EXPLICIT EVALUATIONS OF A RAMANUJAN-SELBERG CONTINUED FRACTION

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To the memory of my father, Professor Guang-Da Zhang

Abstract. This paper gives explicit evaluations for a Ramanujan-Selberg continued fraction in terms of class invariants and singular moduli.

§1. Introduction

Let, for $|q| < 1$,

\begin{equation}
N(q) = 1 + \frac{q}{1 + \frac{q + q^2}{1 + \frac{q^3 + q^4}{1 + \cdots}}}.
\end{equation}

Set

\begin{equation}
(a; q)_\infty := \prod_{k=1}^{\infty} (1 - aq^{k-1}).
\end{equation}

In his notebooks \cite{14}, p. 290, Ramanujan asserted that

\begin{equation}
N(q) = \frac{(-q; q^2)_\infty}{(-q^2; q^2)_\infty}.
\end{equation}

This formula was first proved in print by A. Selberg \cite{18}. Other proofs have been given by K. G. Ramanathan \cite{12}, G. Andrews et al. \cite{11} and the author \cite{21}.

In his "Lost" Notebooks \cite{16}, p. 44, Ramanujan also stated that if $|q| < 1$, and

\begin{equation}
L(q) = 1 + \frac{q^2}{1 + \frac{q + q^3}{1 + \frac{q^4}{1 + \cdots}}},
\end{equation}

then

\begin{equation}
L(q) = \frac{(-q; q^2)_\infty}{(-q^2; q^2)_\infty}.
\end{equation}

Here, we just point out that (1.5) can be proved by using the well-known Heine \cite{10} continued fraction formula in the same fashion as the proof of (1.3) in the author’s paper \cite{21}. Set, for $|q| < 1$,

\begin{equation}
S_1(q) = \frac{q^{1/8}}{1 + \frac{q}{1 + \frac{q + q^2}{1 + \frac{q^3 + q^4}{1 + \cdots}}}.
\end{equation}

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From (1.1), (1.3) and (1.5), we have
\begin{equation}
S_1(q) = \frac{q^{1/8}}{N(q)} = \frac{q^{1/8}(-q^2;q^2)_\infty}{(-q;q^2)_\infty}.
\end{equation}
We call $S_1(q)$ the Ramanujan-Selberg continued fraction.

Also, set
\begin{equation}
S_2(q) = \frac{q^{1/8}}{1 + \frac{-q}{1 + \frac{-q + q^2}{1 + \frac{-q^3}{1 + \frac{q^2 + q^4}{1 + \cdots}}}}},
\end{equation}
Replacing $q$ by $-q$ in (1.1) and (1.3), one can see that
\begin{equation}
S_2(q) = \frac{q^{1/8}}{N(-q)} = \frac{q^{1/8}(-q^2;q^2)_\infty}{(q;q^2)_\infty}.
\end{equation}
The famous Rogers-Ramanujan continued fraction is defined by
\begin{equation}
F(q) = \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \frac{q^4}{1 + \cdots}}}}},
\end{equation}
and let $S(q) = -F(-q)$. In his first letter to G. H. Hardy, Ramanujan asserted that
\begin{equation}
F(e^{-2\pi}) = \sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5} + 1}{2},
\end{equation}
\begin{equation}
S(e^{-\pi}) = \sqrt{\frac{5 - \sqrt{5}}{2}} - \frac{\sqrt{5} - 1}{2},
\end{equation}
and
\begin{equation}
F(e^{-\pi\sqrt{n}}) \text{ can be exactly found if } n \text{ is any positive rational quantity.}
\end{equation}
Identities (1.11) and (1.12) were first proved by G. N. Watson [19]. Watson vaguely discussed (1.13) and merely claimed that $F(e^{-\pi\sqrt{n}})$ is an algebraic number.

Ramanathan [13] computed $F(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for several positive rational numbers $n$ for which the ideal class groups of $K = \mathbb{Q}(\sqrt{-n})$ have the property that each genus contains a single class. By using Weber-Ramanujan’s class invariants and a modular equation of degree 5, Berndt, Chan and the author [4] were able to establish general formulas for $F(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$.

The aim of this note is to establish general formulas for the Ramanujan-Selberg continued fraction and its companion in terms of class invariants, or equivalently in terms of singular moduli.

