PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 2, Pages 419–426
S 0002-9939(01)06161-5
Article electronically published on June 8, 2001

POROUS MEASURES ON $\mathbb{R}^n$: LOCAL STRUCTURE AND DIMENSIONAL PROPERTIES

ESA JÄRVENPÄÄ AND MAARIT JÄRVENPÄÄ

(Communicated by David Preiss)

Abstract. We study dimensional properties of porous measures on $\mathbb{R}^n$. As a corollary of a theorem describing the local structure of nearly uniformly porous measures we prove that the packing dimension of any Radon measure on $\mathbb{R}^n$ has an upper bound depending on porosity. This upper bound tends to $n-1$ as porosity tends to its maximum value.

1. Introduction and preliminaries

Porosity is a quantity that describes irregularities of fractals. The study of dimensional properties featured by porous sets was pioneered by P. Mattila. In [M1] he verified the existence of a non-increasing function which gives an upper bound for Hausdorff dimension of any set in $\mathbb{R}^n$ as a function of porosity. Furthermore, he showed that this upper bound tends to $n-1$ as porosity tends to its maximum value. In [S] A. Salli generalized the corresponding results for packing dimension and established the correct asymptotic behaviour for the upper bound.

For measures the following definition of porosity was introduced in [EJJ].

1.1. Definition. The porosity of a Radon measure μ on $\mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is defined by

$$\operatorname{por}(\mu, x) = \lim_{\epsilon \downarrow 0} \liminf_{r \downarrow 0} \operatorname{por}(\mu, x, r, \varepsilon) \tag{1.2}$$

where for all $r, \varepsilon > 0$

$$\begin{aligned} \operatorname{por}(\mu, x, r, \varepsilon) = \sup\{p \geq 0 \mid &\text{ there is } z \in \mathbb{R}^n \text{ such that } B(z, pr) \subset B(x, r) \\ &\text{and } \mu(B(z, pr)) \leq \varepsilon \mu(B(x, r))\}. \end{aligned} \tag{1.3}$$

The porosity of μ is

$$\begin{aligned} \operatorname{por}(\mu) &= \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \operatorname{por}(\mu, x) \\ &= \inf\{s \geq 0 \mid \operatorname{por}(\mu, x) \leq s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n\}. \end{aligned} \tag{1.4}$$

We will relate porosity of measures to packing dimension defined as follows. (For the definition of the packing dimension of a set see [M2, 5.9 and Theorem 5.11].)

Received by the editors June 13, 2000.
2000 *Mathematics Subject Classification.* Primary 28A12, 28A80.

1.5. Definition. Let μ be a Radon measure on $\mathbb{R}^n$. The packing dimension of μ is defined in terms of upper local dimensions

$$\dim_{\mathrm{p}}(\mu) = \sup\{s \geq 0 \mid \limsup_{i\to\infty} \frac{\log \mu(D_i(x))}{\log 2^{-i}} \geq s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n\} \tag{1.6}$$

where $D_i(x)$ is the closed dyadic cube of side-length 2^{-i} containing x. Equivalently this definition can be given using packing dimensions of Borel sets with positive μ-measure

$$\dim_{\mathrm{p}}(\mu) = \inf\{\dim_{\mathrm{p}}(A) \mid A \text{ is a Borel set with } \mu(A) > 0\}. \tag{1.7}$$

Replacing "liminf" by "limsup" in (1.2) gives the upper porosity of a measure which was studied by M. E. Mera and M. Morán in [MM]. They showed that if μ satisfies the doubling condition, that is,

$$\limsup_{r\to 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty$$

for μ-almost all $x \in \mathbb{R}^n$, then the upper porosity of μ is either 0 or 1/2. (Above $B(x,r)$ is the closed ball with radius r and centre x.) Furthermore, for any non-doubling measure the upper porosity equals 1. Note that the (lower) porosity may obtain any value between 0 and 1/2 for both doubling and non-doubling measures (see [JJM, (2.17)] and [EJJ, Example 4]). The upper porosity is too weak for the purpose of obtaining a non-trivial upper bound for dimension; for any $p = 0, 1/2, 1$ and $0 \leq d \leq n$ there exists a Radon measure μ with the upper porosity equal to p and with both Hausdorff and packing dimension equal to d.

