PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 1, Pages 85–93
S 0002-9939(01)06143-3
Article electronically published on May 22, 2001

ON A SEMILINEAR SCHRÖDINGER EQUATION WITH CRITICAL SOBOLEV EXPONENT

JAN CHABROWSKI AND ANDRZEJ SZULKIN

(Communicated by David S. Tartakoff)

Abstract. We consider the semilinear Schrödinger equation $-\Delta u + V(x)u = K(x)|u|^{2^*-2}u + g(x,u)$, $u \in W^{1,2}(\mathbf{R}^N)$, where $N \geq 4$, V, K, g are periodic in x_j for $1 \leq j \leq N$, $K > 0$, g is of subcritical growth and 0 is in a gap of the spectrum of $-\Delta + V$. We show that under suitable hypotheses this equation has a solution $u \neq 0$. In particular, such a solution exists if $K \equiv 1$ and $g \equiv 0$.

1. Introduction and statement of the main result

In this paper we shall be concerned with the semilinear Schrödinger equation

$$-\Delta u + V(x)u = K(x)|u|^{2^*-2}u + g(x,u), \qquad u \in W^{1,2}(\mathbf{R}^N), \tag{1.1}$$

where $N \geq 4$, $2^* := 2N/(N-2)$ is the critical Sobolev exponent and g is of subcritical growth. More precisely, we make the following assumptions:

- **(A1):** $V, K \in C(\mathbf{R}^N)$, $g \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$, $K(x) > 0$ in $\mathbf{R}^N$ and V, K, g are 1-periodic in x_j for $j = 1, \ldots, N$.
- **(A2):** $|g(x,u)| \leq c_0(1 + |u|^{p-1})$ on $\mathbf{R}^N \times \mathbf{R}$ for some $c_0 > 0$ and $p \in (2, 2^*)$.
- **(A3):** $g(x,u)/u \to 0$ uniformly in x as $u \to 0$.
- **(A4):** $0 \leq 2G(x,u) \leq ug(x,u)$ on $\mathbf{R}^N \times \mathbf{R}$, where $G(x,u) := \int_0^u g(x,s)\,ds$.
- **(A5):** $0 \notin \sigma(-\Delta + V)$ and $\sigma(-\Delta + V) \cap (-\infty, 0) \neq \emptyset$, where σ denotes the spectrum in $L^2(\mathbf{R}^N)$.

Note that we do not exclude the case of $g \equiv 0$. It is well-known that under our hypotheses on V the spectrum of $-\Delta + V$ in $L^2(\mathbf{R}^N)$ is bounded below and is the union of disjoint closed intervals; see e.g. p. 161 and Theorem 4.5.9 in [12]. So (A5) is equivalent to 0 being in a spectral gap of $-\Delta + V$. According to (A3), $g(x,0) \equiv 0$. Hence $u = 0$ is necessarily a solution of (1.1).

Our main result is the following

Theorem 1.1. *Suppose that conditions* (A1)–(A5) *are satisfied,* $N \geq 4$ *and* $K(x_0) = \max_{\mathbf{R}^N} K(x)$. *If* $K(x) - K(x_0) = o(|x - x_0|^2)$ *as* $x \to x_0$ *and* $V(x_0) < 0$, *then equation* (1.1) *has a solution* $u \neq 0$.

Received by the editors May 20, 2000.
2000 *Mathematics Subject Classification.* Primary 35B33, 35J65, 35Q55.
Key words and phrases. Semilinear Schrödinger equation, critical Sobolev exponent, linking.
The second author was supported in part by the Swedish Natural Science Research Council.

Remark 1.2. (i) If $N = 4$, then it suffices that $K(x) - K(x_0) = O(|x - x_0|^2)$ as $x \to x_0$ (see the comment at the end of Section 4). This condition is obviously satisfied if K is of class C^2.

(ii) The flatness condition $K(x) - K(x_0) = o(|x - x_0|^2)$ has been imposed by several authors; see e.g. [7].

As an immediate consequence of Theorem 1.1 we obtain the following:

Corollary 1.3. *If conditions* (A1)–(A5) *are satisfied,* $N \geq 4$ *and* $K(x) \equiv K$ *is a positive constant, then equation* (1.1) *has a solution* $u \neq 0$.

Equation (1.1) with $K \equiv 0$ and V, g satisfying (A1)–(A3), (A5) and a stronger version of (A4) (the subcritical case) has been considered by several authors; see e.g. [1, 3, 5, 9, 11, 13, 16, 17, 18] and the references there. Equation (1.1) under conditions similar to (A1)–(A5) was discussed in [6], and our Theorem 1.1 is an extension of the main result there. We also note that when $g \equiv 0$, (A5) cannot be replaced by the hypothesis that $0 \notin \sigma(-\Delta + V)$. Indeed, as was observed in [4], equation $-\Delta u + \lambda u = |u|^{2^*-2}u$, where $\lambda \neq 0$, has only the trivial solution $u = 0$ in $W^{1,2}(\mathbf{R}^N)$.