\section{Explicit Formulas for $S_1(q)$ and $S_2(q)$}

For $q = \exp(-\pi\sqrt{n})$, where $n$ is positive rational, let
\begin{equation}
G_n := 2^{-1/4}q^{1/24}(-q;q^2)_\infty
\end{equation}
and
\begin{equation}
g_n := 2^{-1/4}q^{1/24}(q;q^2)_\infty.
\end{equation}
We shall refer to $G_n$ and $g_n$ as the \textit{Ramanujan-Weber class invariants}, which can be roughly viewed as generators of the Hilbert class field of the complex quadratic field of $K = \mathbb{Q}(\sqrt{-n})$. The reader is referred to the important paper of B. Birch [7] and the excellent books of Cox [9] and Lang [11]. We also use modular equations in
the sequel, and refer to [2, pp. 213, 214] for this terminology. The singular modulus
\( \alpha := \alpha_n \) is that unique positive number \( \alpha_n \) between 0 and 1 satisfying

\[
\sqrt{n} = \frac{2F_1\left(1; 1; 1; 1 - \alpha_n\right)}{2F_1\left(1; 1; 1; \alpha_n\right)},
\]

where \( 2F_1 \) is the hypergeometric function. Moreover (cf. [2, p. 102]),

\[
2F_1\left(1; 1; 1/2; 1/2; 1; \alpha\right) = \frac{2}{\pi} \int_0^\pi \frac{d\phi}{\sqrt{1 - \alpha \sin^2 \phi}}.
\]

Then we have [3, p. 185]

\[
G_n = (4\alpha_n(1 - \alpha_n))^{-1/24}
\]

and

\[
g_n = (4\alpha_n(1 - \alpha_n)^2)^{-1/24}.
\]

Let \( \alpha \) and \( \beta \) be moduli. We say that \( \beta \) is of degree \( d \) over \( \alpha \) if

\[
\frac{2F_1\left(1; 1; 1; 1 - \beta\right)}{2F_1\left(1; 1; 1; \beta\right)} = d \frac{2F_1\left(1; 1; 1; 1 - \alpha\right)}{2F_1\left(1; 1; 1; \alpha\right)}.
\]

Therefore, if \( \alpha = \alpha_n \) and \( \beta \) is of degree \( d \) over \( \alpha \), then, by (2.3), \( \beta = \alpha d_{4n} \). A modular equation of second degree is an equation connecting \( \alpha = \alpha_n \) and \( \beta = \alpha_{4n} \) which will be used in our proofs.

**Theorem** (modular equations of second degree [2, p. 214]). Let \( \beta \) be of second degree over \( \alpha \) and

\[
m = \frac{2F_1\left(1; 1; 1; \alpha\right)}{2F_1\left(1; 1; 1; \beta\right)}.
\]

Then

\[
m\sqrt{1 - \alpha} + \sqrt{\beta} = 1
\]

and

\[
m^2(1 - \alpha) + \beta = 1.
\]

Now, we state and prove the main theorems.

**Theorem 2.1.** Let \( q = e^{-\pi\sqrt{n}} \) and \( \alpha = \alpha_n \). Then

\[
S_1(q) = \frac{q^{1/8}}{\sqrt{2}}.
\]

**Proof.** First, it is easy to show that (cf. [2, p. 37, (22.3)])

\[
(-q^2; q^2)_{\infty} = \frac{1}{(q^2; q^4)_{\infty}},
\]

which is a very famous theorem of Euler. By (1.7), (2.11), (2.1) and (2.2) we have

\[
S_1(q) = \frac{q^{1/8}}{(-q^2; q^2)_{\infty}} = \frac{1}{\sqrt{2G_n g_{4n}}}.
\]

Set \( \alpha = \alpha_n \) and \( \beta = \alpha_{4n} \). Then \( \beta \) is of second degree over \( \alpha \). From (2.8) and (2.9), we find that

\[
\sqrt{\beta} = \frac{1 - \sqrt{1 - \alpha}}{1 + \sqrt{1 - \alpha}}.
\]
and

\[ 1 - \beta = \frac{4\sqrt{1 - \alpha}}{(1 + \sqrt{1 - \alpha})^2}. \]

It follows that, by (2.6) and (2.14),

\[ g_{4n} = \left( \frac{4\beta}{(1 - \beta)^2} \right)^{-1/24} = \left( \frac{2\sqrt{\beta}}{1 - \beta} \right)^{-1/12} = \left( \frac{2(1 - \sqrt{1 - \alpha})(1 + \sqrt{1 - \alpha})^2}{(1 + \sqrt{1 - \alpha})(4\sqrt{1 - \alpha})} \right)^{-1/12} = \left( \frac{\alpha}{2\sqrt{1 - \alpha}} \right)^{-1/12}. \]

Therefore, from (2.12), (2.5) and (2.15),

\[ S_1(q) = \frac{1}{\sqrt{2}}(4\alpha(1 - \alpha))^{1/24} \left( \frac{\alpha^2}{4(1 - \alpha)} \right)^{1/24} = \frac{\alpha^{1/8}}{\sqrt{2}}. \]

This completes the proof.

