ON A SEMILINEAR SCHRÖDINGER EQUATION WITH CRITICAL SOBOLEV EXPONENT

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ABSTRACT. We consider the semilinear Schrödinger equation $-\Delta u + V(x)u = K(x)|u|^{2^* - 2}u + g(x,u)$, $u \in W^{1,2}(\mathbb{R}^N)$, where $N \geq 4$, $V, K, g$ are periodic in $x_j$ for $1 \leq j \leq N$, $K > 0$, $g$ is of subcritical growth and $0$ is in a gap of the spectrum of $-\Delta + V$. We show that under suitable hypotheses this equation has a solution $u \neq 0$. In particular, such a solution exists if $K \equiv 1$ and $g \equiv 0$.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this paper we shall be concerned with the semilinear Schrödinger equation

$$-\Delta u + V(x)u = K(x)|u|^{2^* - 2}u + g(x,u), \quad u \in W^{1,2}(\mathbb{R}^N),$$

where $N \geq 4$, $2^* := 2N/(N - 2)$ is the critical Sobolev exponent and $g$ is of subcritical growth. More precisely, we make the following assumptions:

- **(A1):** $V, K \in C(\mathbb{R}^N)$, $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $K(x) > 0$ in $\mathbb{R}^N$ and $V, K, g$ are 1-periodic in $x_j$ for $j = 1, \ldots, N$.
- **(A2):** $|g(x,u)| \leq c_0(1 + |u|^{p-1})$ on $\mathbb{R}^N \times \mathbb{R}$ for some $c_0 > 0$ and $p \in (2, 2^*)$.
- **(A3):** $g(x,u)/u \to 0$ uniformly in $x$ as $u \to 0$.
- **(A4):** $0 \leq 2G(x,u) \leq ug(x,u)$ on $\mathbb{R}^N \times \mathbb{R}$, where $G(x,u) := \int_0^u g(x,s)\,ds$.
- **(A5):** $0 \notin \sigma(-\Delta + V)$ and $\sigma(-\Delta + V) \cap (-\infty, 0) \neq \emptyset$, where $\sigma$ denotes the spectrum in $L^2(\mathbb{R}^N)$.

Note that we do not exclude the case of $g \equiv 0$. It is well-known that under our hypotheses on $V$ the spectrum of $-\Delta + V$ in $L^2(\mathbb{R}^N)$ is bounded below and is the union of disjoint closed intervals; see e.g. p. 161 and Theorem 4.5.9 in [12]. So (A5) is equivalent to 0 being in a spectral gap of $-\Delta + V$. According to (A3), $g(x,0) \equiv 0$. Hence $u = 0$ is necessarily a solution of (1.1).

Our main result is the following

**Theorem 1.1.** Suppose that conditions (A1)–(A5) are satisfied, $N \geq 4$ and $K(x_0) = \max_{\mathbb{R}^N} K(x)$. If $K(x) - K(x_0) = o(|x - x_0|^2)$ as $x \to x_0$ and $V(x_0) < 0$, then equation (1.1) has a solution $u \neq 0$. 

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Remark 1.2. (i) If $N = 4$, then it suffices that $K(x) - K(x_0) = O(|x - x_0|^2)$ as $x \to x_0$ (see the comment at the end of Section 4). This condition is obviously satisfied if $K$ is of class $C^2$.

(ii) The flatness condition $K(x) - K(x_0) = o(|x - x_0|^2)$ has been imposed by several authors; see e.g. [7].

As an immediate consequence of Theorem 1.1 we obtain the following:

Corollary 1.3. If conditions (A1)–(A5) are satisfied, $N \geq 4$ and $K(x) \equiv K$ is a positive constant, then equation (1.1) has a solution $u \neq 0$.

Equation (1.1) with $K \equiv 0$ and $V, g$ satisfying (A1)–(A3), (A5) and a stronger version of (A4) (the subcritical case) has been considered by several authors; see e.g. [1, 3, 5, 11, 13, 16, 17, 18] and the references there. Equation (1.1) under conditions similar to (A1)–(A5) was discussed in [6], and our Theorem 1.1 is an extension of the main result there. We also note that when $g \equiv 0$, (A5) cannot be replaced by the hypothesis that $0 \notin \sigma(-\Delta + V)$. Indeed, as was observed in [4], equation $-\Delta u + \lambda u = |u|^{2^*-2}u$, where $\lambda \neq 0$, has only the trivial solution $u = 0$ in $W^{1,2}(R^N)$.

