THE HOPF CONJECTURE FOR MANIFOLDS WITH LOW COHOMOGENEITY OR HIGH SYMMETRY RANK

THOMAS PÜTTMANN AND CATHERINE SEARLE

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Abstract. We prove that the Euler characteristic of an even-dimensional compact manifold with positive (nonnegative) sectional curvature is positive (nonnegative) provided that the manifold admits an isometric action of a compact Lie group $G$ with principal isotropy group $H$ and cohomogeneity $k$ such that $k - (\text{rank } G - \text{rank } H) \leq 5$. Moreover, we prove that the Euler characteristic of a compact Riemannian manifold $M^{4l+4}$ or $M^{4l+2}$ with positive sectional curvature is positive if $M$ admits an effective isometric action of a torus $T^l$, i.e., if the symmetry rank of $M$ is $\geq l$.

The Gauss-Bonnet theorem states that the Euler characteristic of a closed surface $M$ is determined by its total curvature:

$$\chi(M) = 2\pi \int_M K.$$

In particular, if the curvature is positive (nonnegative), the Euler characteristic of the surface is positive (nonnegative). H. Hopf [H] generalized in 1925 the Gauss-Bonnet theorem to even-dimensional hypersurfaces of Euclidean space and posed in the early 1930's (according to Berger [Be]) the question of whether a compact even-dimensional manifold which admits a metric of positive (nonnegative) sectional curvature must have positive (nonnegative) Euler characteristic.

Indications that the Hopf conjecture should be true came from the generalizations of the Gauss-Bonnet theorem: Fenchel [F] and Allendoerfer [A] proved in 1940 independently a Gauss-Bonnet formula for submanifolds of Euclidean space with arbitrary codimension. Three years later Allendoerfer and Weil [AW] (using E. Cartan’s result that any Riemannian manifold can locally be embedded into Euclidean space) established the theorem in its final intrinsic version: For any even-dimensional manifold the Euler characteristic can be obtained by integrating a function derived from the curvature tensor, the so-called Gauss-Bonnet integrand. Chern [C1] gave the first intrinsic proof of this theorem in 1944.

After this, many attempts were made to settle the stronger algebraic Hopf conjecture: A curvature tensor with positive (nonnegative) sectional curvature yields...
a positive (nonnegative) Gauss-Bonnet integrand. Milnor (unpublished, see [C2]) actually proved the algebraic Hopf conjecture in dimension 4, but finally in 1976 Geroch [G] found curvature tensors with positive sectional curvature in all even dimensions \( \geq 6 \) that do not provide a positive Gauss-Bonnet integrand.

A different approach to the Hopf conjecture is to consider first Riemannian manifolds that have a certain amount of symmetry. Hopf himself and Samelson [HS] proved in 1941 that the Euler characteristic of every compact homogeneous space \( G/H \) is nonnegative and positive if and only if \( \text{rank} G = \text{rank} H \) holds. The key observations in their proof are that a regular element in the compact Lie group \( G \) has at most finitely many fixed points in the homogeneous space \( G/H \) and that each of these fixed points has fixed point index 1. In 1972 Wallach [W] then showed that for any even-dimensional homogeneous space of positive sectional curvature one actually has \( \text{rank} G = \text{rank} H \). Therefore, the Hopf conjecture is true for homogeneous spaces. Recently, Podestà and Verdiani [PV] proved among other things that the Hopf conjecture also holds for cohomogeneity one manifolds. We show that much weaker symmetry assumptions are sufficient.

**Theorem 1.** Let \( M \) be a compact even-dimensional Riemannian manifold with positive (nonnegative) sectional curvature. Let \( G \times M \rightarrow M \) be an isometric action of a compact Lie group \( G \) with principal isotropy group \( H \) and cohomogeneity \( k \). If

\[
 k - (\text{rank} G - \text{rank} H) \leq 5,
\]

then \( M \) has positive (nonnegative) Euler characteristic.

**Proof.** If \( M \) is nonorientable, then the action of \( G \) can be lifted to an action by orientation preserving isometries (see [Br, Corollary I.9.4]) on the orientable double covering space of \( M \). We can therefore assume that \( M \) is orientable.