**Corollary 2.2.** Let \( q = e^{-\pi \sqrt{\tau}}, G = G_n \) and \( g = g_n \). Then

\[ S_1(q) = 2^{-5/8} \left( 1 - \sqrt{1 - G^{-24}} \right)^{1/8} \]

and

\[ S_1(q) = 2^{-1/2} \left( (1 + 2g^{24}) - \sqrt{(1 + 2g^{24})^2 - 1} \right)^{1/8}. \]

**Proof.** From (2.5) and (2.6), we have

\[ \alpha = \frac{1}{2} \left( 1 - \sqrt{1 - G^{-24}} \right) \]

and

\[ \alpha = (1 + 2g^{24}) - \sqrt{(1 + 2g^{24})^2 - 1}. \]

Then, by (2.10), Corollary (2.2) follows immediately.

**Theorem 2.3.** Let \( q = e^{-\pi \sqrt{\tau}} \) and \( \alpha = \alpha_n \). Then

\[ S_2(q) = \frac{1}{\sqrt{2}} \left( \frac{\alpha}{1 - \alpha} \right)^{1/8}. \]

**Proof.** By (1.9), (2.11) and (2.2), we have

\[ S_2(q) = \frac{q^{1/8}}{(q; q^2)_\infty(q^2; q^4)_\infty} = \frac{1}{\sqrt{2}g_n g_{4n}}. \]

Then the theorem follows from (2.2), (2.6) and (2.15) immediately.

By (2.18) and (2.19), \( S_2(q) \) can be also expressed either in terms of \( G \) or \( g \).

The Theorems and Corollaries above provide explicit evaluations of the Ramanujan-Selberg continued fraction in terms of the Ramanujan-Weber class invariants or singular moduli. For values of \( G_n \) and \( g_n \), see the paper of Berndt, Chan and the author [5], and the author’s papers [22], [23], for values of \( \alpha_n \), see the paper of Berndt, Chan and the author [5]. Ramanujan calculated numerous class invariants

**Example 1.** We have (cf. [3, p. 282])

\[
\alpha_{58} = (13\sqrt{58} - 99)^2(99 - 70\sqrt{2})^2.
\]

Then by (2.10), we find that

\[ S_1 \left( e^{-\pi\sqrt{58}} \right) = 2^{-1/2}(13\sqrt{58} - 99)^{1/4}(99 - 70\sqrt{2})^{1/4}. \]

**Example 2.** In his first notebook, Ramanujan [14, p. 310] claimed that

\[
\alpha_{10} = (\sqrt{10} - 3)^2(3 - 2\sqrt{2})^2 = \frac{3\sqrt{2} - \sqrt{5} - 2}{3\sqrt{2} + \sqrt{5} + 2}.
\]

For a proof, see [3, p. 282]. Then

\[
\frac{\alpha_{10}}{1 - \alpha_{10}} = \frac{3\sqrt{10} - 1}{2} - 3\sqrt{2},
\]

and, by (2.20),

\[ S_2 \left( e^{-\pi\sqrt{10}} \right) = \frac{1}{\sqrt{2}} \left( \frac{3\sqrt{10} - 1}{2} - 3\sqrt{2} \right)^{1/8}. \]

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**References**

10. E. Heine, *Untersuchungen über die Reihe* 1 + \( \frac{(1-q^n)(1-q^m)}{(1-q)^2} x \) + \( \frac{(1-q^n)(1-q^{n+1})(1-q^m)(1-q^{m+1})}{(1-q)^3(1-q^2)} x^2 \) + \ldots , J. Reine Angew. Math. 34 (1847), 285-328.
15. S. Ramanujan, *Modular equations and approximations to \( \pi \)*, Quart. J. Math. 3 (1914), 81-98.

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