Recall [19] that there is a one-to-one correspondence between solutions of (1.1) and critical points of the functional

$$J(u) := \frac{1}{2} \int_{R^N} (|\nabla u|^2 + Vu^2) \, dx - \frac{1}{2^*} \int_{R^N} K|u|^{2^*} \, dx - \int_{R^N} G(x, u) \, dx.$$ 

Moreover, $J \in C^1(E, R)$, where $E := W^{1,2}(R^N)$. Later we shall see that the functional $J$ has the so-called linking geometry.

In what follows we shall usually abbreviate $L^p(R^N)$ by $L^p$ and the Sobolev space $W^{m,p}(R^N)$ by $W^{m,p}$. The norms will be respectively denoted by $\| \cdot \|_p$ and $\| \cdot \|_{m,p}$.

The open ball centered at $a$ and having radius $r$ will be denoted by $B(a, r)$. The spaces $L^p$ and $W^{m,p}$ are real except in Section 2 where they are complex.

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2. The linear operator

Let $\mathcal{L}_q : D(\mathcal{L}_q) \subset L^q(R^N) \rightarrow L^q(R^N)$, $2 \leq q < \infty$, be the operator given by $\mathcal{L}_qu := -\Delta u + V(x)u$. If $q = 2$, we shall write $\mathcal{L}$ instead of $\mathcal{L}_2$. In this section we assume that $V \in L^\infty(R^N)$, $N \geq 1$, and we do not require $V$ to be periodic.

Lemma 2.1. $\mathcal{L}_q$ is a closed operator with domain $D(\mathcal{L}_q) = W^{2,q}(R^N)$.

Proof. The operator $u \mapsto (V(x) - 1)u$ is bounded in $L^q$. Therefore it suffices to prove the above statement for $-\Delta + 1$. However, this is an immediate consequence of the fact that $(-\Delta + 1)^{-1}$ is an isomorphism of $L^q$ onto $W^{2,q}$ (a property of the Bessel potentials; see formula (41) and Theorem 3 of Chap. V in [14]).  

Recall that in this section the spaces $L^p$ and $W^{m,p}$ are complex. By a result of Hempel and Voigt [8] (see also Arendt [2] Example 5.3) $\sigma(\mathcal{L}_q) = \sigma(\mathcal{L})$ and $(\mathcal{L}_q - \lambda)^{-1}|_{L^q \cap L^2} = (\mathcal{L} - \lambda)^{-1}|_{L^q \cap L^2}$ for all complex $\lambda \notin \sigma(\mathcal{L})$.

Let $(E(\lambda))_{\lambda \in R}$ be the spectral family of $\mathcal{L}$. Then for a fixed $\mu$, $E(\mu)L^2$ is the subspace of $L^2$ corresponding to $\lambda \leq \mu$.

Proposition 2.2. If $V \in L^\infty(R^N)$ satisfies (A5), then $\|u\|_{1,\infty} \leq c_0\|u\|_2$ for some constant $c_0 > 0$ and all $u \in E(0)L^2$. 


Proof. Let $\Gamma$ be a positively oriented smooth Jordan curve (in $\mathbb{C}$) containing $\sigma(L) \cap (-\infty,0)$ in its interior and the remaining part of $\sigma(L)$ in its exterior. Since $L$ is a closed operator,

\begin{equation}
E(0) = -\frac{1}{2\pi i} \int_{\Gamma} (-\Delta + V - \lambda)^{-1} \, d\lambda
\end{equation}

according to formula (III.6.19) in [10]. So

\begin{equation}
u = -\frac{1}{2\pi i} \int_{\Gamma} (-\Delta + V - \lambda)^{-1} u \, d\lambda
\end{equation}

