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EXPLICIT EVALUATIONS OF A RAMANUJAN-SELBERG CONTINUED FRACTION

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To the memory of my father, Professor Guang-Da Zhang

ABSTRACT. This paper gives explicit evaluations for a Ramanujan-Selberg continued fraction in terms of class invariants and singular moduli.

§1. INTRODUCTION

Let, for $|q| < 1$,

$$N(q) = 1 + \frac{q}{1+}\frac{q+q^2}{1}\,\frac{q^3}{+\,1\,+}\,\frac{q^2+q^4}{1}\,+\cdots. \tag{1.1}$$

Set

$$(a;q)_\infty := \prod_{k=1}^{\infty}(1-aq^{k-1}). \tag{1.2}$$

In his notebooks [14, p. 290], Ramanujan asserted that

$$N(q) = \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}. \tag{1.3}$$

This formula was first proved in print by A. Selberg [18]. Other proofs have been given by K. G. Ramanathan [12], G. Andrews et al. [1] and the author [21].

In his "Lost" Notebooks [16, p. 44], Ramanujan also stated that if $|q| < 1$, and

$$L(q) = \frac{1+q}{1}\,\frac{q^2}{+\,1\,+}\,\frac{q+q^3}{1}\,\frac{q^4}{+\,1\,+}\cdots, \tag{1.4}$$

then

$$L(q) = \frac{(-q;q^2)_\infty}{(-q^2;q^2)_\infty}. \tag{1.5}$$

Here, we just point out that (1.5) can be proved by using the well-known Heine [10] continued fraction formula in the same fashion as the proof of (1.3) in the author's paper [21]. Set, for $|q| < 1$,

$$S_1(q) = \frac{q^{1/8}}{1}\,\frac{q}{+\,1+}\,\frac{q+q^2}{1}\,\frac{q^3}{+\,1\,+}\,\frac{q^2+q^4}{1}\,+\cdots. \tag{1.6}$$

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From (1.1), (1.3) and (1.5), we have

$$S_1(q) = \frac{q^{1/8}}{N(q)} = \frac{q^{1/8}}{L(q)} = \frac{q^{1/8}(-q^2;q^2)_\infty}{(-q;q^2)_\infty}. \tag{1.7}$$

We call $S_1(q)$ the Ramanujan-Selberg continued fraction.

Also, set

$$S_2(q) = \frac{q^{1/8}}{1}\,\frac{-q}{+\ 1}\,\frac{-q+q^2}{+\quad 1}\,\frac{-q^3}{+\ 1}\,\frac{q^2+q^4}{+\quad 1}+\cdots. \tag{1.8}$$

Replacing q by $-q$ in (1.1) and (1.3), one can see that

$$S_2(q) = \frac{q^{1/8}}{N(-q)} = \frac{q^{1/8}}{L(-q)} = \frac{q^{1/8}(-q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{1.9}$$

The famous Rogers-Ramanujan continued fraction is defined by

$$F(q) = \frac{q^{1/5}}{1}\,\frac{q}{+1}\,\frac{q^2}{+\ 1}\,\frac{q^3}{+\ 1}\,\frac{q^4}{+\ 1}+\cdots, \tag{1.10}$$

and let $S(q) = -F(-q)$. In his first letter to G. H. Hardy, Ramanujan asserted that

$$F(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}, \tag{1.11}$$

$$S(e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}, \tag{1.12}$$

and

(1.13) $F(e^{-\pi\sqrt{n}})$ can be exactly found if n is any positive rational quantity.

Identities (1.11) and (1.12) were first proved by G. N. Watson [19]. Watson vaguely discussed (1.13) and merely claimed that $F(e^{-\pi\sqrt{n}})$ is an algebraic number.

Ramanathan [13] computed $F(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for several positive rational numbers n for which the ideal class groups of $K = \mathbb{Q}(\sqrt{-n})$ have the property that each genus contains a single class. By using Weber-Ramanujan's class invariants and a modular equation of degree 5, Berndt, Chan and the author [4] were able to establish general formulas for $F(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$.

