A GLOBAL PINCHING THEOREM FOR SURFACES
WITH CONSTANT MEAN CURVATURE IN $S^3$

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Abstract. Let $M$ be a compact immersed surface in the unit sphere $S^3$ with constant mean curvature $H$. Denote by $\phi$ the linear map from $T_p(M)$ into $T_p(M)$, $\phi = A - \frac{H}{n}I$, where $A$ is the linear map associated to the second fundamental form and $I$ is the identity map. Let $\Phi$ denote the square of the length of $\phi$. We prove that if $\|\Phi\|_2 \leq C$, then $M$ is either totally umbilical or an $H(r)$-torus, where $C$ is a constant depending only on the mean curvature $H$.

1. Introduction

Let $M$ be a compact immersed hypersurface in the unit sphere $S^{n+1}$ with constant mean curvature $H$. Denote by $h = [h_{ij}]$ the second fundamental form of $M$ and by $\phi$ the tensor $\phi_{ij} = h_{ij} - \frac{H}{n}\delta_{ij}$. Let $\Phi$ denote the square of the length of $\phi$. It is well known that if $H = 0$ and $0 \leq \Phi \leq n$, then $M$ is either the equatorial sphere or a Clifford torus [3]. Recently, H. Alencar and M. do Carmo extended the above result to a hypersurface $M$ with constant mean curvature $H$ [1]. They proved that $M$ is either totally umbilical or an $H(r)$-torus if $\Phi$ satisfies a certain pointwise pinching condition. In 1989, C. L. Shen proved that a minimal hypersurface $M$ is totally geodesic if $M$ is of nonnegative sectional curvature, and $\Phi$ satisfies a certain global pinching condition [3]. Later, the first author improved a result of Shen in the case of $n = 2$ and found a sharp bound concerning the global pinching condition [6]. The purpose of this paper is to extend our global theorem to a surface $M$ with constant mean curvature $H$ and obtain the best constant.

Before stating our main result, let $B$ be the constant $B = 2 + \frac{H^2}{n}$ and $m(B)$ be the maximum value of the function $q(x) = 2\sqrt{\frac{2(B - 2)x + B\sqrt{B}}{(B + x)^2((x^2 - B)^2 + 8x^2)}}$ on $[0, \infty)$. The following is our main result.

Theorem 1.1. Let $M$ be a compact immersed surface in the unit sphere $S^3$ with constant mean curvature $H$. Then

$$||\Phi||_2 \geq 2\pi \sqrt{\frac{2g}{m(B)}}.$$
where \( g \) is the genus of \( M \) and \( \| \cdot \|_2 \) is the \( L^2 \)-norm. The equality holds if and only if \( M \) is either totally umbilical or an \( H(r) \)-torus. In particular, if \( \| \Phi \|_2 \leq 2\pi \sqrt{\frac{B}{m(B)}} \), then \( M \) is either totally umbilical or an \( H(r) \)-torus.

It should be noted that for \( M \) to be either totally umbilical or an \( H(r) \)-torus, \( \| \Phi \|_2 = 2\pi \sqrt{g(B)} \). It turns out that there exists a constant \( H_0 \) such that \( m(B) = (\frac{B}{2})^2 \) for all \( |H| \leq H_0 \). As a immediate consequence of the above result, we state

**Corollary 1.2.** Let \( M \) be a compact immersed surface in the unit sphere \( S^3 \) with constant mean curvature \( H \), \( |H| \leq H_0 \). Then

\[ \| \Phi \|_2 \geq 2\pi \sqrt{g(B)} \]

where \( g \) is the genus of \( M \) and \( \| \cdot \|_2 \) is the \( L^2 \)-norm. The equality holds if and only if \( M \) is either totally umbilical or an \( H(r) \)-torus. In particular, if \( \| \Phi \|_2 \leq 2\pi \sqrt{g(B)} \), then \( M \) is either totally umbilical or an \( H(r) \)-torus.

For the proof of the main theorem, we shall need the following Bernstein-Hopf theorem (see [2], [5]).

**Theorem 1.3.** Let \( M \) be a compact immersed surface in the unit sphere \( S^3 \) with constant mean curvature \( H \). If \( M \) is a topological sphere, then \( M \) is totally umbilical.