whenever $u \in E(0)L^2$. Since $\Gamma$ is compact and $-\Delta + V - \lambda$ is invertible for each $\lambda \in \Gamma$ (as an operator from $\mathcal{D}(L)$ into $L^2$), it is easy to see from (2.2) and the Sobolev embedding theorem that $\|u\|_{q_1} \leq c_1 \|u\|_{2,q_1} \leq c_2 \|u\|_2$, where $q_1 = 2N/(N-4)$ if $N > 4$ and $q_1$ may be chosen arbitrarily large if $N \leq 4$ (here and in what follows $c_1$, $c_2$, etc. denote positive constants whose numerical values are immaterial). Keeping in mind that $L_q$ is closed and $L_q - \lambda$ is invertible on $\Gamma$ for all $q$, we may employ the usual bootstrap argument: we get $\|u\|_{q_2} \leq c_3 \|u\|_{2,q_1} \leq c_4 \|u\|_{q_1} \leq c_5 \|u\|_2$, where $q_2 = 2N/(N-8)$; after a finite number of iterations $q_k > N$ and by (2.2) again, $\|u\|_{2,q_k} \leq \tilde{c} \|u\|_2$. Now the conclusion follows by the Sobolev embedding $W^{2,q_k} \hookrightarrow W^{1,\infty}$.

Proposition 2.3 (Troestler [17]). If $V \in L^\infty(\mathbb{R}^N)$ satisfies (A5) and $q \in (2,\infty)$, then $E(0)|_{L^2 \cap L^q}$ is $L^q$-continuous. In particular, $E(0)$ and $I - E(0)$ extend to continuous projections of $L^q$ onto the complementary subspaces $c_1 L^q (E(0)L^2)$ and $c_1 L^q ((I - E(0))L^2)$ ($c_1$ denotes the closure).

Proof. By (2.1), $\|E(0)u\|_q \leq \|E(0)u\|_{2,q} \leq c_0 \|u\|_q$ for all $u \in L^2 \cap L^q$ and some $c_0 > 0$. Hence $E(0)$ and $I - E(0)$ may be extended to continuous projections of $L^q$ onto the complementary subspaces as required.

Proposition 2.4. If $V \in L^\infty(\mathbb{R}^N)$, then for each $\mu \in \mathbb{R}$ there exist constants $c_1$ and $c_2 = c_2(\mu)$ such that $\|u\|_q \leq c_1 \|u\|_{2,q} \leq c_2 \|u\|_2$ whenever $u \in E(\mu)L^2$. Here $q = 2N/(N-4)$ if $N > 4$, $q$ may be taken arbitrarily large if $N = 4$ and $q = \infty$ if $N < 4$.

Proof. The operator $L^\mu := L|_{E(\mu)L^2} : E(\mu)L^2 \to E(\mu)L^2$ is bounded. Let $\Gamma$ be a positively oriented smooth Jordan curve enclosing the spectrum of $L^\mu$. Then (2.2) still holds for all $u \in E(\mu)L^2$ (with $(-\Delta + V - \lambda)^{-1}$ replaced by $(L^\mu - \lambda)^{-1}$). Therefore $\|u\|_q \leq c_1 \|u\|_{2,q} \leq c_2 \|u\|_2$.

3. Existence of a Palais-Smale sequence

In this section we assume that the hypotheses (A1)–(A5) are satisfied. Recall $E = W^{1,2}(\mathbb{R}^N)$ and let $E^- := E(0)L^2 \cap E$ and $E^+ := (I - E(0))L^2 \cap E (E(\lambda) \text{ as is in the preceding section})$. Then the quadratic form $\int_{\mathbb{R}^N} (\nabla u^2 + Vu^2) \, dx$ is positive definite on $E^+$ and negative definite on $E^-$ [13, Sections 8 and 9]. Hence we may introduce a new inner product $\langle \cdot, \cdot \rangle$ in $E$ such that the corresponding norm $\|\cdot\|$ is equivalent to $\|\cdot\|_{1,2}$ and $\int_{\mathbb{R}^N} (\nabla u^2 + Vu^2) \, dx = \|u^+\|^2 - \|u^-\|^2$, where $u^\pm \in E^\pm$. 

