ON THE $L^p$ BOUNDEDNESS OF THE NON-CENTERED GAUSSIAN HARDY-LITTLEWOOD MAXIMAL FUNCTION

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Abstract. The purpose of this paper is to prove the $L^p(\mathbb{R}^n, d\gamma)$ boundedness, for $p > 1$, of the non-centered Hardy-Littlewood maximal operator associated with the Gaussian measure $d\gamma = e^{-|x|^2} dx$.

Let $d\gamma = e^{-|x|^2} dx$ be a Gaussian measure in Euclidean space $\mathbb{R}^n$. We consider the non-centered maximal function defined by

$$\mathcal{M}f(x) = \sup_{x \in B} \frac{1}{\gamma(B)} \int_B |f| d\gamma,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$ containing $x$. P. Sjögren [2] proved that $\mathcal{M}$ is not of weak type $(1,1)$ with respect to $d\gamma$ for $n > 1$. A more general result was obtained by A. Vargas [4], who characterized those radial and strictly positive measures for which the corresponding maximal operator is of weak type $(1,1)$. However, these papers leave open the question of the $L^p(d\gamma)$ boundedness of $\mathcal{M}$ for $p > 1$ and $n > 1$.

The main result in this paper is

**Theorem 1.** $\mathcal{M}$ is a bounded operator on $L^p(d\gamma)$ for $p > 1$, that is, there exists a constant $C = C(n, p)$ such that for $f \in L^p(d\gamma)$,

$$\|\mathcal{M}f\|_{L^p(d\gamma)} \leq C\|f\|_{L^p(d\gamma)}.$$

In a forthcoming paper [3], P. Sjögren and F. Soria prove estimates for the maximal operator associated with a more general radial measure with decreasing density.

We denote $S^n_0 = \{x \in \mathbb{R}^n : |x| = r\}$ and $S^n_1 = S^n_1$, and write $d\sigma$ for the area measure on $S^n_1$. The spherical maximal function

$$\mathcal{M}^\sigma f(h) = \sup_{R > 0} \frac{1}{\sigma(\{z' : |z'-h| \leq R\})} \int_{|z'-h| \leq R} |f(z')| d\sigma(z'), \quad h \in S^{n-1},$$

is bounded on $L^p(d\sigma)$. We extend $\mathcal{M}^\sigma$ to functions defined in $\mathbb{R}^n$ by using polar coordinates $x = px'$ with $x' \in S^{n-1}$ and applying $\mathcal{M}^\sigma$ in the $x'$ variable. Then $\mathcal{M}^\sigma$ is bounded on $L^p(d\gamma)$.

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In order to prove Theorem 1, we need the following technical lemma, proved later.

**Lemma 1.** Let $B$ be a closed ball in $\mathbb{R}^n$ of radius $r$. Denote by $q$ the point of $B$ whose distance to the origin is minimal. Assume that $|q| \geq 1$ and that $r \geq 1/|q|$. Then for all $x, y \in B$

\begin{equation}
\gamma(B) \leq C \frac{e^{-|q|^2}}{|q|} \left(1 + \frac{|y - x|^2}{|q|(|x| \vee |y| - |q|)}\right)^{-\frac{n+1}{n}}.
\end{equation}

Here and in the sequel, we write $\gamma$ for various positive finite constants and denote $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

**Proof of Theorem 1.** We assume that $n \geq 2$, since the case $n = 1$ is well known; see, e.g., [2]. Take $0 \leq f \in L^p(d\gamma)$ and $x \in \mathbb{R}^n$. For any ball $B$ containing $x$, we must estimate the average $A f(B) = \frac{1}{\gamma(B)} \int_B f \, d\gamma$. Let $r$ and $q$ be defined as in Lemma 1.

We first consider small balls $B$, and denote by $\mathcal{M}_0 f(x)$ the supremum of $A f(B)$ taken only over balls $B$ containing $x$ and verifying $r < 1 \wedge |q|^{-1}$. Split $\mathbb{R}^n$ into rings $R_k = \{x : \sqrt{k} - 1 \leq |x| < \sqrt{k}\}$, $k = 1, 2, \ldots$. The width of $R_k$ is no larger than $1/\sqrt{k}$, and so the Gaussian density is of constant order of magnitude in each $R_k$. Using Lebesgue measure arguments, one can easily estimate the $L^p(d\gamma)$ norm of $\mathcal{M}_0 f$ in $R_k$ in terms of the $L^p(d\gamma)$ norm of $f$ in $\bigcup R_k$. This takes care of small balls.