2. Notations and auxiliary results

Let \( M \) be a compact connected immersed surface in the unit sphere \( S^3 \). Following the notations of [1] and [3],

**Lemma 2.1.** \( \frac{1}{2} \Delta \Phi = \Phi (B - \Phi) + \sum \phi^2_{ijk} \) where \( \phi_{ijk} \) denote the covariant derivative of \( \phi_i \).

**Lemma 2.2.** \( |\nabla \Phi|^2 = 2\Phi \sum \phi^2_{ijk} \).

**Lemma 2.3.** If \( \Phi \geq B \), then \( \Phi \) is a constant function, \( \Phi \equiv B \), and \( M \) is an \( H(r) \)-torus.

**Proof.** By Lemmas 2.1 and 2.2, we have \( \frac{1}{2} \Delta \log \Phi = B - \Phi \) at the points where \( \Phi \) is positive. It follows that \( \Phi = B \) on \( M \).

According to Lemma 2.1 and Lemma 2.3, we see that if \( \Phi \) is a constant function, then either \( \Phi = 0 \) or \( \Phi = B \).

In the minimal case, H. B. Lawson proved that the set of all zeros of \( \Phi \) is either the whole space \( M \) or at most a finite set of points [7]. We need the following analogous result for the case that \( M \) is with constant mean curvature.

**Lemma 2.4.** The set of all zeros of \( \Phi \) is either the whole \( M \) or at most a finite set of points.

**Proof.** The proof of the lemma is similar to that of Lawson. Let us sketch the proof for completeness. We use an isothermal coordinate \((u, v)\) on a neighborhood \( D \) in \( M \). Denote the position vector of this immersion by \( X \) and the unit normal of \( M \) in \( S^3 \) by \( N \). Then the mean curvature \( H \) and the Gaussian curvature \( K \) are given
by $\frac{1}{2}(X_{uu} + X_{uv}) \cdot N$ and $\frac{1}{2}[(X_{uu} \cdot N)(X_{uv} \cdot N) - (X_{uv} \cdot N)^2]$ respectively, where $F^2 = X_u \cdot X_u = X_v \cdot X_v$.

Since $M$ is of constant mean curvature, the Weingarten equations imply that $X_{uv} \cdot N$ is harmonic in the $(u,v)$ coordinate, $X_{uu} \cdot N$ and $X_{vv} \cdot N$ differ by a constant if $X_{uv} \cdot N$ is constant on $D$, and the zero set of $X_{uv} \cdot N$ and that of $(X_{uu} - X_{vv}) \cdot N$ intersect transversely at the points where the gradient of $X_{uv} \cdot N$ does not vanish. Let $G$ be the set of all points where the gradient of $X_{uv} \cdot N$ vanishes. Since $X_{uv} \cdot N$ is harmonic, $G$ is either isolated or the whole $D$. For $G$ being isolated, the set of all zeros of $\Phi$ is isolated. In the other case, the set of all zeros of $\Phi$ is either empty or the whole $D$.

**Lemma 2.5.** If $M$ is not totally umbilical, then

$$\lim_{\epsilon \to 0} \sum_{i=1}^k \int_{\partial B_\epsilon(p_i)} \frac{\Phi_r}{\Phi} = 16\pi(g - 1)$$

where $p_1, p_2, \ldots, p_k$ constitute all the zeros of $\Phi$ and $\Phi_r$ denotes the derivative of $\Phi$ on $\partial B_\epsilon(p_i)$ in the radial direction from $p_i$. In particular, if $\Phi$ is positive on $M$, then $M$ is a topological torus.

**Proof.** At the points where $\Phi$ is positive, by Lemma 2.2, we get

$$\Delta \log \Phi = B - \Phi. \quad (2.1)$$

Integrating (2.1) over $M = M \setminus \bigcup_{i=1}^k B_\epsilon(p_i)$, we get, from the Gauss equation

$$2K = B - \Phi, \quad (2.2)$$

where $K$ is the Gaussian curvature of $M$, the assertion by Stokes’s theorem and the theorem of Gauss-Bonnet.

