ by induction down the tree. Set $e_Y(t) = 1$ for $t \in l_n(Y) = Y$. For $i < n$ and $t \in l_i(Y)$, set $e_Y(t) = 1$ if $\{\xi \in \kappa : e_Y(t^\frown\xi) = 1\}$ is stationary in κ; set $e_Y(t) = 0$ otherwise.

For each $j \in \{1, \ldots, n\}$, we define $C(j, t)$, club in κ_j, for $t \in l_i(Y)$ by induction from $i = j-1$ to $j = 0$. Let $i = j-1$. If $e_Y(t) = 0$, choose $C(j, t)$ club in κ_j so that $C(j, t) \cap \{\xi \in \kappa_j : e_Y(t^\frown\xi) = 1\} = \emptyset$; if $e_Y(t) = 1$, set $C_t = \kappa$. Next, for $i < j-1$, set $C(j, t) = \bigwedge_{\xi\in N(Y,t)} C(j, t^\frown\xi)$. Then $C(1, \emptyset), \ldots, C(n, \emptyset)$ are the desired club sets. □

Definition 3.8. When c and r are n-tuples of ordinals, let $entw(c, r)$ abbreviate

$$c_1 \leq r_1 < c_2 \leq r_2 < \ldots < c_n \leq r_n.$$

For $C_1, \ldots, C_n$ with each C_j club in κ_j, define

$$E(C_1, \ldots, C_n) = \{r \in \kappa_1 \times \ldots \times \kappa_n : entw(c, r) \text{ for some } c \in C_1 \times \ldots \times C_n\}.$$

The next lemma provides more combinatorics for $(\kappa_1, \ldots, \kappa_n)$-stationary sets. When $\kappa_1, \ldots, \kappa_n$-stationary $= \kappa^n$-stationary, clause 1 is in [4] and [5], clauses 1 and the particular case of 2 are in [8], and clause 2 may be new. Note that when $n = 1$, clause 3 gives the familiar result that an open stationary set contains a final segment.

Theorem 3.9. *Let $Y \subset \{a \in \kappa_1 \times \ldots \times \kappa_n : (\forall i < n)(a_i < a_{i+1})\}$.*

1. *If Y is $(\kappa_1, \ldots, \kappa_n)$-stationary and $f : Y \to S$, where $|S| < \kappa_1$, then there are $s \in S$ and Y', a $(\kappa_1, \ldots, \kappa_n)$-stationary subset of Y, such that $f(a) = s$ for all $a \in Y'$.*
2. *If Y is open in $\kappa_1 \times \ldots \times \kappa_n$ and contains a $(\kappa_1, \ldots, \kappa_n)$-stationary set, A, then there are $C_1, \ldots, C_n$, each C_j club in κ_j, such that $E(C_1, \ldots, C_n) \subset Y$. In particular, $\{a \in C_1 \times \ldots \times C_n : (\forall i < n)(a_i < a_{i+1})\} \subset Y$.*

Proof. Clause 1 follows from Lemma 3.7. Alternately, we could make a direct proof similar to the first paragraph of the proof of Lemma 3.7.

By induction on i from $i = n$ down to $i = 0$, we define b_t for $t \in l_i(A)$ for $0 \leq i \leq n$, also ζ_t and S_t for $t \in l_i(A)$ for $0 \leq i < n$. Because $A \subset Y$ and Y is open in $\kappa_1 \times \ldots \times \kappa_n$, for $t \in l_n(A) = A$ there is $b_t < t$ so that $(b_t, t] \subset Y$. Now let $t \in l_i(A)$, where $i < n$. By applying the Pressing Down Lemma repeatedly, find $b_t \in \kappa^i$, $\zeta_t \in \kappa$, and S_t stationary in κ_{i+1} so that

$$b_{t^\frown\eta} = b_t^\frown \zeta_t \text{ for all } \eta \in S_t.$$

For each $j \in \{1, \ldots, n\}$, we define $C(j,t)$, club in κ_j for $t \in l_i(A)$ by induction from $i = j-1$ to $i = 0$. Let $i = j-1$. For $t \in l_i(A)$, let $C(j,t)$ be the set of $\gamma \in \kappa_j$ satisfying $\gamma = \sup(S_t \cap \gamma)$ and $\zeta_{t^\frown\eta} < \gamma$ for all $\eta < \gamma$. Next, for $i < j-1$, set $C(j,t) = \bigwedge_{\xi \in N(A,t)} C(j, t^\frown\xi)$. Then $(C(1,\emptyset), \ldots, C(n,\emptyset))$ is the desired n-tuple. Note that $\zeta_\emptyset < \min C(1,\emptyset)$.