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Set \( \psi(u) := (2^*)^{-1} \int_{\mathbb{R}^N} K|u|^{2^*} \, dx + \int_{\mathbb{R}^N} G(x, u) \, dx; \) then

\[
J(u) = \begin{cases} 
\frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 + Vu^2 \right) \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} K|u|^{2^*} \, dx - \int_{\mathbb{R}^N} G(x, u) \, dx \\
\frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \psi(u).
\end{cases}
\]

Let \( z_0 \in E^+ \setminus \{0\}, \)

\[ M := \{ u = u^- + sz_0 : u^- \in E^-, \ s \geq 0 \ \text{and} \ \|u\| \leq R \} \]

and denote the boundary of \( M \) in \( E^- \oplus \mathbb{R}_z \) by \( \partial M \). We summarize the properties of \( J \) in the following:

**Proposition 3.1.** (i) There exist \( \alpha, \rho > 0 \) and \( R > \rho \) (\( R \) depending on \( z_0 \)) such that \( J(u) \geq \alpha \) for all \( u \in E^+ \cap \partial B(0, \rho) \) and \( J(u) \leq 0 \) for all \( u \in \partial M \).

(ii) \( \psi \geq 0 \), \( \psi \) is weakly sequentially lower semicontinuous and \( \psi' \) is weakly sequentially continuous.

Functionals satisfying (i) above are said to have the linking geometry.

**Proof.** (i) See e.g. [11, 18, 19]. The proofs given there are for nonlinearities of subcritical growth but the argument remains unchanged in our case (the part showing \( J|_{\partial M} \leq 0 \) is in fact somewhat simpler here; observe only that \( (2^*)^{-1} K(x)|u|^{2^*} + G(x, u) \geq c_0|u|^{2^*} \) for some \( c_0 > 0 \).

(ii) It is obvious that \( \psi \geq 0 \). Let \( u_n \rightharpoonup u \). Then \( u_n \to u \) a.e. in \( \mathbb{R}^N \), possibly after passing to a subsequence. Hence it follows from the Fatou lemma that \( \psi \) is weakly sequentially lower semicontinuous. Moreover, since \( u_n \to u \) in \( L^p_{\text{loc}} \), it is easy to see from (A2) and (A3) that

\[
\int_{\mathbb{R}^N} g(x, u_n)v \, dx \to \int_{\mathbb{R}^N} g(x, u)v \, dx \quad \text{for each} \ v \in E.
\]

Finally, \( u_n \to u \) in \( L^{(N+2)/(N-2)}_{\text{loc}} \); therefore \( K|u_n|^{2^*-2}u_n \rightharpoonup K|u|^{2^*-2}u \) in \( L^1_{\text{loc}} \) and

\[
\int_{\mathbb{R}^N} K|u_n|^{2^*-2}u_n \varphi \, dx \to \int_{\mathbb{R}^N} K|u|^{2^*-2}u \varphi \, dx \quad \text{whenever} \ \varphi \in C_0^\infty.
\]

Taking into account that the sequence \( (K|u_n|^{2^*-1}) \) is bounded in \( L^{2N/(N+2)} \), we may replace \( \varphi \) by \( v \in E \). This completes the proof.

**Proposition 3.2.** If \( J \) is a functional of the form appearing in the second line of (3.1) and if (i), (ii) of Proposition 3.1 are satisfied, then there exists a Palais-Smale sequence \( (u_n) \) for \( J \) such that \( J(u_n) \to c \in [\alpha, \sup_{M} J] \).

This is a special case of Theorem 3.4 in [11]; see also Theorem 6.10 in [19].

We have thus shown that the functional \( J \) associated with (1.1) possesses a Palais-Smale sequence \( (u_n) \) with \( J(u_n) \to c \).

**Proposition 3.3.** The Palais-Smale sequence above is bounded.

**Proof.** It follows from (A2)–(A3) that for each \( \varepsilon > 0 \) there exists \( c_1(\varepsilon) \) such that \( |g(x, u)| \leq \varepsilon |u| + c_1(\varepsilon)|u|^{2^*-1} \). By (A4),