In this paper we will establish a connection between porosity and packing dimension for all Radon measures on $\mathbb{R}^n$. The case $n = 1$ was studied in [JJ]. In [EJJ] the emphasis was given to doubling measures on $\mathbb{R}^n$. For such measures the porosity can be given in terms of porosities of Borel sets with positive measure:

$$\mathrm{por}(\mu) = \sup\{\mathrm{por}(A) \mid A \text{ is a Borel set with } \mu(A) > 0\}.$$

(The doubling condition is necessary here; see [EJJ] for details.)

We will generalize the results of [JJ] to higher dimensions by verifying that the packing dimension of any Radon measure on $\mathbb{R}^n$ is bounded above by a function that depends on porosity and by showing that this upper bound goes to $n-1$ as porosity tends to its maximum value 1/2 (see Corollary 2.9). In particular, the packing dimension of any Radon measure on $\mathbb{R}^n$ having porosity close to 1/2 cannot be much larger than $n-1$.

Our main tools are a dimension estimate obtained from the strong law of large numbers and a description of the local structure of nearly uniformly porous measures. The latter one states that for a given nearly uniformly porous measure any sufficiently small dyadic cube can be divided into three parts, two having small measure and the remaining one being a narrow boundary of a convex set (see Theorem 2.2).

2. Local structure and dimensional properties

We recall the following lemma from [JJ] according to which we may replace any measure by a nearly uniformly porous measure when estimating packing dimension from above.

2.1. Lemma. *Assume that μ is a Radon measure on $\mathbb{R}^n$ such that* $\operatorname{por}(\mu) \geq p$. *Let $0 < \delta < 1$. Then there is a Radon measure μ_δ with compact support* $\operatorname{spt}(\mu_\delta) \subset \operatorname{spt}(\mu)$ *and with* $\dim_{\mathrm{p}}(\mu_\delta) \geq \dim_{\mathrm{p}}(\mu)$ *such that the following property holds: there exists $\varepsilon_\delta > 0$ such that for all $0 < \varepsilon \leq \varepsilon_\delta$ there are a Borel set $B_{\delta,\varepsilon}$ and $r_{\delta,\varepsilon} > 0$ with $\mu_\delta(\mathbb{R}^n \setminus B_{\delta,\varepsilon}) \leq \delta\mu_\delta(\mathbb{R}^n)$ and*

$$\operatorname{por}(\mu_\delta, x, r, \varepsilon) > p - \frac{\delta}{2}$$

for all $x \in B_{\delta,\varepsilon}$ and $0 < r \leq r_{\delta,\varepsilon}$.

Proof. See [JJ, Lemma 2.2]. □

For all positive integers i we use the notation $\mathcal{D}_i$ for the family of closed dyadic cubes in $\mathbb{R}^n$ with side-length 2^{-i}, that is, cubes of the form $\{x \in \mathbb{R}^n \mid k_j 2^{-i} \leq x_j \leq (k_j+1)2^{-i}$ for all $j = 1, \ldots, n\}$ where k_j, $j = 1, \ldots, n$, are integers. For all $Q \in \mathcal{D}_i$ and for all positive integers k, let $\mathcal{N}^k(Q) \subset \mathcal{D}_i$ be the family of the $(2k+1)^n$ neighbouring dyadic cubes of Q with side-length 2^{-i} being located symmetrically around Q (including Q itself).

For all $\delta > 0$, *a δ-plate* is a $\frac{\delta}{2}$-neighbourhood of an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$. *An affine δ-boundary* of a convex polyhedron P is the union of parts of δ-plates glued on all faces of P such that the union of P and its affine δ-boundary is a polyhedron obtained by magnifying P.

The following theorem describes the local structure of porous measures by stating that in all sufficiently small dyadic cubes such measures are essentially concentrated on a narrow boundary of some convex set.