Consider now balls $B$ with $r \geq 1 \wedge |q|^{-1}$. To begin with observe that the case $|q| < 2$ is simple, since then $\gamma(B) \geq C$ and thus

$$Af(B) \leq C \int f \, d\gamma \leq C \| f \|_{L^p(d\gamma)}.$$  

The corresponding part of $\mathcal{M} f$ thus satisfies the $L^p(d\gamma)$ estimate.

It remains to consider $\mathcal{M}_0 f(x) = \sup B A f(B)$, the supremum taken over balls $B$ containing $x$ and with the property that $r \geq |q|^{-1}$ and $|q| \geq 2$. Let $B$ be such a ball, and observe that it satisfies the hypotheses of Lemma 1.

For each $\rho \geq 1$ such that $S_{\rho}^{n-1}$ intersects $B$, let $y_\rho \in S_{\rho}^{n-1} \cap \partial B$ be such that $|y_\rho - x| = \sup_{z \in B \cap S_{\rho}^{n-1}} |z - x|$. Write $x' = x/|x|$.

For each $z' \in S_{\rho}^{n-1}$ such that $\rho z' \in B$ we have

\begin{equation}
|x' - z'| = \frac{1}{\rho} |\rho x' - \rho z'|
\leq \frac{1}{\rho} \left( |x - \rho z'| + |\rho - |x|| \right)
\leq \frac{2}{\rho} |y_\rho - x|,
\end{equation}

and trivially $|x' - z'| \leq 2$. 

Because of (2) and the definition of $M^e$,

\begin{equation}
Af(B) = \int_{|q|}^{q+2r} \frac{1}{\gamma(B)} \int_{S^{n-1}} \chi_B(\rho z') f(\rho z') d\sigma(z') \rho^{n-1} e^{-\rho^2} d\rho \\
\leq \int_{|q|}^{q+2r} \frac{1}{\gamma(B)} \int_{|z'-x| \leq 2(1 + \frac{|y_q - z|}{\rho})} f(\rho z') d\sigma(z') \rho^{n-1} e^{-\rho^2} d\rho \\
\leq C \int_{|q|}^{q+2r} \left\{ \frac{1}{\gamma(B)} \right\} M^e f(\rho x') \rho^{n-1} e^{-\rho^2} d\rho \\
\leq C \int_{|q|}^{q+2r} |q| \left( 1 \vee \left( \frac{|q|(M - |q|)}{|x - y_q|^2} \right)^{\frac{1}{n-1}} \right) \left\{ 1 \wedge \left( \frac{|y_q - x|}{\rho} \right)^{n-1} \right\} \frac{1}{\gamma(B)} \\
M^e f(\rho x') \rho^{n-1} e^{-\rho^2} d\rho,
\end{equation}

where we applied Lemma 1 with $y = y_q$ to get the last inequality.

Write $M = \rho \vee |x|$ and $m = \rho \wedge |x|$, so that $|q| \leq m \leq M$.

**Lemma 2.** For $|q| < \rho < |q| + 2r$ and some $C$,

\begin{equation}
\left| q \right|^2 \left( 1 \vee \left( \frac{|q|(M - |q|)}{|x - y_q|^2} \right)^{\frac{1}{n-1}} \right) \left\{ 1 \wedge \left( \frac{|y_q - x|}{\rho} \right)^{n-1} \right\} \leq C \left( \frac{1}{m^2} \vee \frac{M - m}{m} \right)^{\frac{n-1}{2}}.
\end{equation}

Assuming this lemma for the moment, we conclude from (3) that

\[ Af(B) \leq C \int_{1}^{\infty} m e^{m^2} \left( \frac{1}{m^2} \vee \frac{M - m}{m} \right)^{\frac{n-1}{2}} M^e f(\rho x') \rho^{n-1} e^{-\rho^2} d\rho. \]

We split this integral into five integrals taken over the following intervals:

1. $I_1 = \left[ 1, \frac{|x|}{2} \right]$,
2. $I_2 = \left( \frac{|x|}{2}, |x| - \frac{1}{|x|} \right]$,
3. $I_3 = \left( |x| - \frac{1}{|x|}, |x| + \frac{1}{|x|} \right]$,
4. $I_4 = \left( |x| + \frac{1}{|x|}, \frac{5}{4} |x| \right]$,
5. $I_5 = \left( \frac{5}{4} |x|, +\infty \right]$.

