THE ROOT LATTICE \( A_n^* \) AND RAMANUJAN’S CIRCULAR SUMMATION OF THETA FUNCTIONS

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Abstract. We relate a formula of Ramanujan on the circular summation of the \( n \)th power of theta functions, \( F_n(q) \), to the theta series of the root lattice \( A_n^* \). We then use properties of the lattice to show that \( F_n \) includes an \( \text{SL}_2(\mathbb{Z}) \) modular form when \( n \) is an odd perfect square as well as to derive a very simple expression for \( F_9(q) \).

1. Introduction

Let \( \theta(z, \tau) = \sum_{m=-\infty}^{\infty} q^m e^{2\pi imz} \) be the Jacobi theta function. We assume throughout this paper that \( q = e^{\pi i \tau} \), where \( \tau \) lies in the upper half plane. On page 54 of his “Lost Notebook” [7], Ramanujan claimed (in his own notation) the following result:

**Ramanujan’s Claim.** If \( n \) is a positive integer, then

\[
\sum_{k=0}^{n-1} q^{k^2} e^{2\pi i k z} \theta(z + k\tau, n\tau)^n = \theta(z, \tau) F_n(q^2)
\]

where \( F_n(q^2) = 1 + 2nq^{n-1} + \cdots \).

Ramanujan’s claim was proven by Rangachari [8] in 1994 and also more recently by Son [9] using elementary methods. Ramanujan also stated very simple and elegant formulae for \( F_n(q) \) for \( n = 2, 3, 4, 5, 7 \). In order to prove Ramanujan’s formulae for \( F_n(q) \), Rangachari noticed an interesting relation with the theta series of the root lattice \( A_{n-1}^* \) (the dual of \( A_{n-1} \)). However, because he used the theory of linear codes over a finite field, Rangachari was only able to prove his relation for the case \( n = p \) a prime. Our first result is a generalization of Rangachari’s observation to all \( n \) where we use essentially only the Jacobi Inversion Formula (the reader should refer to Section 2 for the scaling of our lattice):

**Theorem 1.** For all \( n \geq 2 \), \( F_n(q^2) = \theta_{A_{n-1}^*}(n\tau) \).

Theorem 1 has several immediate consequences (see Corollaries 2.1 and 2.2 in Section 2) and a surprising application for efficient computation for the theta series of \( A_{n-1}^* \) (see Lemma 2.2) and even of \( A_{n-1} \) when \( n \) is square free (see Lemma 3.4). The problem of exhibiting ‘simple’ explicit identities for \( F_n(q^2) \) has been the
subject of a number of recent works (see for example [1, 6, 9]). Using Theorem 1
and some properties of $A_{n-1}$, we will derive further results in this direction. One
of our key observations is that if $n = r^2$ is an odd perfect square (note this implies
$n - 1 \equiv 0 \mod 8$), there is an even unimodular lattice midway between $A_{n-1}$ and
$A_{n-1}^*$ (see Lemma 3.1 below). This allows us to determine a part of $F_n(q^2)$ in this
case. More precisely, let

\[ F_n(q^2) = \sum_{m=0}^{\infty} a_n(m)q^m, \]

and set

\[ G_n(\tau) = \sum_{m=0}^{\infty} a_n(2nm)q^{2m}. \]

Then we have the following:

**Theorem 2.** If $n = r^2$ is an odd perfect square, then $G_n(\tau)$ is a holomorphic
modular form for $\text{SL}_2(\mathbb{Z})$ of weight $(n - 1)/2$ and is given by the theta series of
$E_{n-1}$, the unique even unimodular lattice between $A_{n-1}$ and $A_{n-1}^*$.