Recall [19] that there is a one-to-one correspondence between solutions of (1.1) and critical points of the functional

$$J(u) := \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u|^2 + Vu^2)\,dx - \frac{1}{2^*}\int_{\mathbf{R}^N} K|u|^{2^*}\,dx - \int_{\mathbf{R}^N} G(x,u)\,dx.$$

Moreover, $J \in C^1(E, \mathbf{R})$, where $E := W^{1,2}(\mathbf{R}^N)$. Later we shall see that the functional J has the so-called linking geometry.

In what follows we shall usually abbreviate $L^p(\mathbf{R}^N)$ by L^p and the Sobolev space $W^{m,p}(\mathbf{R}^N)$ by $W^{m,p}$. The norms will be respectively denoted by $\| \ \|_p$ and $\| \ \|_{m,p}$. The open ball centered at a and having radius r will be denoted by $B(a, r)$. The spaces L^p and $W^{m,p}$ are real except in Section 2 where they are complex.

The second author would like to thank P. Kurasov for helpful discussions on the topic of Section 2.

2. The linear operator

Let $\mathcal{L}_q : \mathcal{D}(\mathcal{L}_q) \subset L^q(\mathbf{R}^N) \to L^q(\mathbf{R}^N)$, $2 \leq q < \infty$, be the operator given by $\mathcal{L}_q u := -\Delta u + V(x)u$. If $q = 2$, we shall write $\mathcal{L}$ instead of $\mathcal{L}_2$. In this section we assume that $V \in L^\infty(\mathbf{R}^N)$, $N \geq 1$, and we do not require V to be periodic.

Lemma 2.1. $\mathcal{L}_q$ *is a closed operator with domain* $\mathcal{D}(\mathcal{L}_q) = W^{2,q}(\mathbf{R}^N)$.

Proof. The operator $u \mapsto (V(x) - 1)u$ is bounded in L^q. Therefore it suffices to prove the above statement for $-\Delta + 1$. However, this is an immediate consequence of the fact that $(-\Delta + 1)^{-1}$ is an isomorphism of L^q onto $W^{2,q}$ (a property of the Bessel potentials; see formula (41) and Theorem 3 of Chap. V in [14]). □

Recall that in this section the spaces L^p and $W^{m,p}$ are complex. By a result of Hempel and Voigt [8] (see also Arendt [2, Example 5.3]) $\sigma(\mathcal{L}_q) = \sigma(\mathcal{L})$ and $(\mathcal{L}_q - \lambda)^{-1}|_{L^q \cap L^2} = (\mathcal{L} - \lambda)^{-1}|_{L^q \cap L^2}$ for all complex $\lambda \notin \sigma(\mathcal{L})$.

Let $(E(\lambda))_{\lambda \in \mathbf{R}}$ be the spectral family of $\mathcal{L}$. Then for a fixed μ, $E(\mu)L^2$ is the subspace of L^2 corresponding to $\lambda \leq \mu$.

Proposition 2.2. *If* $V \in L^\infty(\mathbf{R}^N)$ *satisfies* (A5), *then* $\|u\|_{1,\infty} \leq c_0\|u\|_2$ *for some constant* $c_0 > 0$ *and all* $u \in E(0)L^2$.

Proof. Let Γ be a positively oriented smooth Jordan curve (in $\mathbf{C}$) containing $\sigma(\mathcal{L})\cap(-\infty,0)$ in its interior and the remaining part of $\sigma(\mathcal{L})$ in its exterior. Since $\mathcal{L}$ is a closed operator,

$$E(0) = -\frac{1}{2\pi i}\int_\Gamma (-\Delta + V - \lambda)^{-1}\, d\lambda \tag{2.1}$$

according to formula (III.6.19) in [10]. So

$$u = -\frac{1}{2\pi i}\int_\Gamma (-\Delta + V - \lambda)^{-1} u\, d\lambda \tag{2.2}$$

whenever $u \in E(0)L^2$. Since Γ is compact and $-\Delta+V-\lambda$ is invertible for each $\lambda\in\Gamma$ (as an operator from $\mathcal{D}(\mathcal{L})$ into L^2), it is easy to see from (2.2) and the Sobolev embedding theorem that $\|u\|_{q_1} \le c_1\|u\|_{2,2} \le c_2\|u\|_2$, where $q_1 = 2N/(N-4)$ if $N>4$ and q_1 may be chosen arbitrarily large if $N\le 4$ (here and in what follows c_1, c_2, etc. denote positive constants whose numerical values are immaterial). Keeping in mind that $\mathcal{L}_q$ is closed and $\mathcal{L}_q-\lambda$ is invertible on Γ for all q, we may employ the usual bootstrap argument: we get $\|u\|_{q_2} \le c_3\|u\|_{2,q_1} \le c_4\|u\|_{q_1} \le c_5\|u\|_2$, where $q_2 = 2N/(N-8)$; after a finite number of iterations $q_k > N$ and by (2.2) again, $\|u\|_{2,q_k} \le \tilde{c}\|u\|_2$. Now the conclusion follows by the Sobolev embedding $W^{2,q_k} \hookrightarrow W^{1,\infty}$. □