Let for $i = 1, ..., 5$

\[ \mathcal{M}_i f(x) = \int_{I_i} m e^{m^2} \left( \frac{1}{m^2} \vee \frac{M - m}{m} \right)^{\frac{n-1}{2}} M^e f(\rho x') \rho^{n-1} e^{-\rho^2} d\rho. \]

Then $\mathcal{M} f \leq C \sum_{i=1}^{5} \mathcal{M}_i f$.

**Bound for $\mathcal{M}_1 f(x)$**. One finds that

\[ \mathcal{M}_1 f(x) \leq \left| x \right|^n \int_{1}^{\left| x \right|} M^e f(\rho x') d\rho. \]
Hölder’s inequality and the $L^p(\sigma)$ boundedness of $M^e$ imply
\[
\| M_1 f \|_{L^p(\sigma)}^p \leq \int_1^{+\infty} \int s^{n-1} \left( \int_1^s f(x') d\sigma(x') s^{n-1} e^{-s^2} ds \right)^p d\sigma(x') \leq \int_1^{+\infty} s^{n-1} \left( \int_1^s [\mathcal{M}^e f(x')]^{p'} e^{s^2} d\sigma(x') s^{n-1} e^{-s^2} ds \right)^{\frac{p}{p'}} \leq \int_1^{+\infty} \left( s e^{-\frac{1}{2}s^2} \right)^p \| f \|_{L^p(\sigma)} \| f \|_{L^p(\sigma)} \cdot
\]

Bound for $M_2 f(x)$. Making the change of variable $\rho = |x| - \frac{t}{|x|}$, we get
\[
M_2 f(x) \leq |x|^{n+1} \int_0^{|x|/2} (|x| - |x|) \mathcal{M}^e f(x') \, d\rho \leq \int_1^{|x|^2/2} t^{n+1} \mathcal{M}^e f \left( (|x| - \frac{t}{|x|})x' \right) \, dt.
\]

From Minkowski’s integral inequality and the $L^p(\sigma)$ boundedness of $M^e$, we obtain
\[
\| M_2 f \|_{L^p(\sigma)} \leq \int_1^{+\infty} t^{n+1} \left( \int |x|^{n+1} \mathcal{M}^e f \left( (|x| - \frac{t}{|x|})x' \right) t^{n+1} \mathcal{M}^e f \left( (|x| - \frac{t}{|x|})x' \right) \right)^{\frac{p}{p}} \, dt.
\]

We now make the change of variables $s \to \rho = s - t/s$, observing that $s \leq 2\rho$ and $-s^2 = -\rho^2 - 2t + t^2/s^2 \leq -\rho^2 - 2t/2$ and $d\rho/ds \geq 1$. Thus
\[
\| M_2 f \|_{L^p(\sigma)} \leq C \int_1^{+\infty} \int_{\mathbb{R}_+} \left( \int_{|x| - 1/|x|}^{+\infty} \int_{|x| - 1/|x|}^{+\infty} f(x') |x|^{n+1} e^{-3t^2/4} d\rho d\sigma(x') \right)^{\frac{p}{p'}} \, dt \leq C \| f \|_{L^p(\sigma)} \left( \int_1^{+\infty} \int_{\mathbb{R}_+} e^{-\frac{3t^2}{4}} \, dt \right) \leq C \| f \|_{L^p(\sigma)}.
\]

Bound for $M_3 f(x)$. Let $d\mu = \rho^{n-1} e^{-\rho^2} d\rho$ in $\mathbb{R}_+$. We have
\[
M_3 f(x) \leq C |x| \int_{|x| - 1/|x|}^{+\infty} \mathcal{M}^e f(x') d\rho \leq C (\mu(|x| - 1/|x|, |x| + 1/|x|))^{-1} \int_{|x| - 1/|x|}^{+\infty} \mathcal{M}^e f(x') d\mu(\rho).
\]

Let $M^\mu$ denote the one-dimensional centered maximal operator defined in terms of $\mu$, acting in the $\rho$ variable. Then
\[
M_3 f(x) \leq C M^\mu M^e f(|x|x').
\]
But $M^n$ is known to be bounded on $L^p(d\mu)$; see [1] or [2]. The $L^p(d\gamma)$ boundedness of $M_4$ follows.