The $G_n(\tau)$ above are effectively computable as a polynomial in the Eisenstein
series and $\Delta(\tau)$ the unique cusp form of weight 12, for any given $n$, and we will
give explicit examples in Section 3. Theorem 2 however, only determines a part of
$F_n(q^2)$. We can sometimes determine the other part (which corresponds to the theta
series of the translates of the intermediate even unimodular lattice). We give as one
example an explicit evaluation of $F_9(q^2)$ where a different expression has been given
by S. Ahlgren in ([1], Theorem 1, (1.5)). We will also indicate in Section 3 how we
are led to our formula. As has been remarked several times earlier [1, 6], the proof
of such formula, once it is found is a trivial verification by computing enough terms
of the expansion since they belong to a space of some finite-dimensional modular
forms. It is therefore the process of discovering the formula, via lattices in our case,
which is more interesting. The problem of evaluating the next interesting case of
$F_{25}(q^2)$ is apparently still open. We have already determined a part of it, namely
$G_{25}(\tau)$.

2. THE ROOT LATTICE $A_n^*$

It follows from Ramanujan’s claim (1.1) that

\[ F_n(q^2) = \sum_{k=0}^{n-1} q^{k^2} \theta(k\tau, n\tau)^n / \theta_3(\tau) = \sum_{m=-\infty}^{\infty} \left( \sum_{n=-\infty}^{\infty} q^{(nm+k)^2/n} \right)^n / \sum_{m=-\infty}^{\infty} q^{m^2}. \]

We will now state precisely some relevant facts on lattices following the standard
reference [5]. The root lattice $A_n$ is defined by

\[ A_n = \{(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} : \sum x_i = 0\} \]

and its dual is given by

\[ A_n^* = \left\{ y = (y_0, \ldots, y_n) \in \mathbb{R}^{n+1} : \sum y_i = 0, y^T x \in \mathbb{Z}, x \in A_n \right\}. \]
We note that $A_n$ has determinant $n + 1$ and so $A_n^*$ has determinant $1/(n + 1)$. For an $n$-dimensional lattice $\Lambda$, its theta series is defined by
\begin{equation}
\theta_\Lambda(\tau) = \sum_{x \in \Lambda} e^{\pi i \tau (x \cdot x)}.
\end{equation}

The theta series of a lattice and its dual is related by the following well-known Jacobi Transformation formula (see [5])
\begin{equation}
\theta_\Lambda^*(\tau) = \sqrt{\det(\Lambda)}(i/\tau)^{n/2}\theta_\Lambda(-1/\tau).
\end{equation}

We also recall the following well-known transformation rule for the Jacobi theta function $\theta(z, \tau)$ (see [4]):
\begin{equation}
\begin{split}
\theta(z + \mu + \lambda \tau, \tau) &= e^{-\pi i (\lambda^2 \tau + 2\lambda z)}\theta(z, \tau), \quad (\mu, \lambda) \in \mathbb{Z}^2, \\
\theta \left( \frac{z}{\tau}, \frac{1}{\tau} \right) &= \sqrt{\frac{-\tau}{i}} \theta(z, \tau), \\
\theta(z, \tau + 2) &= \theta(z, \tau).
\end{split}
\end{equation}

These show incidentally that $\theta(z, \tau)$ ought to be a singular Jacobi form of weight 1/2 and index 1/2 for the subgroup $G(2)$. We note that Rangachari’s proof of Ramanujan’s claim essentially consists of verifying that the left-hand side of (1.1) also satisfies (2.6a) and hence its quotient by $\theta(z, \tau)$ is an elliptic function of order $\leq 1$ and hence must be independent of $z$. We also need the following explicit formula for the theta series of $A_{n-1}$ (see [5], pg. 110, formula (56)).

**Lemma 2.1.** $\theta_{A_{n-1}}(\tau) = \sum_{k=1}^{n-1} \left( \sum_{m=-\infty}^{n-1} q^{\frac{m^2}{2} \zeta^{km}} \right)^n / n! \theta_3(n\tau)$, where $\zeta = e^{2\pi i/n}$ and $\theta_3(\tau) = \sum_{m=-\infty}^{\infty} q^{m^2}.$