Proposition 2.3 (Troestler [17]). *If $V \in L^\infty(\mathbf{R}^N)$ satisfies* (A5) *and $q\in(2,\infty)$, then $E(0)|_{L^2\cap L^q}$ is L^q-continuous. In particular, $E(0)$ and $I-E(0)$ extend to continuous projections of L^q onto the complementary subspaces* $\mathrm{cl}_{L^q}\big(E(0)L^2\big)$ *and* $\mathrm{cl}_{L^q}\big((I-E(0))L^2\big)$ *(*cl *denotes the closure).*

Proof. By (2.1), $\|E(0)u\|_q \le \|E(0)u\|_{2,q} \le c_0\|u\|_q$ for all $u\in L^2\cap L^q$ and some $c_0>0$. Hence $E(0)$ and $I-E(0)$ may be extended to continuous projections of L^q onto the complementary subspaces as required. □

Proposition 2.4. *If $V\in L^\infty(\mathbf{R}^N)$, then for each $\mu\in\mathbf{R}$ there exist constants c_1 and $c_2 = c_2(\mu)$ such that $\|u\|_q \le c_1\|u\|_{2,2} \le c_2\|u\|_2$ whenever $u\in E(\mu)L^2$. Here $q = 2N/(N-4)$ if $N>4$, q may be taken arbitrarily large if $N=4$ and $q=\infty$ if $N<4$.*

Proof. The operator $\mathcal{L}^\mu := \mathcal{L}|_{E(\mu)L^2} : E(\mu)L^2 \to E(\mu)L^2$ is bounded. Let Γ be a positively oriented smooth Jordan curve enclosing the spectrum of $\mathcal{L}^\mu$. Then (2.2) still holds for all $u\in E(\mu)L^2$ (with $(-\Delta+V-\lambda)^{-1}$ replaced by $(\mathcal{L}^\mu-\lambda)^{-1}$). Therefore $\|u\|_q \le c_1\|u\|_{2,2} \le c_2\|u\|_2$. □

3. Existence of a Palais-Smale sequence

In this section we assume that the hypotheses (A1)–(A5) are satisfied. Recall $E = W^{1,2}(\mathbf{R}^N)$ and let $E^- := E(0)L^2\cap E$ and $E^+ := (I-E(0))L^2\cap E$ ($E(\lambda)$ is as in the preceding section). Then the quadratic form $\int_{\mathbf{R}^N}(|\nabla u|^2 + Vu^2)\,dx$ is positive definite on E^+ and negative definite on E^- [15, Sections 8 and 9]. Hence we may introduce a new inner product $\langle\ ,\ \rangle$ in E such that the corresponding norm $\|\ \|$ is equivalent to $\|\ \|_{1,2}$ and $\int_{\mathbf{R}^N}(|\nabla u|^2+Vu^2)\,dx = \|u^+\|^2 - \|u^-\|^2$, where $u^\pm\in E^\pm$.

Set $\psi(u) := (2^*)^{-1} \int_{\mathbf{R}^N} K|u|^{2^*}\,dx + \int_{\mathbf{R}^N} G(x,u)\,dx$; then

(3.1)
$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbf{R}^N} (|\nabla u|^2 + Vu^2)\,dx - \frac{1}{2^*}\int_{\mathbf{R}^N} K|u|^{2^*}\,dx - \int_{\mathbf{R}^N} G(x,u)\,dx \\ &= \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \psi(u). \end{aligned}$$

Let $z_0 \in E^+ \setminus \{0\}$,
$$M := \{u = u^- + sz_0 : u^- \in E^-,\ s \geq 0 \text{ and } \|u\| \leq R\}$$
and denote the boundary of M in $E^- \oplus \mathbf{R}z_0$ by ∂M. We summarize the properties of J in the following:

Proposition 3.1. (i) *There exist $\alpha, \rho > 0$ and $R > \rho$ (R depending on z_0) such that $J(u) \geq \alpha$ for all $u \in E^+ \cap \partial B(0,\rho)$ and $J(u) \leq 0$ for all $u \in \partial M$.*

(ii) *$\psi \geq 0$, ψ is weakly sequentially lower semicontinuous and ψ' is weakly sequentially continuous.*

Functionals satisfying (i) above are said to have the linking geometry.