\[
c + 1 + \|u_n\| \geq J(u_n) - \frac{1}{2} \langle J'(u_n), u_n \rangle \geq \frac{1}{N} \int_{\mathbb{R}^N} K|u_n|^{2^*} \, dx
\]
for almost all \( n \), and since \( K(x) \) is bounded below by a positive constant,
\[
(3.2) \quad \|u_n\|^{2^*}_{2^*} \leq c_2 + c_3\|u_n\|.
\]
Using the Hölder and Sobolev inequalities we obtain, for large \( n \),
\[
\|u_n^+\|^2 = \langle J'(u_n), u_n^+ \rangle + \int_{\mathbb{R}^N} K|u_n|^{2^*-2}u_nu_n^+ \, dx + \int_{\mathbb{R}^N} g(x, u_n)u_n^+ \, dx
\leq \|u_n^+\| + c_4\|u_n\|^{2^*-1}\|u_n^+\| + c_5(\varepsilon\|u_n\| + c_1(\varepsilon)\|u_n\|^{2^*-1})\|u_n^+\|.
\]
Hence by \((3.2)\),
\[
\|u_n^+\| \leq c_6(\varepsilon) + c_7(\varepsilon)\|u_n\|^{(2^*-1)/2} + c_5\varepsilon\|u_n\|
\]
and a similar inequality holds for \( \|u_n^-\| \). Choosing \( \varepsilon \) sufficiently small, we see that \((u_n)\) must be bounded.

\[\square\]

4. Proof of Theorem 1.1

In the preceding section we have shown that there exists a bounded Palais-Smale sequence \((u_n)\) such that \( J(u_n) \to c \in [\alpha, \sup_M J] \). Clearly, \((u_n)\) is either

(i) **Vanishing:** For each \( r > 0 \), \( \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, r)} u_n^2 \, dx = 0 \), or

(ii) **Non-vanishing:** There exists \( r, \eta > 0 \) and a sequence \((y_n) \subset \mathbb{R}^N\) such that
\[
\lim_{n \to \infty} \int_{B(y_n, r)} u_n^2 \, dx \geq \eta.
\]

In (ii) we may assume \( y_n \in \mathbb{Z}^N \) by taking a larger \( r \) if necessary. Suppose (ii) holds and let \( \tilde{u}_n(x) := u_n(x + y_n) \). Since \( J \) is invariant with respect to the translation of \( x \) by elements of \( \mathbb{Z}^N \) (i.e. \( J(u(.)) = J(u(\cdot + y)) \) whenever \( y \in \mathbb{Z}^N \), \( \|\tilde{u}_n\| = \|u_n\| \) and \( \|J'(\tilde{u}_n)\| = \|J'(u_n)\| \)). Hence \( \tilde{u}_n \to \tilde{u} \) after passing to a subsequence, \( J'(\tilde{u}) = 0 \) and since \( \limsup_{n \to \infty} \int_{B(0, r)} \tilde{u}_n^2 \, dx \geq \eta \), \( \tilde{u} \neq 0 \). So \( \tilde{u} \) is a nontrivial solution of \((1.1)\).

To complete the proof of Theorem 1.1 it remains therefore to show that vanishing cannot occur. This will be done in the following two propositions. Let
\[
(4.1) \quad S := \inf_{u \in E \setminus \{0\}} \frac{\|\nabla u\|_{2^*}^2}{\|u\|_{2^*}^2}.
\]

**Proposition 4.1.** If \( 0 < c < c^* := \frac{S^{N/2}}{N\|K\|_{\infty}^{(N-2)/2}} \), then \((u_n)\) cannot be vanishing.

**Proof.** If \((u_n)\) is vanishing, then it follows from P.L. Lions’ lemma [19, Lemma 1.21] that \( u_n \to 0 \) in \( L^r \) whenever \( 2 < r < 2^* \). Let \((z_n)\) be a bounded sequence in \( E \). Since for each \( \varepsilon > 0 \) there is \( c_1(\varepsilon) \) such that \( |g(x, u)| \leq \varepsilon |u| + c_1(\varepsilon) |u|^{p-1} \),
\[
\int_{\mathbb{R}^N} |g(x, u_n)| |z_n| \, dx \leq c_2 \varepsilon \|u_n\| \|z_n\| + c_3(\varepsilon)\|u_n\|^{p-1}\|z_n\|.
\]