**Proof.** We have $\left( \sum_{m=-\infty}^{n-1} q^{m^2} \zeta^{km} \right)^n = \sum_{x \in \mathbb{Z}^n} (q^{\sum x_i^2} \zeta^{\sum x_i})$, so
\begin{equation}
\sum_{k=0}^{n-1} \left( \sum_{m \in \mathbb{Z}} q^{m^2} \zeta^{km} \right)^n = \sum_{x \in \mathbb{Z}^n} \left( \sum_{k=0}^{n-1} \zeta^{k \sum x_i} \right) q^{\sum x_i^2} = n \sum_{x \in \mathbb{Z}^n} \sum_{\sum x_i = mn} q^{\sum x_i^2},
\end{equation}
where we have used the well-known orthogonality relation
\begin{equation}
\sum_{k=0}^{n-1} \zeta^{ka} = \begin{cases} n & a = mn, \\
0 & \text{otherwise}
\end{cases}
\end{equation}
in the last equality. Setting $y_i = x_i - m$ in the inner sum on the right-hand side of (2.7) gives $\sum_{y \in \mathbb{Z}^n} \sum_{y_i = 0} (q^{\sum y_i^2 + 2m(\sum y_i) + mn^2} = q^{nm^2} \theta_{A_{n-1}}(\tau)$ and substituting back into (2.7) we are done. \hfill $\square$

We can now prove Theorem 1.

**Proof of Theorem 1.** By Lemma 2.1 we can rewrite
\begin{equation}
\theta_{A_{n-1}}(\tau) = \frac{1}{n! \theta_3(n\tau)} \sum_{k=0}^{n-1} \theta^n \left( \frac{k}{n}, \frac{1}{n} \right).
\end{equation}

By the Jacobi Inversion Formula (2.5), we get
\begin{equation}
\theta_{A_{n-1}^*}(\tau) = \sqrt{n} \left( \frac{i}{\tau} \right)^{(n-1)/2} \sum_{k=0}^{n-1} \theta^n \left( \frac{k}{n}, \frac{1}{n} \right) / n! \theta_3 \left( -\frac{n}{\tau} \right).
\end{equation}
From (2.6b), we get by setting $z = 0$, $\tau' = \tau/n$ and $z = k\tau/n$ respectively that

\[(2.9a)\quad \theta_3(-n/\tau) = \sqrt{\tau/\tau'} \theta_3(n/\tau)\]

and

\[(2.9b)\quad \theta\left(\frac{k}{n} - \frac{1}{\tau}\right) = \sqrt{\frac{\tau}{\tau'}} e^{2\pi i n k/\tau} \theta\left(\frac{k\tau}{n}, \tau\right).\]

Substituting (2.9a), (2.9b) into (2.8) gives

\[A_{n^*}^\prime(\tau) = \sum_{k=0}^{n-1} q^{k^2/n} \theta^n(k\tau/n, \tau)/\theta_3(\tau/n) = F_n(q^{2/n})\quad \text{by (2.10).} \quad \square\]

Theorem 1 has many immediate corollaries. First, as Rangachari noted, by utilising the explicit theta series for the translate of $A_{n^*}^\prime$ on (pg. 110 of \[5\], formula (57)), we have immediately

**Corollary 2.1.** For every positive integer $n$ and $\zeta = e^{2\pi i/n}$, we have

\[(2.10)\quad A_{n^*}^\prime(n\tau) = \sum_{k=0}^{n-1} q^{k^2/n} \theta^n(k\tau/n, n\tau)/\theta_3(\tau/n) = \sum_{l=0}^{n-1} \sum_{k=0}^{n-1} \frac{e^{-kl/n} \theta^n(k/n, n\tau)}{nq^{l^2} \theta(nl\tau, n^2\tau)},\]

where as usual $\theta_3(\tau) = \theta(0, \tau)$.

Remark. Formula (2.10) is easily verified, for small $n = 2, 3, \ldots$ using the standard theta identities in \[5\], pg. 104. It seems to be nontrivial in general.