Proof. (i) See e.g. [11, 18, 19]. The proofs given there are for nonlinearities of subcritical growth but the argument remains unchanged in our case (the part showing $J|_{\partial M} \leq 0$ is in fact somewhat simpler here; observe only that $(2^*)^{-1}K(x)|u|^{2^*} + G(x,u) \geq c_0|u|^{2^*}$ for some $c_0 > 0$).

(ii) It is obvious that $\psi \geq 0$. Let $u_n \rightharpoonup u$. Then $u_n \to u$ a.e. in $\mathbf{R}^N$, possibly after passing to a subsequence. Hence it follows from the Fatou lemma that ψ is weakly sequentially lower semicontinuous. Moreover, since $u_n \to u$ in L^p_{loc}, it is easy to see from (A2) and (A3) that
$$\int_{\mathbf{R}^N} g(x,u_n)v\,dx \to \int_{\mathbf{R}^N} g(x,u)v\,dx \text{ for each } v \in E.$$
Finally, $u_n \to u$ in $L^{(N+2)/(N-2)}_{loc}$; therefore $K|u_n|^{2^*-2}u_n \to K|u|^{2^*-2}u$ in L^1_{loc} and
$$\int_{\mathbf{R}^N} K|u_n|^{2^*-2}u_n\varphi\,dx \to \int_{\mathbf{R}^N} K|u|^{2^*-2}u\varphi\,dx \text{ whenever } \varphi \in C_0^\infty.$$
Taking into account that the sequence $(K|u_n|^{2^*-1})$ is bounded in $L^{2N/(N+2)}$, we may replace φ by $v \in E$. This completes the proof. □

Proposition 3.2. *If J is a functional of the form appearing in the second line of* (3.1) *and if* (i), (ii) *of Proposition 3.1 are satisfied, then there exists a Palais-Smale sequence (u_n) for J such that $J(u_n) \to c \in [\alpha, \sup_M J]$.*

This is a special case of Theorem 3.4 in [11]; see also Theorem 6.10 in [19].

We have thus shown that the functional J associated with (1.1) possesses a Palais-Smale sequence (u_n) with $J(u_n) \to c$.

Proposition 3.3. *The Palais-Smale sequence above is bounded.*

Proof. It follows from (A2)–(A3) that for each $\varepsilon > 0$ there exists $c_1(\varepsilon)$ such that $|g(x,u)| \leq \varepsilon|u| + c_1(\varepsilon)|u|^{2^*-1}$. By (A4),
$$c + 1 + \|u_n\| \geq J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle \geq \frac{1}{N}\int_{\mathbf{R}^N} K|u_n|^{2^*}\,dx$$

for almost all n, and since $K(x)$ is bounded below by a positive constant,

$$\|u_n\|_{2^*}^{2^*} \le c_2 + c_3\|u_n\|. \tag{3.2}$$

Using the Hölder and Sobolev inequalities we obtain, for large n,

$$\begin{aligned}\|u_n^+\|^2 &= \langle J'(u_n), u_n^+\rangle + \int_{\mathbf{R}^N} K|u_n|^{2^*-2}u_n u_n^+\,dx + \int_{\mathbf{R}^N} g(x,u_n)u_n^+\,dx\\ &\le \|u_n^+\| + c_4\|u_n\|_{2^*}^{2^*-1}\|u_n^+\| + c_5(\varepsilon\|u_n\| + c_1(\varepsilon)\|u_n\|_{2^*}^{2^*-1})\|u_n^+\|.\end{aligned}$$

Hence by (3.2),

$$\|u_n^+\| \le c_6(\varepsilon) + c_7(\varepsilon)\|u_n\|^{(2^*-1)/2^*} + c_5\varepsilon\|u_n\|,$$

and a similar inequality holds for $\|u_n^-\|$. Choosing ε sufficiently small, we see that (u_n) must be bounded. □

4. Proof of Theorem 1.1

In the preceding section we have shown that there exists a bounded Palais-Smale sequence (u_n) such that $J(u_n) \to c \in [\alpha, \sup_M J]$. Clearly, (u_n) is either

(i) *Vanishing:* For each $r > 0$, $\lim_{n\to\infty} \sup_{y\in\mathbf{R}^N} \int_{B(y,r)} u_n^2\,dx = 0$, or