**Bound for $M_4 f(x)$.** Making the change of variable $\rho = |x| + \frac{t}{|x|}$, we have

$$M_4 f(x) \leq C |x|^{\frac{n+2}{2}} e^{\frac{|x|^2}{2}} \int_{|x| + \frac{t}{|x|}}^{\infty} (\rho - |x|)^{\frac{n-1}{2}} M^e f(\rho x') e^{-\rho^2} d\rho$$

$$\leq C \int_{1}^{\infty} t^{\frac{n-2}{2}} M^e f \left( \left( |x| + \frac{t}{|x|} \right)x' \right) e^{-2t} e^{-\frac{2t^2}{s^2}} dt.$$

Minkowski’s integral inequality implies

$$\|M_4 f\|_{L^p(d\gamma)} \leq C \int_{1}^{\infty} t^{\frac{n-2}{2}} \left\| M^e f \left( \left( |x| + \frac{t}{|x|} \right)x' \right) e^{-\frac{2t^2}{s^2}} \chi_{\{1 \leq t \leq \frac{s^2}{t} \}} \right\|_{L^p(d\gamma)} e^{-2t} dt.$$

But $M^e$ is bounded on $L^p(d\sigma)$, so that

$$\|M^e f\|_{L^p(d\gamma)} \leq C \int_{1}^{\infty} t^{\frac{n-2}{2}} \left\| f \left( \frac{t}{s} x' \right) e^{-\frac{2t^2}{s^2}} \chi_{\{1 \leq t \leq \frac{s^2}{t} \}} \right\|_{L^p(d\gamma)} e^{-2t} dt.$$

Almost as in the case of $M_2$, we make the change of variable $\rho = s + t/s$ and observe that $s \leq \rho$ and $-s^2 = -\rho^2 + 2t + t^2/s^2$ and $d\rho/ds \geq 1/2$. Since $e^{-\rho^2/s^2} e^{2t/s^2} < 1$, it follows that the above double integral is at most

$$C \int_{s-1}^{s+1} \int_{1}^{\infty} |f(\rho x')|^p \rho^{-n-1} e^{-\rho^2} d\rho d\sigma(x') e^{-2t} \leq C \|f\|_{L^p(d\gamma)}^p e^{-2t}.$$

Thus

$$\|M_4 f\|_{L^p(d\gamma)} \leq C \int_{1}^{\infty} t^{\frac{n-2}{2}} \|f\|_{L^p(d\gamma)} e^{\frac{2t}{s^2}} e^{-2t} dt \leq C \|f\|_{L^p(d\gamma)}.$$

**Bound for $M_5 f(x)$.** Observe that

$$M_5 f(x) \leq |x|^{\frac{n-2}{2}} e^{\frac{|x|^2}{2}} \int_{\frac{t}{|x|}}^{\infty} M^e f(\rho x') \rho^{\frac{n-1}{2}} \rho^{n-1} e^{-\rho^2} d\rho.$$

We take the $L^p$ norm and then apply Hölder’s inequality, getting

$$\|M_5 f\|_{L^p(d\gamma)} \leq \int_{1}^{\infty} \int_{s-1}^{s+1} \left( \int_{|x| + \frac{t}{|x|}}^{\infty} M^e f(\rho x') \rho^{\frac{n-1}{2}} \rho^{n-1} e^{-\rho^2} d\rho \right)^p d\sigma(x') s^{n-1} e^{-s^2} ds$$

$$\leq \int_{1}^{\infty} \int_{s-1}^{s+1} \left[ M^e f(\rho x') \right]^p \rho^{n-1} e^{-\rho^2} d\rho \left( \int_{1}^{\infty} \rho^{\frac{n-1}{2}(n-1)} e^{-\rho^2} d\rho \right)^{\frac{-p}{2}} d\sigma(x') s^{n-1} e^{-s^2} ds$$

$$\leq \|f\|_{L^p(d\gamma)} \left( \int_{1}^{\infty} s^C \rho^{-p-1} s^2 e^{-\rho^2} d\rho \right)^{\frac{p}{4}}$$

$$\leq C \|f\|_{L^p(d\gamma)}.$$

To finish the proof of Theorem [1], it now only remains to prove the two lemmas.
Proof of Lemma 1. Consider the hyperplane orthogonal to \( q \) whose distance from the origin is \( |q| + t \), with \( 1/(2|q|) < t < 1/|q| \). Its intersection with \( B \) is an \((n-1)\)-dimensional ball whose radius is at least \( C \sqrt{r/t} \geq C \sqrt{r/|q|} \). Integrating the Gaussian density first along this \((n-1)\)-dimensional ball and then in \( t \), we get