One can also view Theorem 1 as stating that $F_n(q^2)$ is the theta series of the scaled lattice $qA_{n^*}^\prime$ (with determinant $n^{n^2}$). Using the known quadratic form (eq. (77) in pg. 115 of \[5\]) we have the following multi-dimensional sum:

**Corollary 2.2.** $F_n(q^2) = \sum_{x \in \mathbb{Z}^{n-1}} q^{(n-1)\sum_i x_i^2 - \sum_{i<j} x_i x_j}$.

The multi-dimensional sum in Corollary 2.2 is useless as a formula for computing the theta series for $A_{n^*}^\prime$ as it runs in exponential time. It seems the only known way previously for actually computing this series is via the formula (57) on p. 110 of \[5\] where one actually has to compute over the ring $\mathbb{Z}[\zeta]$ (see the analogous algorithm of Robert Harley for computing $A_{11}$ in sequence A023923 of \[10\]). A surprising consequence of Ramanujan’s formula and Theorem 1 is that it gives an efficient way to compute theta series of $A_{n^*}^\prime$ via the formula on the RHS of (2.1).

We state it as a lemma.

**Lemma 2.2.** The theta series of $qA_{n^*}^\prime (= F_n(q^2))$ can be computed efficiently (involving only polynomial operations over the integers) using the following expressions:

\[(2.11)\quad \theta_3(\tau) F_n(q^2) = \left\{ \begin{array}{ll}
\left(\sum_m q^{nm^2}\right)^n & + 2 \sum_{k=0}^{(n-1)/2} q^{k^2} \left(\sum_m q^{n^2+2km}\right)^n & n \text{ odd},
\sum_m q^{nm^2} & + q^{n^2/4} \left(\sum_m q^{nm^2+nm}\right)^n & \sum_{k=1}^{(n-2)/2} q^{k^2} \left(\sum_m q^{n^2+2km}\right)^n & n \text{ even},
\end{array} \right.\]

where the summation of $m$ is over all integers.
Proof. Equation (2.11) follows easily from the right-hand side of (2.1) and the fact that \( \sum_n q^{(nm+k)/n} = \sum_n q^{(n+m+k)/n} \). We also note that in our expression (2.11), the summation in \( m \) is only over nonnegative powers of \( q \) so the right-hand side is easily computed to any accuracy using only polynomial series expansion and the same holds for the division by \( \theta_3 \).

As was noted by Rangachari, one can also use Theorem 1 to evaluate \( F_n(q^2) \). For example if \( n = 4 \), since we know \( A_4^* \approx D_4^* \) and the theta series of \( D_4^* \) has a simple expression ((96) of [5]), one gets immediately \( F_4(q^2) = \theta_3^3(q^4) + \theta_2^3(q^4) \), as was noted by Ramanujan. We mention also that for \( n = 3 \), since \( A_2^* \approx A_2 \) is the hexagonal lattice and determinant \( (\sqrt{3}A_2^*) = 1 \), we must have \( F_3(q) = \sum_{m,n} q^{m^2+mn+n^2} = a(q) \) and the Ramanujan expression for \( n = 3 \) as given in [6]:

\[
F_3^3(q) = \left( \frac{f^3(-q)}{f(-q^3)} \right)^3 + \left( 3q \frac{f^3(-q^3)}{f(-q)} \right)^3
\]

is exactly the Cubic Identity of [3] \( a(q)^3 = b(q)^3 + c(q)^3 \). The fact that \( b(q) = f^3(-q)/f(-q^3) \) and \( c(q) = 3q^{1/3} f^3(-q^3)/f(-q) \) has been noted by Berndt [2]. It is however possible to derive further properties of \( F_n(q^2) \) using Theorem 1 and more detail structure of \( A_{n-1}^* \) as we will see in Section 3.

3. Explicit evaluation of \( F_n(q^2) \)

Since \( \text{determinant}(A_{n-1}) = n \), \( A_{n-1} \) is a subgroup of \( A_{n-1}^* \) of index \( n \) and the glue group \( A_{n-1}^*/A_{n-1} \) is cyclic of order \( n \) (see [3]). By (2.2) and (2.4), the vector

\[
w = (1, \ldots, 1, -(n-1))/n \in R^n
\]

is clearly in \( A_{n-1}^*/A_{n-1} \) and it is indeed a generator. For \( 0 \leq j \leq n-1 \), let \( C_j \) be the coset \( A_{n-1} + jw \), so that

\[
A_{n-1}^* = \bigcup_{j=0}^{n-1} C_j.
\]