Now assume $entw(c,r)$ for some $c \in C(1,\emptyset) \times \ldots \times C(n,\emptyset)$. We define t_j by induction from $j = 1$ to $j = n$. For $j = 1$, choose $t_1 \in S_\emptyset$ satisfying $r_1 < t_1 < c_2$. Note that $b_{t_1} = \zeta_\emptyset < c_1 \leq r_1$. If $j < n$ and t_j has been defined, choose $t_{j+1} \in S_{t_1,\ldots,t_j}$ satisfying $r_{j+1} < t_{j+1}$ (and $t_{j+1} < c_{j+2}$ if $j < n-1$). Because $(t_1, \ldots, t_j) \in c^j_{j+1}$, we have $\zeta_{t_1,\ldots,t_j} < c_{j+1} \leq r_{j+1}$. Set $t = (t_1, \ldots, t_n)$. Then $t \in A$ and $r \in (b_t, t] \subset Y$, as claimed. □

Lemma 3.10. *If Y is a $(\kappa_1, \ldots, \kappa_n)$-stationary set, then Y is not λ-metacompact for $\lambda > \kappa_1$.*

Proof. Via Lemma 3.4 and passing to a closed subspace, we may assume that $\kappa_1 < \kappa_2$. For $\theta < \kappa_1$, set $U_\theta = \{y \in Y : y_1 \leq \theta\}$. Then $\mathcal{U} = \{U_\theta : \theta \in \kappa_1\}$ is an open cover of X with $|\mathcal{U}| = \kappa_1$. Let $\mathcal{V} = \{V_\theta : \theta \in \kappa_1\}$ be an open refinement of $\mathcal{U}$. We will show that $\mathcal{V}$ is not point-$< \kappa_1$. For each $\zeta^\frown q \in Y$, choose $\theta(\zeta, q) < \kappa_1$, $\eta_{\zeta,q} < \kappa_1$, and $p_{\zeta,q} \in \kappa_2 \times \ldots \times \kappa_n$ so that

$$((\eta_{\zeta^\frown q}, \zeta] \times (p_{\zeta^\frown q}, q]) \cap Y \subset V_{\theta(\zeta^\frown q)}.$$

Apply Theorem 3.9 for each $\zeta < \kappa_1$ to find $\theta(\zeta) < \kappa_1$, $\eta_\zeta < \kappa_1$, and $C_{2,\zeta}, \ldots, C_{n,\zeta}$ with each $C_{j,\zeta}$ club in κ_j, so that

$$(\eta_\zeta, \zeta] \times E(C_{2,\zeta}, \ldots, C_{n,\zeta}) \cap Y \subset V_{\theta(\zeta)}.$$

Next, find $\eta < \kappa_1$ and B stationary in κ_1 so that $\eta = \eta_\zeta$ for all $\zeta \in B$. For each j, set $C_j = \bigcap_{\zeta \in B} C_{j,\zeta}$, and inductively choose $q_j \in C_j$ with $\eta < q_2 < \ldots < q_n$. Then $\eta^\frown q \in V_{\theta(\zeta)}$ for all $\zeta \in B$. Recall that $\zeta \leq \theta(\zeta)$, so $|\{\theta(\zeta) : \zeta \in B\}| = \kappa_1$. □

Not all results about stationary sets generalize to $(\kappa_1, \ldots, \kappa_n)$-stationary sets.

Example 3.11. Let κ be an uncountable regular cardinal. For each $(\zeta, \nu) \in \overline{\kappa}^2$, define $f(\zeta, \nu) = (0, \zeta)$. Then f "presses down" on a κ^2-stationary set, but f is not constant on a κ^2-stationary set.

To apply κ^n-stationary techniques to arbitrary subspaces of κ^n (instead of subspaces of $\overline{\kappa}^n$), we need another technique from [8]. We decompose κ^n into finitely many pieces, each homeomorphic to $\overline{\kappa}^m$ for some $m \leq n$. For $n = 1$, κ^1 is $\overline{\kappa}^1$. For $n = 2$, the decomposition is $\kappa^2 = Z_0 \cup Z_1 \cup Z_2$, where Z_0 is the diagonal, homeomorphic to $\overline{\kappa}^1$, Z_1 is above the diagonal, which is $\overline{\kappa}^2$, and Z_2 is below the diagonal, which is homeomorphic to $\overline{\kappa}^2$.