The aim of this note is to establish general formulas for the Ramanujan-Selberg continued fraction and its companion in terms of class invariants, or equivalently in terms of singular moduli.

§2. Explicit formulas for $S_1(q)$ and $S_2(q)$

For $q = \exp(-\pi\sqrt{n})$, where n is positive rational, let

$$G_n := 2^{-1/4}q^{1/24}(-q;q^2)_\infty \tag{2.1}$$

and

$$g_n := 2^{-1/4}q^{1/24}(q;q^2)_\infty. \tag{2.2}$$

We shall refer to G_n and g_n as the *Ramanujan-Weber class invariants*, which can be roughly viewed as generators of the Hilbert class field of the complex quadratic field of $K = \mathbb{Q}(\sqrt{-n})$. The reader is referred to the important paper of B. Birch [7] and the excellent books of Cox [9] and Lang [11]. We also use modular equations in the sequel, and refer to [2, pp. 213, 214] for this terminology. The singular modulus $\alpha := \alpha_n$ is that unique positive number α_n between 0 and 1 satisfying

$$\sqrt{n} = \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha_n)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha_n)}, \tag{2.3}$$

where ${}_2F_1$ is the hypergeometric function. Moreover (cf. [2, p. 102]),

$${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-\alpha\sin^2\phi}}. \tag{2.4}$$

Then we have [3, p. 185]

$$G_n = (4\alpha_n(1-\alpha_n))^{-1/24} \tag{2.5}$$

and

$$g_n = (4\alpha_n(1-\alpha_n)^{-2})^{-1/24}. \tag{2.6}$$

Let α and β be moduli. We say that β is of degree d over α if

$$\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \beta)} = d\,\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha)}. \tag{2.7}$$

Therefore, if $\alpha = \alpha_n$ and β is of degree d over α, then, by (2.3), $\beta = \alpha_{d^2n}$. A modular equation of second degree is an equation connecting $\alpha = \alpha_n$ and $\beta = \alpha_{4n}$ which will be used in our proofs.

Theorem (modular equations of second degree [2, p. 214]). *Let β be of second degree over α and*

$$m = \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \beta)}.$$

Then

$$m\sqrt{1-\alpha} + \sqrt{\beta} = 1 \tag{2.8}$$

and

$$m^2\sqrt{1-\alpha} + \beta = 1. \tag{2.9}$$

Now, we state and prove the main theorems.

Theorem 2.1. *Let $q = e^{-\pi\sqrt{n}}$ and $\alpha = \alpha_n$. Then*

$$S_1(q) = \frac{\alpha^{1/8}}{\sqrt{2}}. \tag{2.10}$$

Proof. First, it is easy to show that (cf. [2, p. 37, (22.3)])

$$(-q^2; q^2)_\infty = \frac{1}{(q^2; q^4)_\infty}, \tag{2.11}$$

which is a very famous theorem of Euler. By (1.7), (2.11), (2.1) and (2.2) we have

$$S_1(q) = \frac{q^{1/8}}{(-q; q^2)_\infty (q^2; q^4)_\infty} = \frac{1}{\sqrt{2}G_n g_{4n}}. \tag{2.12}$$

Set $\alpha = \alpha_n$ and $\beta = \alpha_{4n}$. Then β is of second degree over α. From (2.8) and (2.9), we find that

$$\sqrt{\beta} = \frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}} \tag{2.13}$$

and

$$1-\beta=\frac{4\sqrt{1-\alpha}}{(1+\sqrt{1-\alpha})^2}. \tag{2.14}$$

It follows that, by (2.6) and (2.14),

$$\begin{aligned} g_{4n} &= \left(\frac{4\beta}{(1-\beta)^2}\right)^{-1/24} = \left(\frac{2\sqrt{\beta}}{1-\beta}\right)^{-1/12} \\ &= \left(2\,\frac{(1-\sqrt{1-\alpha})}{(1+\sqrt{1-\alpha})}\,\frac{(1+\sqrt{1-\alpha})^2}{(4\sqrt{1-\alpha})}\right)^{-1/12} = \left(\frac{\alpha}{2\sqrt{1-\alpha}}\right)^{-1/12}. \end{aligned} \tag{2.15}$$