(ii) *Non-vanishing:* There exist $r, \eta > 0$ and a sequence $(y_n) \subset \mathbf{R}^N$ such that

$$\limsup_{n\to\infty} \int_{B(y_n,r)} u_n^2\,dx \ge \eta.$$

In (ii) we may assume $y_n \in \mathbf{Z}^N$ by taking a larger r if necessary. Suppose (ii) holds and let $\tilde{u}_n(x) := u_n(x + y_n)$. Since J is invariant with respect to the translation of x by elements of $\mathbf{Z}^N$ (i.e. $J(u(.)) = J(u(. + y))$ whenever $y \in \mathbf{Z}^N$), $\|\tilde{u}_n\| = \|u_n\|$ and $\|J'(\tilde{u}_n)\| = \|J'(u_n)\|$. Hence $\tilde{u}_n \rightharpoonup \tilde{u}$ after passing to a subsequence, $J'(\tilde{u}) = 0$ and since $\limsup_{n\to\infty} \int_{B(0,r)} \tilde{u}_n^2\,dx \ge \eta$, $\tilde{u} \ne 0$. So $\tilde{u}$ is a nontrivial solution of (1.1). To complete the proof of Theorem 1.1 it remains therefore to show that vanishing cannot occur. This will be done in the following two propositions. Let

$$S := \inf_{u\in E\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}. \tag{4.1}$$

Proposition 4.1. *If* $0 < c < c^* := \dfrac{S^{N/2}}{N\|K\|_\infty^{(N-2)/2}}$, *then* (u_n) *cannot be vanishing.*

Proof. If (u_n) is vanishing, then it follows from P.L. Lions' lemma [19, Lemma 1.21] that $u_n \to 0$ in L^r whenever $2 < r < 2^*$. Let (z_n) be a bounded sequence in E. Since for each $\varepsilon > 0$ there is $c_1(\varepsilon)$ such that $|g(x,u)| \le \varepsilon|u| + c_1(\varepsilon)|u|^{p-1}$,

$$\int_{\mathbf{R}^N} |g(x,u_n)|\,|z_n|\,dx \le c_2\varepsilon\|u_n\|\,\|z_n\| + c_3(\varepsilon)\|u_n\|_p^{p-1}\|z_n\|.$$

Using this and a similar argument for G we see that

$$\int_{\mathbf{R}^N} g(x,u_n)z_n\,dx \to 0 \qquad \text{and} \qquad \int_{\mathbf{R}^N} G(x,u_n)\,dx \to 0. \tag{4.2}$$

Hence

$$J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle = \frac{1}{N}\int_{\mathbf{R}^N} K|u_n|^{2^*}\,dx + o(1) \to c. \tag{4.3}$$

Recall $(E(\lambda))_{\lambda\in\mathbf{R}}$ is the spectral family of $-\Delta+V$ in L^2. Let $u=u^++u^-\in E^+\oplus E^-$ and write $u^+=w+z$, where $w\in E(\mu)L^2$, $z\in(I-E(\mu))L^2$, $\mu>0$ large (to be determined). By Proposition 2.4, $w\in E$, hence also $z\in E$; moreover, $\|u_n^-\|_q\le c_4\|u_n^-\|_2\le c_5\|u_n\|$ and $\|w_n\|_q\le c_4\|w_n\|_2\le c_5\|u_n\|$, where $q=2N/(N-4)$ if $N>4$ and q may be taken arbitrarily large if $N=4$. Let r be such that $(2^*-1)/r+1/q=1$. Then $2<r<2^*$ (for $N=4$, q needs to be larger than 4). Since $\|u_n^-\|_q$ is bounded and $u_n\to 0$ in L^r, we obtain using (4.2) and the Hölder inequality that

$$\begin{aligned}\|u_n^-\|^2 &= -\langle J'(u_n),u_n^-\rangle-\int_{\mathbf{R}^N}K|u_n|^{2^*-2}u_nu_n^-\,dx-\int_{\mathbf{R}^N}g(x,u_n)u_n^-\,dx\\ &\le \|K\|_\infty\|u_n\|_r^{2^*-1}\|u_n^-\|_q+o(1)\to 0.\end{aligned}$$

Similarly,

$$\|w_n\|^2=\int_{\mathbf{R}^N}K|u_n|^{2^*-2}u_nw_n\,dx+o(1)\to 0.$$

Hence

$$u_n-z_n=w_n+u_n^-\to 0, \tag{4.4}$$

and therefore

$$\begin{aligned}\|z_n\|^2 &= \int_{\mathbf{R}^N}(|\nabla z_n|^2+Vz_n^2)\,dx \;=\; \int_{\mathbf{R}^N}K|u_n|^{2^*-2}u_nz_n\,dx+o(1)\\ &= \int_{\mathbf{R}^N}K|u_n|^{2^*}\,dx+o(1).\end{aligned} \tag{4.5}$$