Note that we can rewrite Theorem 1 as \( F_n(q^2) = \theta_{\sqrt{n} A_{n-1}^*}(\tau) \), where \( \sqrt{n} A_{n-1}^* \) is the lattice \( A_{n-1}^* \) scaled by a factor of \( \sqrt{n} \). It follows from (3.1) and (3.2) that \( \sqrt{n} A_{n-1}^* \) is an integral lattice and is even integral if \( n \) is odd. One of our key observations is the following:

Lemma 3.1. If \( n = r^2 \) is an odd perfect square (note this means \( n - 1 \equiv 0 \pmod{8} \), there is an even unimodular \((n-1)\)-dimensional lattice \( E_{n-1} \) midway between \( A_{n-1} \) and its dual, i.e. \( A_{n-1} \subset E_{n-1} \subset A_{n-1}^* \). We also have

\[
E_{n-1} = \bigcup_{j=0}^{r-1} C_{jr}
\]

and

\[
A_{n-1}^* = \bigcup_{j=0}^{r-1} E_{n-1} + jw.
\]

Proof. We define \( E_{n-1} \) by (3.3a) which is clearly the unique subgroup of index \( r \) in \( A_{n-1}^* \) since \( A_{n-1} \) is of index \( n = r^2 \). It follows that \( \text{determinant}(E_{n-1}) = 1 \). The fact that \( E_{n-1} \) is integral and even follows from Lemma 3.2 below.

□
The lattice $E_{n-1}$ is easily identified for small $n$. For $n = 9$, it must be the lattice $E_8$ since there is only one such lattice. For $n = 25$, $E_{24}$ is clearly the 24-dimensional Niemeier lattice of type $A_{24}$ with Coxeter number 25 (see [5], table 16.1, pg. 407). Now a standard observation of Hecke states that the theta series of an even unimodular $m$-dimensional lattice is a modular form of weight $m/2$ for $SL_2(\mathbb{Z})$ and it follows from (3.4) that we may hope to see an $SL_2(\mathbb{Z})$ modular form to appear in the theta series of $A_{n-1}^*$. That this is indeed the case follows from the next two lemmas.

**Lemma 3.2.** Let $y = \sqrt{n}(x+jw)$, where $x \in A_{n-1}$, be a vector in the coset $\sqrt{n}C_j$. Then we have

\[(3.4)\quad \|y\|^2 \equiv j^2(n-1) \mod 2n.\]

**Proof.** By (2.2) and (3.1) we have clearly $\langle x, w \rangle = \sum_{j=1}^{n-1} x_j$ is integral for any $x \in A_{n-1}$ and $\|w\|^2 = (n-1)/n$. So we have $\|y\|^2 = n\|x\|^2 + 2jn(x, w) + j^2(n-1)$. (3.3) follows from this and the fact that $A_{n-1}$ is even integral ($\sum x_i^2 \equiv \sum x_i = 0 \mod 2$).

It follows from Lemma 3.2 and (3.2) that $\alpha_n(m)$ is only nonzero for certain arithmetic progressions of $m \mod 2n$. In particular, we have the following

**Lemma 3.3.** If $n = r^2$ is odd and $y \in \sqrt{n}A_{n-1}^*$, then $\|y\|^2 \equiv 0 \mod 2n$ iff $y \in \sqrt{n}E_{n-1}$.

**Proof.** The “if” part follows from equations (3.4) and (3.3a). Conversely, since $\gcd(n-1, 2n) = 2$, (3.4) implies that $\|y\|^2 \equiv 0 \mod 2n$ can only occur for $y \in A_{n-1}^*$, a coset $C_j$ where $r$ divides $j$ and the result follows again from (3.3a).

We can now prove Theorem 2.

**Proof of Theorem 2.** By Lemma 3.3, a vector in $\sqrt{n}A_{n-1}^*$ with norm $2nm$ must lie in the scaled even unimodular sublattice $\sqrt{n}E_{n-1}$ and only such vectors can have norm dividing $2n$. It follows that \( \sum_{m=0}^{\infty} \alpha_n(2nm)q^{2nm} = \theta_{\sqrt{n}E_{n-1}}(\tau) = \theta_{E_{n-1}}(n\tau) \).