For the general case, we define the partition via an equivalence relation. For a, $b \in \kappa^n$, let $a \sim b$ iff (for all i, $j < n$, $a_i < a_j$ iff $b_i < b_j$). For example, if $a = (\omega^2, 6, 6) \in \kappa^3$, then the equivalence class $[a] = \{b \in \kappa^3 : b_1 = b_2 < b_0\}$. Clearly, $a \mapsto (a_1, a_0)$ is a homeomorphism of $[a]$ onto $\overline{\kappa}^2$ (there are details on p. 69 of [8]).

Next, we must confine the closure of the equivalence class of a. Note that $\{b \in \kappa^3 : b_1 \neq b_2\}$, $\{b \in \kappa^3 : b_0 < b_1\}$, and $\{b \in \kappa^3 : b_0 < b_2\}$ are open and disjoint from $[a]$. Hence $\overline{[a]} \subset \{b \in \kappa^3 : b_1 = b_2 \leq b_0\}$. In the general case, if $b \in \overline{[a]} \setminus [a]$, then $|\text{range } b| < |\text{range } a|$. List the equivalence classes as Z_k, $k < K$, so that if $|\text{range } b| < |\text{range } a|$, $Z_{k'} = [b]$, and $Z_k = [a]$, then $k' < k$. We have proved

Lemma 3.12. *For all $n \in \omega$, κ^n can be decomposed into finitely many pieces, Z_k, $k < K$, so that:*

1. *Each Z_k is homeomorphic to $\overline{\kappa}^m$, for some $m \leq n$,*
2. *$\bigcup_{k'<k} Z_{k'}$ is closed in κ^n for all $k < K$.*

4. Proof of Theorem 1.3

One direction of Theorem 1.3 follows from Lemma 3.10. For the other direction we will use the following abbreviations. We let $S_\lambda(Y)$ abbreviate "Y is homeomorphic to some $(\kappa_1, \ldots, \kappa_n)$-stationary subset of $\kappa_1 \times \ldots \times \kappa_n$ with $\kappa_1 < \lambda$". We let $N_\lambda(X)$ abbreviate "X has no closed subset Y with $S_\lambda(Y)$". We let $H_\lambda(Z)$ abbreviate "For all $X \subset Z$, if $N_\lambda(X)$, then X is λ-metacompact".

The special cases $\lambda = \omega_1$ and $\lambda = \infty$ deserve discussion. There is no uncountable cardinal less than ω_1. Hence, $S_{\omega_1}(Y)$ is true of no Y, $N_{\omega_1}(X)$ holds for all X, and $H_{\omega_1}(Z)$ means that every subspace X of Z is countably metacompact. For $\lambda = \infty$, recall Definition 2.3 and Lemma 3.4.

Our proof of Theorem 1.3 will be by induction. The next theorem lists some methods to prove $H_\lambda(Z)$ for "big" spaces from $H_\lambda(Z')$ for "small" spaces.

Lemma 4.1. *Each of the following are sufficient to imply $H_\lambda(Z)$:*

1. *Z is homeomorphic to a subset of Z' and $H_\lambda(Z')$.*
2. *$Z = \bigoplus_{i \in I} Z_i$ and $(\forall i \in I) H_\lambda(Z_i)$.*
3. *for every $X \subset Z$ satisfying $N_\lambda(X)$, for every open cover $\mathcal{U}$ of X with $|\mathcal{U}| < \lambda$, there is $\mathcal{V}$, a point-finite open partial refinement of $\mathcal{U}$, such that $H_\lambda(Z \setminus \bigcup \mathcal{V})$.*
4. *$Z = \bigcup_{k<K} Z_k$, $H_\lambda(Z_k)$ for all $k < K$, and $\bigcup_{k'<k} Z_k$ is closed in Z for all $k < K$.*

Proof. Clauses 1 and 2 are obvious. Towards clause 3, let X, $\mathcal{U}$, and $\mathcal{V}$ be as hypothesized. Note that $X' = X \setminus \bigcup \mathcal{V}$ is closed in X, so we have $N(X')$. Next, $H_\lambda(Z \setminus \bigcup \mathcal{V})$ gives that X' is λ-metacompact. Let $\mathcal{V}'$ be a point-finite open refinement of $\{U \cap X' : U \in \mathcal{U}\}$. By Corollary 2.6, there is $\mathcal{V}''$, a point-finite open partial refinement of $\mathcal{U}$ with $X' \subset \bigcup \mathcal{V}''$. Then $\{V(U) \cup V''(U) : U \in \mathcal{U}\}$ is the desired refinement of $\mathcal{U}$.