Using this and a similar argument for \( G \) we see that
\[
(4.2) \quad \int_{\mathbb{R}^N} g(x, u_n)z_n \, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} G(x, u_n) \, dx \to 0.
\]
Hence
\[
(4.3) \quad J(u_n) - \frac{1}{2} \langle J'(u_n), u_n \rangle = \frac{1}{N} \int_{\mathbb{R}^N} K|u_n|^{2^*} \, dx + o(1) \to c.
\]
Recall that \((E(\lambda))_{\lambda \in \mathbb{R}}\) is the spectral family of \(-\Delta + V\) in \(L^2\). Let \(u = u^+ + u^- \in E^+ \oplus E^-\) and write \(u^+ = w + z\), where \(w \in E(\mu)L^2\), \(z \in (I - E(\mu))L^2\), \(\mu > 0\) large (to be determined). By Proposition 2.4, \(w\) is large enough. Combining (4.4), (4.1), (4.6) and (4.5) gives

\[
\|u_n^-\| = -\langle J'(u_n), u_n^- \rangle - \int_{\mathbb{R}^N} K|u_n|^{2^*-2}u_n u_n^- dx - \int_{\mathbb{R}^N} g(x, u_n) u_n^- dx \\
\leq \|K\|_\infty \|u_n\|^{2^*-1}_{2} \|u_n^-\|_q + o(1) \to 0.
\]

Similarly,

\[
\|w_n\|^2 = \int_{\mathbb{R}^N} K|u_n|^{2^*-2}u_n w_n dx + o(1) \to 0.
\]

Hence

\[
(4.4) \quad u_n - z_n = w_n + u_n^- \to 0,
\]

and therefore

\[
(4.5) \quad \|z_n\|^2 = \int_{\mathbb{R}^N} (|\nabla z_n|^2 + V z_n^2) dx = \int_{\mathbb{R}^N} K|u_n|^{2^*-2}u_n z_n dx + o(1)
\]

\[
= \int_{\mathbb{R}^N} K|u_n|^{2^*-2} dx + o(1).
\]

Furthermore, for each \(\delta > 0\) we may find \(\mu > 0\) such that

\[
(4.6) \quad (1 - \delta) \int_{\mathbb{R}^N} |\nabla z_n|^2 dx \leq \int_{\mathbb{R}^N} (|\nabla z_n|^2 + V z_n^2) dx.
\]

Indeed, since \(z_n \in (I - E(\mu))L^2 \cap E\), we have \(\int_{\mathbb{R}^N} (|\nabla z_n|^2 + V z_n^2) dx \geq \mu \|z_n\|_2^2\) and

\[
\delta \int_{\mathbb{R}^N} |\nabla z_n|^2 dx \geq \delta(\mu - \|V\|_\infty) \|z_n\|_2^2 \geq -\int_{\mathbb{R}^N} V z_n^2 dx
\]

whenever \(\mu\) is large enough. Combining (4.4), (4.1), (4.6) and (4.5) gives

\[
(1 - \delta)S\|K\|_\infty^{-2^*/2}\left(\int_{\mathbb{R}^N} K|u_n|^{2^*} dx\right)^{2^*/2^*} \leq (1 - \delta)S\|u_n\|_2^2.
\]

\[
= (1 - \delta)S\|z_n\|_2^2 + o(1) \leq (1 - \delta) \int_{\mathbb{R}^N} |\nabla z_n|^2 dx + o(1)
\]

\[
\leq \int_{\mathbb{R}^N} K|u_n|^{2^*-2} dx + o(1).
\]

Passing to the limit and using (4.3) we obtain

\[
(1 - \delta)S\|K\|_\infty^{-2^*/2} (cN)^{2^*/2^*} \leq cN;
\]

hence either \(c = 0\) which is impossible or \((1 - \delta)N^*/c^* \leq c < c^*\) which is also impossible because \(\delta\) may be chosen arbitrarily small.

Let

\[
\varphi_\varepsilon(x) := \frac{c_N \psi_\varepsilon(x)\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{N-2)/2}},
\]
where \( c_N = (N(N-2))(N-2)/4 \), \( \varepsilon > 0 \) and \( \psi \in C^\infty_0(\mathbb{R}^N, [0, 1]) \) is such that \( \psi(x) = 1 \) for \( |x| \leq r/2 \) and \( \psi(x) = 0 \) for \( |x| \geq r \) (\( r \) to be determined). We shall need the following asymptotic estimates as \( \varepsilon \to 0^+ \) (see e.g. pp. 35 and 52 in [19]):

