ON THE COHOMOLOGY
OF GENERALIZED HOMOGENEOUS SPACES

J. P. MAY AND F. NEUMANN

(Communicated by Ralph Cohen)

Abstract. We observe that work of Gugenheim and May on the cohomology
of classical homogeneous spaces $G/H$ of Lie groups applies verbatim to the
calculation of the cohomology of generalized homogeneous spaces $G/H$, where
$G$ is a finite loop space or a $p$-compact group and $H$ is a “subgroup” in the
homotopical sense.

We are interested in the cohomology $H^*(G/H; R)$ of a generalized homogeneous
space $G/H$ with coefficients in a commutative Noetherian ring $R$. Here $G$ is a
“finite loop space” and $H$ is a “subgroup”. More precisely, $G$ and $H$ are homotopy
equivalent to $\Omega BG$ and $\Omega BH$ for path connected spaces $BG$ and $BH$, and $G/H$
is the homotopy fiber of a based map $f : BH \to BG$. We always assume this
much, and we add further hypotheses as needed. Such a framework of generalized
homogeneous spaces was first introduced by Rector [10], and a more recent frame-
work of $p$-compact groups has been introduced and studied extensively by Dwyer
and Wilkerson [4] and others.

We ask the following question: How similar is the calculation of $H^*(G/H; R)$ to
the calculation of the cohomology of classical homogeneous spaces of compact Lie
groups? When $R = \mathbb{F}_p$ and $H$ is of maximal rank in $G$, in the sense that $H^*(H; \mathbb{Q})$
and $H^*(G; \mathbb{Q})$ are exterior algebras on the same number of generators, the second
author has studied the question in [8, 9]. There, the fact that $H^*(BG; R)$ need not
be a polynomial algebra is confronted and results similar to the classical theorems
of Borel and Bott [2, 3] are nevertheless proven. The purpose of this note is to
begin to answer the general question without the maximal rank hypothesis, but
under the hypothesis that $H^*(BG; R)$ and $H^*(BH; R)$ are polynomial algebras.

In fact, we shall not do any new mathematics. Rather, we shall merely point
out that work of the first author [7] that was done before the general context
was introduced goes far towards answering the question. Essentially the following
theorem was announced in [7] and proven in [5]. We give a brief sketch of its proof
and then return to a discussion of its applicability to the question on hand. Let
$BT^n$ be a classifying space of an $n$-torus $T^n$.

Theorem 1. Assume the following hypotheses.
(i) $\pi_1(BG)$ acts trivially on $H^*(G/H; R)$.

Received by the editors May 19, 2000.

2000 Mathematics Subject Classification. Primary 55T20, 57T15, 57T35; Secondary 55P35,
55P45.

The first author was partially supported by the NSF.

©2001 American Mathematical Society

267
(ii) \( R \) is a PID and \( H_*(BG; R) \) is of finite type over \( R \).
(iii) \( H^*(BG; R) \) is a polynomial algebra.
(iv) There is a map \( e : BT^n \longrightarrow BH \) such that \( H^*(BT^n; R) \) is a free \( H^*(BH; R) \)-module via \( e^* \).

Then \( H^*(G/H; R) \) is isomorphic as an \( R \)-module to \( \text{Tor}^H_{*(BG; R)}(R, H^*(BH; R)) \), regarded by total degree. Moreover, there is a filtration on \( H^*(G/H; R) \) such that its associated bigraded \( R \)-algebra is isomorphic to \( \text{Tor}^H_{*(BG; R)}(R, H^*(BH; R)) \).

Proof. The first two hypotheses ensure that \( H^*(G/H; R) \) is isomorphic to the differential torsion product \( \text{Tor}^C_{*(BG; R)}(R, C^*(BH; R)) \). (See, for example, [5] pp. 21–25. The second hypothesis allows Lemma 3.2 there to be applied with \( \mathbb{Z} \) replaced by \( R \), thus allowing the finite type over \( \mathbb{Z} \) hypothesis assumed there to be replaced by the finite type over \( R \) hypothesis assumed here.) Therefore there is an Eilenberg-Moore spectral sequence that converges from \( \text{Tor}^H_{*(BG; R)}(R, H^*(BH; R)) \) to \( H^*(G/H; R) \). The conclusion of the theorem is that this spectral sequence collapses at \( E_2 \) with trivial additive extensions, but not necessarily trivial multiplicative extensions. The last hypothesis and a comparison of spectral sequences argument essentially due to Baum [1] shows that the conclusion holds in general if it holds when \( BH = BT^n \). (See [5] pp. 37–38.) Here the strange result [5, 4.1] gives that there is a morphism

\[ g : C^*(BT^n; R) \longrightarrow H^*(BT^n; R) \]

of differential algebras such that \( g \) induces the identity map on cohomology and annihilates all \( \cup_1 \)-products.