Furthermore, for each $\delta>0$ we may find $\mu>0$ such that

$$(1-\delta)\int_{\mathbf{R}^N}|\nabla z_n|^2\,dx\le\int_{\mathbf{R}^N}(|\nabla z_n|^2+Vz_n^2)\,dx. \tag{4.6}$$

Indeed, since $z_n\in(I-E(\mu))L^2\cap E$, we have $\int_{\mathbf{R}^N}(|\nabla z_n|^2+Vz_n^2)\,dx\ge\mu\|z_n\|_2^2$ and

$$\delta\int_{\mathbf{R}^N}|\nabla z_n|^2\,dx\ge\delta(\mu-\|V\|_\infty)\|z_n\|_2^2\ge-\int_{\mathbf{R}^N}Vz_n^2\,dx$$

whenever μ is large enough. Combining (4.4), (4.1), (4.6) and (4.5) gives

$$\begin{aligned}(1-\delta)S\|K\|_\infty^{-2/2^*}\left(\int_{\mathbf{R}^N}K|u_n|^{2^*}\,dx\right)^{2/2^*} &\le (1-\delta)S\|u_n\|_{2^*}^2\\ &= (1-\delta)S\|z_n\|_{2^*}^2+o(1)\le (1-\delta)\int_{\mathbf{R}^N}|\nabla z_n|^2\,dx+o(1)\\ &\le \int_{\mathbf{R}^N}K|u_n|^{2^*}\,dx+o(1).\end{aligned}$$

Passing to the limit and using (4.3) we obtain

$$(1-\delta)S\|K\|_\infty^{-2/2^*}(cN)^{2/2^*}\le cN;$$

hence either $c=0$ which is impossible or $(1-\delta)^{N/2}c^*\le c<c^*$ which is also impossible because δ may be chosen arbitrarily small. □

Let

$$\varphi_\varepsilon(x):=\frac{c_N\psi(x)\varepsilon^{(N-2)/2}}{(\varepsilon^2+|x|^2)^{(N-2)/2}},$$

where $c_N = (N(N-2))^{(N-2)/4}$, $\varepsilon > 0$ and $\psi \in C_0^\infty(\mathbf{R}^N, [0,1])$ is such that $\psi(x) = 1$ for $|x| \le r/2$ and $\psi(x) = 0$ for $|x| \ge r$ (r to be determined). We shall need the following asymptotic estimates as $\varepsilon \to 0^+$ (see e.g. pp. 35 and 52 in [19]):

(4.7)
$$\|\nabla\varphi_\varepsilon\|_2^2 = S^{N/2} + O(\varepsilon^{N-2}), \quad \|\nabla\varphi_\varepsilon\|_1 = O(\varepsilon^{(N-2)/2}),$$
$$\|\varphi_\varepsilon\|_{2^*}^{2^*} = S^{N/2} + O(\varepsilon^N), \quad \|\varphi_\varepsilon\|_{2^*-1}^{2^*-1} = O(\varepsilon^{(N-2)/2}), \quad \|\varphi_\varepsilon\|_1 = O(\varepsilon^{(N-2)/2})$$

and

$$\|\varphi_\varepsilon\|_2^2 = \begin{cases} b\varepsilon^2|\log\varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \\ b\varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \ge 5, \end{cases} \tag{4.8}$$

where b is a positive constant. Finally, let

$$Z_\varepsilon := E^- \oplus \mathbf{R}\varphi_\varepsilon \equiv E^- \oplus \mathbf{R}\varphi_\varepsilon^+.$$

We may assume without loss of generality that $K(0) = \|K\|_\infty$ and $V(0) < 0$. Moreover, r in the definition of φ_ε may be chosen so that $V(x) \le -\beta$ for some $\beta > 0$ and all x with $|x| \le r$.

Proposition 4.2. *If $\varepsilon > 0$ is small enough, then $\sup_{Z_\varepsilon} J < c^*$. So in particular, if $z_0 = \varphi_\varepsilon^+$ with ε small enough, then $c \le \sup_M J < c^*$.*

Proof. Let

$$I(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{2^*}\int_{\mathbf{R}^N} K|u|^{2^*}\,dx.$$

Since $I(u) \ge J(u)$ for all u, it suffices to show that $\sup_{Z_\varepsilon} I < c^*$.

In what follows we adapt the argument on pp. 52-53 in [19]. If $u \ne 0$, then

$$\max_{t\ge 0} I(tu) = \frac{1}{N}\frac{\left(\int_{\mathbf{R}^N}(|\nabla u|^2 + Vu^2)\,dx\right)^{N/2}}{\left(\int_{\mathbf{R}^N} K|u|^{2^*}\,dx\right)^{(N-2)/2}} \tag{4.9}$$

whenever the integral in the numerator above is positive, and the maximum is 0 otherwise. Let $\|u\|_{2^*,K}^{2^*} := \int_{\mathbf{R}^N} K|u|^{2^*}\,dx$. It is easy to see from (4.9) that if

$$m_\varepsilon := \sup_{\substack{u\in Z_\varepsilon \\ \|u\|_{2^*,K}=1}} \int_{\mathbf{R}^N}(|\nabla u|^2 + Vu^2)\,dx < \frac{S}{\|K\|_\infty^{(N-2)/N}}, \tag{4.10}$$

then $\sup_{Z_\varepsilon} J \le \sup_{Z_\varepsilon} I < c^*$. So it remains to show (4.10) is satisfied for all small $\varepsilon > 0$.