\[
\begin{align*}
\|\nabla \varphi_\varepsilon\|_2^2 &= S^{N/2} + O(\varepsilon^{N-2}), \quad \|\nabla \varphi_\varepsilon\|_1 = O(\varepsilon^{(N-2)/2}), \\
\|\varphi_\varepsilon\|_{2^*}^2 &= S^{N/2} + O(\varepsilon^N), \quad \|\varphi_\varepsilon\|_{2^*-1}^2 = O(\varepsilon^{(N-2)/2}), \quad \|\varphi_\varepsilon\|_1 = O(\varepsilon^{(N-2)/2})
\end{align*}
\]

and

\[
\|\varphi_\varepsilon\|_2^2 = \begin{cases} b\varepsilon^2 \log \varepsilon + O(\varepsilon^2) & \text{if } N = 4, \\
    b\varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \geq 5,
\end{cases}
\]

where \( b \) is a positive constant. Finally, let

\[
Z_\varepsilon := E^- \oplus R\varphi_\varepsilon \equiv E^- \oplus R\varphi_\varepsilon^+.
\]

We may assume without loss of generality that \( K(0) = \|K\|_\infty \) and \( V(0) < 0 \). Moreover, \( r \) in the definition of \( \varphi_\varepsilon \) may be chosen so that \( V(x) \leq -\beta \) for some \( \beta > 0 \) and all \( x \) with \( |x| \leq r \).

**Proposition 4.2.** If \( \varepsilon > 0 \) is small enough, then \( \sup_{Z_\varepsilon} J < c^* \). So in particular, if \( z_0 = \varphi_\varepsilon^+ \) with \( \varepsilon \) small enough, then \( c \leq \sup_{M} J < c^* \).

**Proof.** Let

\[
I(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{2\varepsilon} \int_{\mathbb{R}^N} K|u|^2^* \ dx.
\]

Since \( I(u) \geq J(u) \) for all \( u \), it suffices to show that \( \sup_{Z_\varepsilon} I < c^* \).

In what follows we adapt the argument on pp. 52-53 in [19]. If \( u \neq 0 \), then

\[
\max_{\ell \geq 0} I(tu) = \frac{1}{N} \left( \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2) \ dx}{\int_{\mathbb{R}^N} K|u|^2^* \ dx} \right)^{N/2} \left( \frac{\int_{\mathbb{R}^N} K|u|^2^* \ dx}{\int_{\mathbb{R}^N} K|u|^2 \ dx} \right)^{(N-2)/2}
\]

whenever the integral in the numerator above is positive, and the maximum is 0 otherwise. Let \( \|u\|_{2^*,K} := \int_{\mathbb{R}^N} K|u|^2^* \ dx \). It is easy to see from (4.9) that if

\[
\text{m}_\varepsilon := \sup_{u \in Z_\varepsilon} \int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2) \ dx < \frac{S}{\|K\|_\infty^{(N-2)/N}},
\]

then \( \sup_{Z_\varepsilon} J \leq \sup_{Z_\varepsilon} I < c^* \). So it remains to show (4.10) is satisfied for all small \( \varepsilon > 0 \).

Below we shall repeatedly use (4.7) and (4.8). Since \( \int_{\mathbb{R}^N} (|\nabla \varphi_\varepsilon|^2 + V(\varphi_\varepsilon)^2) \ dx \leq 0 \), \( \int_{\mathbb{R}^N} |\nabla \varphi_\varepsilon|^2 \ dx \leq c_1\|\varphi_\varepsilon\|_2^2 \leq c_1\|\varphi_\varepsilon\|_2^2 \to 0 \) as \( \varepsilon \to 0 \); therefore \( \|\varphi_\varepsilon\|_{2^*} \leq c_2\|\varphi_\varepsilon\|_2 \to 0 \) and \( \|\varphi_\varepsilon^*\|_{2^*-1} \to S^{N/2} \). Suppose \( \|u\|_{2^*,K} = 1 \) and write \( u = u^- + s\varphi_\varepsilon = (u^+ + s\varphi_\varepsilon^+) + s\varphi_\varepsilon^+ \). It follows from Proposition 2.3 and the argument above that \( \|u^-\|_{2^*} \leq c_3 \) and \( |s| \leq c_3 \) for some constant \( c_3 \) independent of \( \varepsilon \). By Proposition 2.2 and convexity of \( \| \cdot \|_{2^*,K} \) we obtain