\[
\gamma(B) \geq \int_{1/(2|q|)}^{1/|q|} e^{-(|q|+t)^2} dt \int_{|v|<C \sqrt{r/|q|}} e^{-|v|^2} dv,
\]

where \( v \) is an \((n-1)\)-dimensional variable. The inner integral here is at least \( C \min(1, (r/|q|)^{(n-1)/2}) \), and \( e^{-((|q|+t)^2)} \geq Ce^{-|q|^2} \) for these \( t \); therefore

\[
\gamma(B) \geq C e^{-|q|^2} \left( 1 \wedge \left( \frac{r}{|q|} \right)^{\frac{n-1}{2}} \right).
\] (4)

To estimate \( r \) from below, we let \( z \) be the center of \( B \) and \( w \) the projection of \( x \) onto the line passing through 0, \( q \) and \( z \). Write \( h = |x - w| \) and \( a = |w - q| \). Applying the Pythagorean Theorem twice, we get

\[
|x - z|^2 - (r - a)^2 = h^2 = |x - q|^2 - a^2.
\]

Since \( |x - z| \leq r \), we conclude that \( 2ar \geq |x - q|^2 \). Clearly \( a \leq |x| - |q| \) so that

\[
r \geq \frac{|x - q|^2}{2(|x| - |q|)} \geq \frac{|x - q|^2}{2(|x| \vee |y| - |q|)}.
\]

Since \( x \) and \( y \) are arbitrary points of \( B \), the same argument also implies

\[
r \geq \frac{|y - q|^2}{2(|x| \vee |y| - |q|)}.
\]

From the triangle inequality we conclude that \( 2|x - q| \vee |y - q| \geq |x - y| \), and so

\[
r \geq \frac{|x - y|^2}{8(|x| \vee |y| - |q|)}.
\]

Combining this with (4), we obtain the inequality of Lemma 1.

Proof of Lemma 2. We write LHS for the left-hand side of the inequality to be proved. Assume first that

\[
\left( \frac{|q|(M - |q|)}{|x - y_\rho|^2} \right)^{\frac{n-1}{2}} \leq 1.
\] (5)

Then LHS \( \leq e^{\rho^2(|x - y_\rho|/\rho)^n} \). The angles at \( q \) of the triangles \( 0qx \) and \( 0qy_\rho \) are obtuse, so that \( |x|^2 \geq |q|^2 + |x - q|^2 \) and \( |y_\rho|^2 \geq |q|^2 + |y_\rho - q|^2 \). But \( |x - y_\rho| \leq |x - q| + |y_\rho - q| \), and this implies

\[
|x - y_\rho|^2 \leq 4 \max(|x - q|^2, |y_\rho - q|^2) \leq 4 \max(|x|^2 - |q|^2, |y_\rho|^2 - |q|^2) = 4(M^2 - |q|^2).
\]

If \( |x| \leq 2\rho \), this last quantity is at most \( 16\rho(M - |q|) \), and then

\[
\text{LHS} \leq Ce^{|q|^2} \left( \frac{M - |q|}{\rho} \right)^{\frac{n-1}{2}}.
\] (6)

In the contrary case \( |x| > 2\rho \), we simply observe that LHS \( \leq Ce^{|q|^2} \) whereas the right-hand side is at least \( Ce^{n^2} \). This case of the lemma is thus trivial.
Assume now that (5) is false. Then

\[
LHS \leq e^{q^2 (|q|(M - |q|))^{\frac{n-1}{2}}} \rho^{n-1}
\]

and we arrive again at (6).

It thus only remains to see that (6) implies Lemma 2. This would follow from the estimate

\[
q^2 (M - |q|)^{\frac{n-1}{2}} \leq C((1/m) \vee (M - m))^{\frac{n-1}{2}}.
\]

To prove (7), we use the fact that

\[
q^2 (M - |q|)^{\frac{n-1}{2}} \leq C \left( (M - m)^{\frac{n-1}{2}} + (m - |q|)^{\frac{n-1}{2}} \right)
\]

and when \( m - |q| > 1/m \) also

\[
e^{q^2 - m^2} e^{-m(m-|q|)(m+|q|)} \leq \frac{C}{(m - |q|)^{\frac{n-1}{2}} m^{\frac{n-1}{2}}}
\]

Now (7) and Lemma 2 follow.

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