

ACTIONS OF FINITELY GENERATED GROUPS ON THE UNIVERSAL Menger CURVE

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ABSTRACT. We prove that every finitely generated group acts effectively on the universal Menger curve.

The purpose of this note is to prove the following theorem on the universal Menger curve μ^1 . A description of the n -dimensional universal Menger compactum μ^n ($n \geq 1$) is given below. For more information, the reader is referred to [1], [4], [5], [8], [9], etc.

Theorem. *Every finitely generated group acts on the universal Menger curve μ^1 , and, in particular, is isomorphic to a subgroup of the homeomorphism group of μ^1 .*

The algebraic structure of the homeomorphism group $H(\mu^n)$ ($n \geq 1$) has been studied by several authors. For example, $H(\mu^n)$ is simple ([3], [7]), and has the trivial homology ([12]). The above theorem presents a certain “universality” of $H(\mu^1)$. The result was announced in [8].

Here we briefly describe a construction (called the *Lefschetz construction*) of the n -dimensional universal Menger compactum μ^n ($n \geq 1$). Let D^{2n+1} be the $(2n+1)$ -dimensional ball with a combinatorial triangulation L . We define a sequence of compact PL manifolds $(M_i)_{i \geq 0}$ as follows. Let $M_0 = D^{2n+1}$, $L_0 = L$, and $M_1 = \text{st}(|L_0^{(n)}|, \beta^2 L_0)$, where $\beta^2 L_0$ is the second barycentric subdivision of L_0 and $|L_0^{(n)}|$ is the polyhedron associated with the n -skeleton of L_0 . Further, let $L_1 = \beta^2 L_0|_{M_1}$, the restriction of $\beta^2 L_0$ to the submanifold M_1 . Inductively, let $M_{i+1} = \text{st}(|L_i^{(n)}|, \beta^2 L_i)$ and $L_{i+1} = \beta^2 L_i|_{M_{i+1}}$. The intersection $\bigcap_{i=1}^{\infty} M_i$ is denoted by μ^n and is called the *n -dimensional universal Menger compactum*. It has been known that every compact 0-dimensional group acts freely on μ^n for each $n \geq 1$ ([2], [6], [10], [11], etc.).

The following result is crucial for the proof of the above theorem, the proof of which is outlined on p. 58 of [5]. A locally compact separable metric space, each point of which has a neighbourhood homeomorphic to μ^n , is called a *μ^n -manifold*.

Proposition. *The Freudenthal compactification of each (noncompact) connected μ^1 -manifold is homeomorphic to μ^1 .*

Proof of the theorem. Let G be a group with a finite set S of generators. We may assume at the outset that G is an infinite group, $S = \{s^{-1} | s \in S\}$ and the

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unit element does not belong to S . Consider the Cayley graph $K = K(G, S)$ of G with respect to S : The set of vertices of K is equal to G , and two vertices $g_1, g_2 \in G \subset K$ are joined by an edge of K if and only if $g_2^{-1}g_1 \in S$. It is a connected one-dimensional simplicial complex and the left multiplication of G on the vertex set G induces the canonical simplicial and effective action on K . The set of all edges of K is denoted by $E(K)$.

Let \tilde{K} be a 3-manifold which collapses onto K such that:

- (1) there exists a handle-body decomposition $\tilde{K} = \bigcup_{g \in G} h_g^0 \cup \bigcup_{e \in E(K)} h_e^1$, where each h_g^0 is a 0-handle and each h_e^1 is a 1-handle ($g \in G, e \in E(K)$), and
- (2) $h_g^0 \cap h_e^1 \neq \emptyset$ if and only if g is an end point of e (see Figure 1).

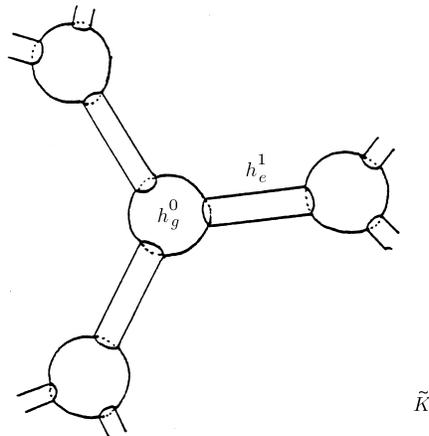
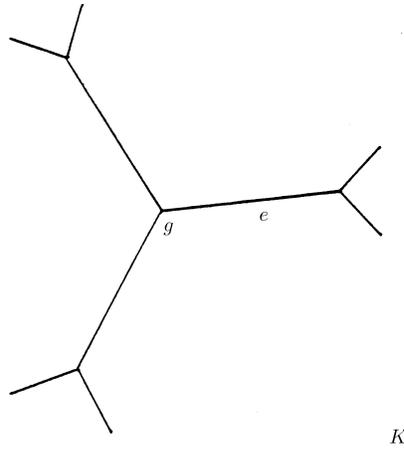


FIGURE 1.

It is obvious that \tilde{K} can be constructed in such a way that

(3) there exists a triangulation L of \tilde{K} such that G acts effectively on \tilde{K} by simplicial isomorphisms with respect to L (which is “induced” by the action of G on K).

Start the Lefschetz construction of a connected μ^1 -manifold X with $M_0 = \tilde{K}$ and $L_0 = L$. Then G naturally acts effectively on X and the action extends to the one on the Freudenthal compactification \overline{X} . By the Proposition, \overline{X} is homeomorphic to μ^1 . This completes the proof.

REFERENCES

- [1] R. D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math. **67** (1958), 313–324. MR **20**:2675
- [2] ———, *One dimensional continuous curves and a homogeneity theorem*, Ann. of Math. **68** (1958), 1–16. MR **20**:2676
- [3] ———, *The algebraic simplicity of certain groups of homeomorphisms*, Amer. J. Math. **80** (1958), 955–963. MR **20**:4607
- [4] M. Bestvina, *Characterizing k -dimensional universal Menger compacta*, Mem. Amer. Math. Soc. **71** (1988), no. 380. MR **89g**:54083
- [5] A. Chigogidze, K. Kawamura and E. D. Tymchatyn, *Menger manifolds in Continua with the Houston problem book*, Marcel Dekker (1995), 37–88. MR **96a**:57052
- [6] A. Dranishnikov, *On free actions of zero-dimensional compact groups*, Izv. Akad. Nauk USSR **32** (1988), 217–232. MR **90e**:57065
- [7] Y. Iwamoto and K. Sakai, *The homeomorphism group of the universal Menger compactum is simple*, preprint.
- [8] K. Kawamura, *A survey on Menger manifold theory -Update-*, Top. Appl. **101** (2000), 83–91. MR **2000i**:54064
- [9] J. C. Mayer, L. G. Oversteegen and E. D. Tymchatyn, *The Menger curve: characterization and extension of homeomorphisms of non-locally-separating closed subsets*, Diss. Math. **252** (1986). MR **87m**:54106
- [10] J. C. Mayer and C. W. Stark, *Group actions on Menger manifolds*, preprint.
- [11] K. Sakai, *Free actions of zero dimensional compact groups on Menger manifolds*, Proc. Amer. Math. Soc., **122** (1994), 647–648. MR **95c**:57057
- [12] V. Sergiescu and T. Tsuboi, *Acyclicity of the groups of homeomorphisms of the Menger compact spaces*, Amer. J. Math. **118** (1996), 1299–1312. MR **97g**:54053

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