Now the general theory of differential torsion products of [5] kicks in. In modern language, implicit in the discussion of [6] p. 70], there is a model category structure on the category of \( A \)-modules for any \( DGA A \) over \( R \) such that every right \( A \)-module \( M \) admits a cofibrant approximation of a very precise sort. Namely, for any \( HA \)-free resolution \( X \otimes_R HA \longrightarrow HM \) of \( HM \), there is a cofibrant approximation \( P = X \otimes_R A \longrightarrow M \). Grading is made precise in the cited sources. The essential point is that \( P \) is not a bicomplex but rather has differential with many components. When \( HA \) is a polynomial algebra and \( M = R \), we can take \( X \) to be an exterior algebra with one generator for each polynomial generator of \( HA \). Here, assuming that \( A \) has a \( \cup_1 \)-product that satisfies the Hirsch formula (\( \cup_1 \) is a graded derivation), [5, 2.2] specifies the required differential explicitly in terms of \( \cup_1 \)-products. Using \( g \) to replace \( C^*(BT^n; R) \) by \( H^*(BT^n; R) \) in our differential torsion product, we see that the differential torsion product \( \text{Tor}^C_{*(BG; R)}(R, H^*(BT^n; R)) \) is computed by exactly the same chain complex as the ordinary torsion product \( \text{Tor}^H_{*(BG; R)}(R, H^*(BT^n; R)) \). (See [5, 2.3] for the conclusion follows.)

Hypotheses (i) and (ii) in Theorem 1 are reasonable and not very restrictive. Hypothesis (iii) is intrinsic to the method at hand. Note that \( H^*(BG; R) \) can have infinitely many polynomial generators, so that \( G \) need not be finite. The key hypothesis is (iv). Here the following homotopical version of a theorem of Borel is relevant. It was first noticed by Rector [10, 2.2] that Baum’s proof [1] of Borel’s theorem is purely homotopical. A generalized variant of Baum’s proof is given in [5] pp. 40–42. That proof applies directly to give the following theorem. We state it for \( H \) and \( G \) as in the first paragraph. However, we are interested in its applicability to \( T^n \) and \( H \) in Theorem 1, and we restate it as a corollary in that special case.
Theorem 2. Let $R$ be a field and assume the following hypotheses.

(i) $\tau_1(BG)$ acts trivially on $H^*(G/H; R)$.
(ii) $H^*(BH; R)$ and $H^*(BG; R)$ are polynomial algebras on the same finite number of generators.
(iii) $H^*(G/H; R)$ is a finite dimensional $R$-module.

Then $H^*(G/H; R) \cong R \otimes_{H^*(BG; R)} H^*(BH; R)$ as an algebra and

$$H^*(BH; R) \cong H^*(BG; R) \otimes_R H^*(G/H; R)$$

as a left $H^*(BG; R)$-module. In particular, $H^*(BH; R)$ is $H^*(BG; R)$-free.

Corollary 3. Let $R$ be a field and assume given a map $e : BT^n \to BH$ that satisfies the following properties, where $H/T^n$ is the fiber of $e$.

(i) $\tau_1(BH)$ acts trivially on $H^*(H/T^n; R)$.
(ii) $H^*(BH; R)$ is a polynomial algebra on $n$ generators.
(iii) $H^*(H/T^n; R)$ is a finite dimensional $R$-module.

Then $H^*(H/T^n; R) \cong R \otimes_{H^*(BH; R)} H^*(BT^n; R)$ as an algebra and

$$H^*(BT^n; R) \cong H^*(BH; R) \otimes_R H^*(H/T^n; R)$$

as a left $H^*(BH; R)$-module. In particular, $H^*(BT^n; R)$ is $H^*(BH; R)$-free.

When Corollary 3 applies, its conclusion gives hypothesis (iv) of Theorem 1. We comment briefly on applications to the integral and $p$-compact settings for the study of generalized homogeneous spaces.

Remark 4. A counterexample of Rector [10] shows that not all finite loop spaces $H$ have (integral) maximal tori. When $H$ does have a maximal torus, hypothesis (iii) of the corollary holds by definition. Assuming that $H$ is simply connected, [9, 3.11] describes for which primes $p$ $H^*(BH; Z)$ is $p$-torsion free, so that $H^*(BH; F_p)$ is a polynomial algebra. If $R$ is the localization of $Z$ at the primes $p$ for which $H^*(H; Z)$ is $p$-torsion free, then $H^*(BH; R)$ is also a polynomial algebra, and $H^*(BT; R)$ is a free $H^*(BH; R)$-module. That is, hypothesis (iv) of Theorem 1 holds for the localization of $Z$ away from the finitely many “bad primes” for which $H^*(BH; F_p)$ is not a polynomial algebra on $n$ generators.

Remark 5. In the $p$-compact setting, taking $R = F_p$, Dwyer and Wilkerson [4] 8.13, 9.7] prove that if $H$ is connected, $BH$ is $F_p$-complete, $H^*(H; F_p)$ is finite dimensional, and $H^*(H; Z_p) \otimes_{Z_p} Q$ is an exterior algebra on $n$ generators, then there is a map $e : BT^n \to BH$ such that $H^*(H/T^n; F_p)$ is finite dimensional. Here Corollary 3 applies whenever $H^*(BH; F_p)$ is a polynomial algebra on $n$ generators.

REFERENCES


Department of Mathematics, The University of Chicago, Chicago, Illinois 60637
E-mail address: may@math.uchicago.edu

Mathematisches Institut der Georg-August-Universität, Göttingen, Germany
E-mail address: neumann@uni-math.gwdg.de