Below we shall repeatedly use (4.7) and (4.8). Since $\int_{\mathbf{R}^N}(|\nabla\varphi_\varepsilon^-|^2 + V(\varphi_\varepsilon^-)^2)\,dx \le 0$, $\int_{\mathbf{R}^N}|\nabla\varphi_\varepsilon^-|^2\,dx \le c_1\|\varphi_\varepsilon^-\|_2^2 \le c_1\|\varphi_\varepsilon\|_2^2 \to 0$ as $\varepsilon \to 0$; therefore $\|\varphi_\varepsilon^-\|_{2^*} \le c_2\|\varphi_\varepsilon^-\| \to 0$ and $\|\varphi_\varepsilon^+\|_{2^*}^{2^*} \to S^{N/2}$. Suppose $\|u\|_{2^*,K} = 1$ and write $u = u^- + s\varphi_\varepsilon = (u^- + s\varphi_\varepsilon^-) + s\varphi_\varepsilon^+$. It follows from Proposition 2.3 and the argument above that $\|u^-\|_{2^*} \le c_3$ and $|s| \le c_3$ for some constant c_3 independent of ε. By Proposition 2.2 and convexity of $\|\ \|_{2^*,K}$ we obtain

$$\begin{aligned} 1 = \|u\|_{2^*,K}^{2^*} &\ge \|s\varphi_\varepsilon\|_{2^*,K}^{2^*} + 2^*\int_{\mathbf{R}^N}(s\varphi_\varepsilon)^{2^*-1}u^-\,dx \\ &\ge \|s\varphi_\varepsilon\|_{2^*,K}^{2^*} - c_4\|\varphi_\varepsilon\|_{2^*-1}^{2^*-1}\|u^-\|_2. \end{aligned} \tag{4.11}$$

Moreover, by Proposition 2.2 again,

$$\int_{\mathbf{R}^N} (\nabla\varphi_\varepsilon \cdot \nabla u^- + V\varphi_\varepsilon u^-)\,dx \;\le\; c_5(\|\nabla\varphi_\varepsilon\|_1 + \|\varphi_\varepsilon\|_1)\|u^-\|_2 = O(\varepsilon^{(N-2)/2})\|u^-\|_2. \tag{4.12}$$

Since $V(x) \le -\beta < 0$ for $x \in \operatorname{supp}\varphi_\varepsilon$ and $K(x) - K(0) = o(|x|^2)$ as $x \to 0$,

$$\int_{\mathbf{R}^N} V\varphi_\varepsilon^2\,dx \le \begin{cases} -d\varepsilon^2 & \text{if } N \ge 5, \\ -d\varepsilon^2|\log\varepsilon| & \text{if } N = 4, \end{cases} \tag{4.13}$$

for some $d > 0$ and

$$\|\varphi_\varepsilon\|_{2^*,K}^{2^*} = \|K\|_\infty \int_{\mathbf{R}^N} \varphi_\varepsilon^{2^*}\,dx + \int_{\mathbf{R}^N} (K(x) - K(0))\varphi_\varepsilon^{2^*}\,dx = \|K\|_\infty S^{N/2} + o(\varepsilon^2). \tag{4.14}$$

Let $N \ge 5$. Using (4.12), (4.14), (4.11), (4.13) and the fact that

$$-\|u^-\|_2^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \le O(\varepsilon^{N-2}),$$

we obtain

$$\begin{aligned} m_\varepsilon &\le -\|u^-\|^2 + \frac{\int_{\mathbf{R}^N}(|\nabla\varphi_\varepsilon|^2 + V\varphi_\varepsilon^2)\,dx}{\|\varphi_\varepsilon\|_{2^*,K}^2}\|s\varphi_\varepsilon\|_{2^*,K}^2 + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &\le -c_6\|u^-\|_2^2 + \frac{\int_{\mathbf{R}^N}(|\nabla\varphi_\varepsilon|^2 + V\varphi_\varepsilon^2)\,dx}{\|K\|_\infty^{(N-2)/N} S^{(N-2)/2} + o(\varepsilon^2)}(1 + c_4\|\varphi_\varepsilon\|_{2^*-1}^{2^*-1}\|u^-\|_2)^{2/2^*} \\ &\quad + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &= -c_6\|u^-\|_2^2 + \frac{S^{N/2} - d\varepsilon^2 + O(\varepsilon^{N-2})}{\|K\|_\infty^{(N-2)/N} S^{(N-2)/2} + o(\varepsilon^2)} + O(\varepsilon^{(N-2)/2})\|u^-\|_2 \\ &\le \frac{S}{\|K\|_\infty^{(N-2)/N}} - d_0\varepsilon^2 + o(\varepsilon^2), \end{aligned}$$