Therefore, from (2.12), (2.5) and (2.15),

$$\begin{aligned} S_1(q) &= \frac{1}{\sqrt{2}}(4\alpha(1-\alpha))^{1/24}\left(\frac{\alpha^2}{4(1-\alpha)}\right)^{1/24} \\ &= \frac{\alpha^{1/8}}{\sqrt{2}}. \end{aligned}$$

This completes the proof.

Corollary 2.2. *Let $q=e^{-\pi\sqrt{n}}, G=G_n$ and $g=g_n$. Then*

$$S_1(q)=2^{-5/8}\left(1-\sqrt{1-G^{-24}}\right)^{1/8} \tag{2.16}$$

and

$$S_1(q)=2^{-1/2}\left((1+2g^{24})-\sqrt{(1+2g^{24})^2-1}\right)^{1/8}. \tag{2.17}$$

Proof. From (2.5) and (2.6), we have

$$\alpha=\frac{1}{2}\left(1-\sqrt{1-G^{-24}}\right) \tag{2.18}$$

and

$$\alpha=(1+2g^{24})-\sqrt{(1+2g^{24})^2-1}. \tag{2.19}$$

Then, by (2.10), Corollary (2.2) follows immediately.

Theorem 2.3. *Let $q=e^{-\pi\sqrt{n}}$ and $\alpha=\alpha_n$. Then*

$$S_2(q)=\frac{1}{\sqrt{2}}\left(\frac{\alpha}{1-\alpha}\right)^{1/8}. \tag{2.20}$$

Proof. By (1.9), (2.11) and (2.2), we have

$$S_2(q)=\frac{q^{1/8}}{(q;q^2)_\infty(q^2;q^4)_\infty}=\frac{1}{\sqrt{2}g_ng_{4n}}. \tag{2.21}$$

Then the theorem follows from (2.2), (2.6) and (2.15) immediately.

By (2.18) and (2.19), $S_2(q)$ can be also expressed either in terms of G or g.

The Theorems and Corollaries above provide explicit evaluations of the Ramanujan-Selberg continued fraction in terms of the Ramanujan-Weber class invariants or singular moduli. For values of G_n and g_n, see the paper of Berndt, Chan and the author [6], and the author's papers [22], [23], for values of α_n, see the paper of Berndt, Chan and the author [5]. Ramanujan calculated numerious class invariants and singulor moduli [14]. The Borweins [8] and Ramanathan [13] also calculated some singular moduli.

Example 1. We have (cf. [3, p. 282])

$$\alpha_{58} = (13\sqrt{58} - 99)^2 (99 - 70\sqrt{2})^2.$$

Then by (2.10), we find that

$$S_1\left(e^{-\pi\sqrt{58}}\right) = 2^{-1/2}(13\sqrt{58} - 99)^{1/4}(99 - 70\sqrt{2})^{1/4}.$$

Example 2. In his first notebook, Ramanujan [14, p. 310] claimed that

$$\alpha_{10} = (\sqrt{10} - 3)^2 (3 - 2\sqrt{2})^2 = \frac{3\sqrt{2} - \sqrt{5} - 2}{3\sqrt{2} + \sqrt{5} + 2}.$$

For a proof, see [3, p. 282]. Then

$$\frac{\alpha_{10}}{1 - \alpha_{10}} = \frac{3\sqrt{10} - 1}{2} - 3\sqrt{2},$$

and, by (2.20),

$$S_2\left(e^{-\pi\sqrt{10}}\right) = \frac{1}{\sqrt{2}}\left(\frac{3\sqrt{10} - 1}{2} - 3\sqrt{2}\right)^{1/8}.$$

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