\[
\begin{align*}
1 &= \|u\|_{2^*,K}^2 \geq \|s\varphi_\varepsilon\|_{2^*,K}^2 + 2^* \int_{\mathbb{R}^N} (s\varphi_\varepsilon)^{2^*-1}u^- \ dx \\
&\geq \|s\varphi_\varepsilon\|_{2^*,K}^2 - c_4\|\varphi_\varepsilon\|_{2^*-1}^2\|u^-\|_2^2.
\end{align*}
\]
Moreover, by Proposition 2.2 again,
\[
(4.12) \quad \int_{\mathbb{R}^N} (\nabla \varphi_\varepsilon \cdot \nabla u^- + V \varphi_\varepsilon u^-) \, dx \leq \ c_5(\|\nabla \varphi_\varepsilon\|_1 + \|\varphi_\varepsilon\|_1)\|u^-\|_2
= \ O(\varepsilon^{(N-2)/2})\|u^-\|_2.
\]
Since \(V(x) \leq -\beta < 0\) for \(x \in \text{supp} \varphi_\varepsilon\) and \(K(x) - K(0) = o(|x|^2)\) as \(x \to 0\),
\[
(4.13) \quad \int_{\mathbb{R}^N} \varphi_\varepsilon^2 \, dx \leq \ \left\{ \begin{array}{cl}
-d\varepsilon^2 & \text{if } N \geq 5, \\
-d\varepsilon^2 \log \varepsilon & \text{if } N = 4,
\end{array} \right.
\]
for some \(d > 0\) and
\[
(4.14) \quad \|\varphi_\varepsilon\|_{2^*,K}^2 = \|K\|_\infty \int_{\mathbb{R}^N} \varphi_\varepsilon^2 \, dx + \int_{\mathbb{R}^N} (K(x) - K(0))\varphi_\varepsilon^2 \, dx
= \|K\|_\infty S_{N/2} + o(\varepsilon^2).
\]
Let \(N \geq 5\). Using (4.12), (4.14), (4.11), (4.13) and the fact that
\[
-\|u^-\|_2^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \leq O(\varepsilon^{N-2}),
\]
we obtain
\[
m_\varepsilon \leq -\|u^-\|_2^2 + \int_{\mathbb{R}^N} (\nabla \varphi_\varepsilon)^2 + V \varphi_\varepsilon^2 \, dx \frac{\|s\varphi_\varepsilon\|_{2^*,K}^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2}{\|s\varphi_\varepsilon\|_{2^*,K}^2}
\leq -c_6\|u^-\|_2^2 + \int_{\mathbb{R}^N} (\nabla \varphi_\varepsilon)^2 + V \varphi_\varepsilon^2 \, dx \frac{\|K\|_\infty S_{N/2} + o(\varepsilon^2)}{(1 + c_4\|\varphi_\varepsilon\|_{2^*,K}^2)\|u^-\|_2^{2^*} + o(\varepsilon^2)} + O(\varepsilon^{N-2})\|u^-\|_2
\leq -c_6\|u^-\|_2^2 + \frac{S}{\|K\|_\infty^{(N-2)/N} - d_0\varepsilon^2 + o(\varepsilon^2)} + O(\varepsilon^{N-2})\|u^-\|_2
\leq \frac{S}{\|K\|_\infty^{(N-2)/N} - d_0\varepsilon^2 + o(\varepsilon^2)},
\]
where \(d_0 > 0\). If \(N = 4\), then in a similar way,
\[
m_\varepsilon \leq \frac{S}{\|K\|_\infty^{(N-2)/N} - d_0\varepsilon^2 \log \varepsilon} + o(\varepsilon^2).
\]
Hence (4.10) holds provided \(\varepsilon\) is sufficiently small. \(\square\)

Note that if \(K(x) - K(0) = O(|x|^2)\) as \(x \to 0\), then (4.14) holds with \(O(\varepsilon^2)\) replacing \(o(\varepsilon^2)\). This does not affect the estimate of \(m_\varepsilon\) if \(N = 4\). Hence for such \(N\) the conclusion of Theorem 1.1 remains valid under the weaker hypothesis on \(K\) as in Remark 1.2(i).

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