where $d_0 > 0$. If $N = 4$, then in a similar way,

$$m_\varepsilon \le \frac{S}{\|K\|_\infty^{(N-2)/N}} - d_0\varepsilon^2|\log\varepsilon| + o(\varepsilon^2).$$

Hence (4.10) holds provided ε is sufficiently small. □

Note that if $K(x) - K(0) = O(|x|^2)$ as $x \to 0$, then (4.14) holds with $O(\varepsilon^2)$ replacing $o(\varepsilon^2)$. This does not affect the estimate of m_ε if $N = 4$. Hence for such N the conclusion of Theorem 1.1 remains valid under the weaker hypothesis on K as in Remark 1.2(i).

References

[1] S. Alama and Y.Y. Li, *On "Multibump" bound states for certain semilinear elliptic equations*, Indiana J. Math. 41 (1992), 983–1026. MR **94d:**35044

[2] W. Arendt, *Gaussian estimates and interpolation of the spectrum in L^p*, Diff. Int. Eq. 7 (1994), 1153–1168. MR **95e:**47066

[3] T. Bartsch, Y. Ding, *On a nonlinear Schrödinger equation with periodic potential*, Math. Ann. 313 (1999), 15–37. MR **99m:**35218

[4] V. Benci and G. Cerami, *Existence of positive solutions of the equation $-\Delta u + a(x)u = u^{(N+2)/(N-2)}$ in $\mathbf{R}^N$*, J. Func. Anal. 88 (1990), 90–117. MR **91f:**35097

[5] B. Buffoni, L. Jeanjean and C.A. Stuart, *Existence of nontrivial solutions to a strongly indefinite semilinear equation*, Proc. Amer. Math. Soc. 119 (1993), 179–186. MR **93k:**35086

[6] J. Chabrowski and J. Yang, *On Schrödinger equation with periodic potential and critical Sobolev exponent*, Topol. Meth. Nonl. Anal. 12 (1998), 245–261. MR **2000f:**35027

[7] J.F. Escobar, *Positive solutions for some semilinear elliptic equations with critical Sobolev exponents*, Comm. Pure Appl. Math. 40 (1987), 623–657. MR **88k:**35013

[8] R. Hempel and J. Voigt, *The spectrum of a Schrödinger operator in $L_p(\mathbb{R}^\nu)$ is p-independent*, Comm. Math. Phys. 104 (1986), 243–250. MR **87h:**35247

[9] L. Jeanjean, *Solutions in spectral gaps for a nonlinear equation of Schrödinger type*, J. Diff. Eq. 112 (1994), 53–80. MR **95h:**35068

[10] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1980. MR **96a:**47025 (Reprint of 1980 ed.)

[11] W. Kryszewski and A. Szulkin, *Generalized linking theorem with an application to semilinear Schrödinger equation*, Adv. Diff. Eq. **3** (1998), 441–472. CMP 2000:11

[12] P. Kuchment, *Floquet Theory for Partial Differential Equations*, Birkhäuser, Basel, 1993. MR **94h:**35002

[13] A.A. Pankov and K. Pflüger, *On a semilinear Schrödinger equation with periodic potential*, Nonl. Anal. TMA 33 (1998), 593–609. MR **2001a:**35167

[14] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970. MR **44:**7280

[15] C.A. Stuart, *Bifurcation into spectral gaps*, Bull. Belg. Math. Soc., Supplement, 1995. MR **96m:**47115

[16] A. Szulkin and W. Zou, *Homoclinic orbits for asymptotically linear Hamiltonian systems*, Preprint.

[17] C. Troestler, *Bifurcation into spectral gaps for a noncompat semilinear Schrödinger equation with nonconvex potential*, Preprint.

[18] C. Troestler and M. Willem, *Nontrivial solution of a semilinear Schrödinger equation*, Comm. P.D.E. 21 (1996), 1431–1449. MR **98i:**35034

[19] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996. MR **97h:**58037

Department of Mathematics, University of Queensland, St. Lucia 4072, Queensland, Australia

E-mail address: `jhc@maths.uq.edu.au`

Department of Mathematics, Stockholm University, 106 91 Stockholm, Sweden

E-mail address: `andrzejs@matematik.su.se`