By (1.3) we must have $G_n(\tau) = \theta_{E_{n-1}}(\tau)$, and so it must be a modular form of weight $(n-1)/2$ for $SL_2(\mathbb{Z})$.

**Remark 1.** Let $m = (n-1)/2$ and $k = [m/12]$, the space of modular form of weight $m$ is spanned by $\{ \Delta^j(\tau)E_{m-12j}(\tau) : 0 \leq j \leq k \}$ and $\theta_{E_{n-1}}$ can be computed as a linear combination of these by computing the coefficients $\alpha_n(2nj)$ for $0 \leq j \leq k$ using Lemma 2.2. For $n = 9$, the weight 4 modular form is unique and $G_9(\tau) = E_4(\tau) = 1 + 240 \sum_{j=0}^{\infty} \sigma_3(j)q^{2j}$. For $n = 25$, one needs only two coefficients $\alpha_{25}(0) = 1$ and $\alpha_{25}(50) = 600$ to get $G_{25}(\tau) = E_{12}(\tau) + (600 - 25520 \Delta(\tau))\Delta(\tau)$.

**Remark 2.** Theorem 2 determines only a part of $F_n(q^2)$, one still needs the theta series for the $r-1$ translates of $E_{n-1}$ in $A_{n-1}^*$. For $n = 9$, computing $F_9(q^2) = E_4(9\tau)$ by Lemma 2.2 gives us the series

\[18q^8 + 72q^{14} + 252q^{20} + 504q^{26} + 1026q^{32} + 1512q^{38} + 2664q^{44} + 3528q^{50} + \cdots.\]

Dividing out by 18, we search for the sequence 1, 4, 14, 28, 57, 84, 148, ... on Neil Sloane’s on line Encyclopedia of Integer Sequence [10] and discover it as the theta
series of the direct sum of 4 copies of the translate of the Hexagonal lattice (sequence A033690), more exactly, if we set \( f(q) = 1 + 4q + 14q^2 + 28q^3 + 57q^4 + 84q^5 + \cdots \), we have, in the notation of [5], pg. 111,

\[
f(q) = (1 + q + 2q^2 + 2q^4 + \cdots)^4 = \left( \frac{\theta_{\text{hex}}[1](\tau)}{3q^{1/3}} \right)^4.
\]

The lattice \( \text{Hex +[1]} \) above is just the translate of the Hexagonal lattice = \( A_2 \) and it is easy to see that the series of this translate is given by

\[
c(q) = \sum_{m,n = -\infty}^{\infty} q^{(m-1/3)^2 + (m-1/3)(n-1/3) + (n-1/3)^2}
\]

which appears in the cubic identity [3]. We have arrived at the rather simple formula

\[
F_9(q^2) = E_4(9\tau) + \frac{2}{9}(c(q^6))^4
\]

which can be easily verified.

Finally, we can also determine \( G_n(\tau) \) in some cases when \( n \) is not a square. One observes that \( A_{n-1} \) is a coset of \( A^*_n \), and try to recover the theta series of \( A_{n-1} \) from that of \( A^*_n \).

**Lemma 3.4.** Let \( \alpha_n(m) \) and \( G_n(\tau) \) be as defined in (1.2) and (1.3). If \( n \) is square free, \( G_n(\tau) = \theta_{A_{n-1}}(\tau) \).

**Proof.** The coset \( C_0 \) gives \( A_{n-1} \) but conversely (3.4) of Lemma 3.2 implies that a vector \( y \) in any other coset cannot have norm which is a multiple of \( 2n \) since \( j^2(n - 1) \neq 0 \mod 2n \) for any \( j \neq 0 \) for square free \( n \). So the vectors in \( \sqrt{n}A_{n-1}^* \) with norm a multiple of \( 2n \) are exactly those in the lattice \( \sqrt{n}A_{n-1} \).

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**References**


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