For clause 4, it suffices to do the case $K = 2$ because the general case then follows by induction. Let $X \subset Z$ satisfy $N(X)$, and let $\mathcal{U}$ be an open cover of X with $|\mathcal{U}| < \lambda$. Note that $X_1 = X \cap Z_1$ is closed in X, so $H(Z_1)$ gives $\mathcal{V}_1$, a point-finite open refinement of $\{U \cap X_1 : U \in \mathcal{U}\}$. By Corollary 2.6, $\mathcal{V}_1$ can be expanded to $\mathcal{V}$ which satisfies the hypotheses of clause 3, which we apply to complete the proof. $\square$

Preparations complete, let us begin the proof of Theorem 1.3. Fix λ and omit the subscript on $S(Y)$, $N(X)$, and $H(Z)$. Because every product of finitely many ordinals is a subspace of α^n for some n and α, to prove the theorem, it suffices to prove $(\forall n \in \omega)H(\alpha^n)$. We proceed by induction. The base step is easy: if α is countable, then α^n is metrizable, hence $(\forall n \in \omega)H(\alpha^n)$. For the induction steps, we will use Lemma 4.3 when α is an uncountable regular cardinal, and Lemma 4.2 otherwise.

Lemma 4.2. *Let α be either a successor ordinal or a singular limit ordinal. If $(\forall m \in \omega)H(\beta^m)$ for all $\beta < \alpha$, then $(\forall n \in \omega)H(\alpha^n)$.*

Proof. We prove $(\forall m \in \omega)H(\beta^m \times \alpha^n)$ by induction on n. The case $n = 0$ follows from hypothesis. Let $n = k + 1$.

If α is a successor, $\alpha = \gamma + 1$, say, then set $C = \{\gamma\}$. If α is a limit, let $C = \{\gamma_j : j < \operatorname{cof} \alpha\}$ be increasing, closed, and cofinal in α, with $\gamma_0 = 0$. Set $Z_1 = \beta^m \times \alpha^k \times C$. Then Z_1 is closed in $\beta^m \times \alpha^n$, and $H(Z_1)$ because Z_1 is homeomorphic to a subspace of $\zeta^{m+1} \times \alpha^k$, where $\zeta = \max\{\beta, \operatorname{cof} \alpha\}$. Set $Z_2 = (\beta^m \times \alpha^n) \setminus Z_1$. If $\alpha = \gamma + 1$, then $H(Z_2)$ follows from $H(\gamma^{m+n})$. If α is a singular limit ordinal, then $Z_2 = \bigoplus_{j<\operatorname{cof}\alpha} \beta^m \times \alpha^k \times (\gamma_j, \gamma_{j+1})$. In this case, $H(Z_2)$ holds by the induction hypothesis and Lemma 4.1.2. We conclude $H(\beta^m \times \alpha^n)$ from $H(Z_1)$, $H(Z_2)$, and Lemma 4.1.4. □

Lemma 4.3. *Assume that α is regular. If $(\forall m \in \omega)H(\beta^m)$ for all $\beta < \alpha$, then $(\forall n \in \omega)H(\alpha^n)$.*

Proof. We prove $(\forall m \in \omega)H(\beta^m \times \alpha^n)$ by induction on n. The case $n = 0$ follows from hypothesis. Let $n = k+1$. Our plan is to prove $H(\beta^m \times \overline{\alpha}^n)$ via Lemma 4.1.3. There are two tasks: find $\mathcal{V}$ and prove $H(\beta^m \times \overline{\alpha}^n \setminus \bigcup \mathcal{V})$. We do the easier first.

Sublemma 4.4. *Let $\beta < \alpha$, let $m < \omega$, and let C be club in α. Set $Z = \beta^m \times (\alpha^n \setminus C^n)$. Then $H(Z)$.*

Proof. Let $(\gamma_\nu : \nu < \alpha)$ be the increasing enumeration of C. For $j \leq n$ and $0 < \nu < \alpha$, set

$$S_{j,\nu} = \beta^m \times \{a \in \alpha^n : \gamma_\nu < a_j \leq \gamma_{\nu+1}\}$$

and set $S_{j,0} = \beta^m \times \{a \in \alpha^n : a_j \leq \gamma_1\}$. Then $\mathcal{S} = \{S_{j,\nu} : j \leq n \wedge \nu < \alpha\}$ has the following properties:

1. Each $S \in \mathcal{S}$ is clopen in $\beta^m \times \alpha^n$.
2. $\bigcup \mathcal{S} \supset Z$.
3. $\mathcal{S}$ is point finite.
4. Each $S \in \mathcal{S}$ is homeomorphic to a subset of $\zeta^m \times \alpha^k$ for some $\zeta < \alpha$.