We consider the fixed point set

\[
 M^T = \{ p \in M \mid \psi(p) = p \text{ for all } \psi \in T \}
\]

of a maximal torus \( T \) of \( G \). Note that \( M^T \) is equal to the fixed point set of a generating element \( \psi \in T \), i.e., of an element \( \psi \) with \( \{\psi^m \mid m \in \mathbb{Z}\} = T \). If the fixed point set \( M^T \) is empty, then there exists a Killing field without zeros. This implies that \( M \) cannot have positive sectional curvature (by Berger’s theorem, see e.g. [W]) and that the Euler characteristic is zero. We can therefore assume that \( M^T \) is nonempty. Now each of the finitely many components of \( M^T \) is a totally geodesic submanifold of \( M \) with even codimension and the Euler characteristic of \( M \) is the sum of the Euler characteristics of the components (see [K] Chapter II).

By Theorem IV.5.3 of [Br] each component \( N \) of \( M^T \) satisfies

\[
 \dim N \leq k - (\text{rank} G - \text{rank} H) \leq 5.
\]

Since \( N \) is even-dimensional and the Hopf conjecture holds in dimensions 2 and 4 we are done.

Note that \( k - (\text{rank} G - \text{rank} H) \leq \dim M - 2 \text{rank} G \) (see [Br, Corollary IV.5.4]) if the action of \( G \) is effective. Hence we get as a special case of Theorem 1 that any compact even-dimensional Riemannian manifold \( M^{2l+4} \) with positive (nonnegative) sectional curvature has positive (nonnegative) Euler characteristic if \( M^{2l+4} \) admits an effective isometric torus action \( T^l \times M \rightarrow M \). Using a result from [GS] we can improve this result in the case of positive sectional curvature.
Theorem 2. Let $M^{4l+2}$ or $M^{4l+4}$ be a Riemannian manifold with positive sectional curvature that admits an almost effective isometric $T^l$-action. Then for any $T^1 \subset T^l$ the Euler characteristics of all the components of the fixed point set $\text{Fix}(M; T^1)$ are positive. In particular, $\chi(M) > 0$.

Proof. As above we can assume that $M$ is orientable in order to have even-dimensional fixed point sets. The proof is done by induction. For $l = 0$ note that the Hopf conjecture is true in dimensions 2 and 4. For the induction step consider $M^{4l+6}$ or $M^{4l+8}$ with an almost effective $T^{l+1}$-action. Consider any circle $T^1 \subset T^{l+1}$ and any component $N$ of its fixed point set in $M$. We will show that $\chi(N) > 0$. Choose a $T^1 \subset T^{l+1}$ such that $N \subset \text{Fix}(M; T^1)$ and such that the component $\tilde{N}$ of $\text{Fix}(M; T^1)$ contains $N$ has maximal dimension. It follows from the slice theorem and from the representation theory of tori that $T^1 = T^{l+1}/\tilde{T}$ acts almost effectively on $\tilde{N}$. If $\text{codim} \tilde{N} \geq 4$, then we know from the induction assumption that in particular $N$ as a component of $\text{Fix}(\tilde{N}; T^1)$ has positive Euler characteristic and hence we are done. In the case where $\text{codim} \tilde{N} = 2$ we know from \cite{GS} that $M$ is differentiably covered by a sphere or a complex projective space. From results of Bredon \cite{Br} Chapters III and VII it follows that all the components of the fixed point set of any circle action on $M$ have positive Euler characteristic. Thus in particular $N$ has positive Euler characteristic.

After this paper was accepted for publication we were informed that Xiaochun Rong obtained Theorem 2 independently (see \cite{R}). In his paper he gives many more results on the topology of positively curved manifolds with high symmetry rank.

References


Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany
E-mail address: puttmann@math.ruhr-uni-bochum.de

Instituto de Matemáticas, Unidad Cuernavaca-UNAM, Apartado Postal 273-3, Aduana 3, Cuernavaca, Morelos, 62251, Mexico
E-mail address: csearle@matcuer.unam.mx