**Lemma 2.6.**

$$\int_M \left( \frac{\pi}{2} + \tan^{-1} \frac{H - \sqrt{2\Phi}}{2} - \tan^{-1} \frac{H + \sqrt{2\Phi}}{2} \right)(B - \Phi) + 2\sqrt{2\Phi} \geq 4\pi^2(1 + g).$$

**Proof.** Regard $M$ as an immersed surface of $\mathbb{R}^4$. Then the total absolute curvature of $M$ in the sense of [4] is given by

$$T(M) = \frac{\pi}{2} \int_0^{2\pi} \left[ \sin \theta + \frac{H + \sqrt{2S - H^2}}{2} \cos \theta \right] \left( \sin \theta + \frac{H - \sqrt{2S - H^2}}{2} \cos \theta \right) d\theta dV$$

$$= \frac{\pi}{2} \int_0^{2\pi} \left[ \tan^{-1} \frac{H - \sqrt{2\Phi}}{2} - \tan^{-1} \frac{H + \sqrt{2\Phi}}{2} \right](B - \Phi) + 2\sqrt{2\Phi}.$$ 

By the well-known inequality of Chern-Lashof [4], we have

$$T(M) \geq \frac{\pi^2}{2}(b_0 + b_1 + b_2),$$

where $b_i$ is the $i$th Betti number relative to the real field, for $i = 0, 1, 2$. Since $M$ is two-dimensional, $b_0 = 1$, $b_1 = 2g$ and $b_2 = 1$. 

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3. PROOF OF MAIN RESULTS

We are now in position to prove the main result of Theorem 1.1. We may assume that \( \Phi \) is positive except possibly at a finite set of points (see Lemma 2.4). By Lemmas 2.2 and 2.6, we get

\[
\int_M 2\sqrt{2\mathcal{H}} + m(B) \sum \phi^2_{ijk} - (\frac{\pi}{2} + \tan^{-1} \frac{H - \sqrt{2\mathcal{H}}}{2} - \tan^{-1} \frac{H + 2\mathcal{H}}{2})(B - \Phi) - 2\sqrt{2\mathcal{H}} \geq 0.
\]

\[
= \int_M m(B) \sum \phi^2_{ijk} - (\frac{\pi}{2} + \tan^{-1} \frac{H - \sqrt{2\mathcal{H}}}{2} - \tan^{-1} \frac{H + 2\mathcal{H}}{2})(B - \Phi)
\]

\[
= \lim_{\epsilon \to 0} \int_{M,} m(B) \frac{\nabla \Phi^2}{2\Phi} + \frac{1}{2} \frac{2\sqrt{2g}}{\sqrt{\mathcal{B} + \sqrt{\Phi}}} - (\frac{\pi}{2} + \tan^{-1} \frac{H - \sqrt{2\mathcal{H}}}{2} - \tan^{-1} \frac{H + 2\mathcal{H}}{2}) \Delta \log \Phi \geq 0.
\]

\[
= 8\pi(1 - g)(\frac{2\sqrt{2g}}{\sqrt{\mathcal{B} + \sqrt{\Phi}}} - \frac{\pi}{2}) + \int_M [m(B) - 2\sqrt{2g}(\sqrt{\mathcal{B} + \sqrt{\Phi}})(\Phi^2 - 2) + 8\Phi^2] \sum \phi^2_{ijk}
\]

\[
\geq 8\pi(1 - g)(\frac{2\sqrt{2g}}{\sqrt{\mathcal{B} + \sqrt{\Phi}}} - \frac{\pi}{2}),
\]

where the equality holds if and only if \( \Phi \) is constant. On the other hand, according to Lemma 2.6, we get

\[
2\sqrt{2\mathcal{H}} \text{Area}(M) + m(B) \int_M \sum \phi^2_{ijk} \geq 2\pi g - 2\sqrt{\frac{2}{\mathcal{B}}} 8\pi(g - 1).
\]

By combining (2.2) with the inequality (3.1), it follows from Lemma 2.1 that

\[
m(B) \int_M \Phi^2 \geq 2\pi g + (m(B) - 2\sqrt{\frac{2}{\mathcal{B}}} \int_M \Phi \geq 8\pi g.
\]

It remains to show that the second assertion holds. Suppose now that \( \|\Phi\|_2 \leq 2\pi \sqrt{\frac{2}{m(B)}} \). The first assertion implies that \( g = 0 \) or 1. If \( g = 1 \), then \( \|\Phi\|_2 = 2\pi \sqrt{\frac{2}{m(B)}} \) and \( \Phi \) is a constant function. If \( g = 0 \), then by Theorem 1.3, \( M \) is totally umbilical. This completes the proof of Theorem 1.1.

REFERENCES


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