Let $\mathcal{U}$ be an open cover of $X \subset Z$ such that $|\mathcal{U}| < \lambda$ and $N(X)$. For each $j \leq n$ and $\nu < \alpha$, $X_{j,\nu} = X \cap S_{j,\nu}$ satisfies $N(X_{j,\nu})$. Therefore, by $H(\zeta^m \times \alpha^k)$ there is $\mathcal{V}_{j,\nu}$, a precise, point-finite, open refinement of $\mathcal{U}|X_{j,\nu}$. Set $V(U) = \bigcup\{V_{j,\nu}(U) : j \leq n \wedge \nu < \alpha\}$; then $\{V(U) : U \in \mathcal{U}\}$ is the desired point-finite open refinement of $\mathcal{U}$. Hence X is λ-metacompact, establishing $H(Z)$. □

Sublemma 4.5. *Let $X \subset (\beta^m \times \overline{\alpha}^n)$. If $N(X)$, then X is λ-metacompact.*

Proof. Set $T = \{t \in \beta^m : X_t \text{ is } \alpha^n\text{-stationary}\}$. For each $s \in \beta^m \setminus T$, choose C_s, club in α, so that $X_s \cap C_s^n = \emptyset$. If $\alpha < \lambda$, then T is empty and in this case we set

$X'' = X$ and skip to the last paragraph of this proof. We continue, assuming that $\lambda \leq \alpha$.

We claim that $N(T)$ holds. Towards a contradiction, assume that $S(Y)$ for some Y closed in T. Then $(Y \times \overline{\alpha}^n) \cap X$ is stationary by Lemma 3.3. Set $E = \bigcap_{s \notin T} C_s$, and set $Y' = (Y \times E^n) \cap X$. Then Y' is closed in X and satisfies $S(Y')$, contradicting $N(X)$.

Now let $\mathcal{U}$ be an open cover of X with $|\mathcal{U}| < \lambda$. Note that $|\mathcal{U}| \cdot |\beta^m| < \alpha$. Hence, by Theorem 3.9, for each $t \in T$ we can find $U_t \in \mathcal{U}$, B_t basic open in β^m, and X'_t a stationary subset of X_t such that for each $x \in X'_t$ there is $R_{t,x}$, basic open in $\overline{\alpha}^n$ satisfying

$$t^\frown x \in (B_t \times R_{t,x}) \subset U_t.$$

Set $U^*_t = \bigcup\{R_{t,x} : x \in X'_t\}$. Now U^*_t is open and stationary, so by Theorem 3.9, there is a club C_t satisfying $(C^n_t \cap \overline{\alpha}^n) \subset U^*_t$. Hence $(B_t \times (\overline{\alpha}^n)) \cap (X \setminus U_t) = \emptyset$.

For each $U \in \mathcal{U}$, define $U' = \bigcup\{B_t : U_t = U\}$. Then $\mathcal{U}' = \{U' : U \in \mathcal{U}\}$ is an open cover of $T \subset \beta^m$. From $H(\beta^m)$ and $N(T)$, we conclude that T is λ-metacompact; hence there is a point-finite, open refinement $\mathcal{V}'$ of $\mathcal{U}'$. For $U \in \mathcal{U}$, set $V''(U) = \pi^{\leftarrow}[V'(U)] \cap U$ (where $\pi : \beta^m \times \alpha^n \to \beta^m$ is the natural projection). Then $\mathcal{V}'' = \{V''(U) : U \in \mathcal{U}\}$ is an open partial refinement of $\mathcal{U}$. Notice that $X'' = X \setminus \bigcup \mathcal{V}''$ is a subset of $\beta^m \times (\alpha^n \setminus \bigcap_{t \in T} C^n_t)$.

Set $C = \bigcap_{t \in \beta^m} C_t$. Now we have $N(X'')$, $X'' \subset Z = \beta^m \times (\alpha^n \setminus C^n)$, and $H(Z)$ (by Lemma 4.4). Hence X is λ-metacompact. □

With $H(\beta^m \times \overline{\alpha}^n)$ established, $H(\beta^m \times \alpha^n)$ follows from Lemma 3.12 and Lemma 4.1. This completes the proof of Lemma 4.3. □

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