2.2. Theorem. *Assume that μ is a Radon measure on $[0,1]^n$ such that* $\operatorname{por}(\mu) \geq \frac{1}{2}(1-\beta)$ *for $0 \leq \beta \leq \frac{1}{18}$. Let K be a positive integer. For all $0 < \delta < \frac{1}{18}$ let μ_δ and ε_δ be as in Lemma 2.1. Let $0 < \varepsilon < \varepsilon_\delta$. Then there is a positive integer i_0 depending on K, δ, and ε such that for all $i \geq i_0$ any cube $Q \in \mathcal{D}_i$ can be divided into three disjoint (not necessarily non-empty) parts*

$$Q = E \cup P \cup I \tag{2.3}$$

where

$$\mu_\delta(E) \leq C_Q^K N\varepsilon$$

for an integer N depending on K, δ, and β and for $C_Q^K = \max_{D \in \mathcal{N}^K(Q)} \mu_\delta(D)$, P is an affine $C_{\beta,\delta}2^{-i}$-boundary of a convex polyhedron with $C_{\beta,\delta} = 6K(\beta+\delta) + \frac{2n}{K(1-\beta-\delta)}$, and $I \subset Q \setminus B_{\delta,\varepsilon}$. Here $B_{\delta,\varepsilon}$ is as in Lemma 2.1.

Proof. Let $r_{\delta,\varepsilon}$ be as in Lemma 2.1 and let i_0 be the smallest integer such that $(1+K)2^{-i_0} < r_{\delta,\varepsilon}$. Consider an integer $i \geq i_0$. Note that for those $Q \in \mathcal{D}_i$ which do not intersect $B_{\delta,\varepsilon}$ equality (2.3) is trivial. Let $Q \in \mathcal{D}_i$ such that $Q \cap B_{\delta,\varepsilon} \neq \emptyset$. Setting $r_x = \operatorname{dist}(x, \partial Q) + K2^{-i}$ for all $x \in Q \cap B_{\delta,\varepsilon}$, we have $B(x, r_x) \subset \bigcup_{D \in \mathcal{N}^K(Q)} D$. Lemma 2.1 implies that for all $x \in Q \cap B_{\delta,\varepsilon}$ there is a ball B_x with radius $qr_x = \frac{1}{2}(1-\beta-\delta)r_x$ such that $B_x \subset B(x, r_x)$ and

$$\mu_\delta(B_x) \leq (2K+1)^n C_Q^K \varepsilon. \tag{2.4}$$

Set $a^{1/n} = 2K(1-2q)\alpha^{1/n}$ where $\alpha = \mathcal{L}^n(B(0,1))$ is the Lebesgue measure of the unit ball. Then there are integers $N(a) \geq N(a, Q) \geq 0$ for which there are

$x_1, \dots, x_{N(a,Q)} \in Q \cap B_{\delta,\varepsilon}$ such that

$$\mathcal{L}^n\Big((B_{x_j} \cap Q) \setminus \bigcup_{k=1}^{j-1} B_{x_k}\Big) \geq a\mathcal{L}^n(Q)$$

for all $j = 1, \dots, N(a,Q)$ and

$$\mathcal{L}^n\Big((B_x \cap Q) \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}\Big) < a\mathcal{L}^n(Q) \tag{2.5}$$

for all $x \in Q \cap B_{\delta,\varepsilon}$ such that $x \neq x_j$ for all $j = 1, \dots, N(a,Q)$. (We use the interpretation $\bigcup_{k=1}^{0} B_{x_k} = \emptyset$. In the case $N(a,Q) = 0$ we have $\mathcal{L}^n(B_x \cap Q) < a\mathcal{L}^n(Q)$ for all $x \in Q \cap B_{\delta,\varepsilon}$.)

Define

$$I_1 = \Big\{y \in Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k} \mid \operatorname{dist}\Big(y, \partial\Big(Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}\Big)\Big) > 2(\frac{a}{\alpha})^{1/n}2^{-i}\Big\}.$$

Then

$$B_x \cap I_1 = \emptyset \tag{2.6}$$

for all $x \in I_1 \cap B_{\delta,\varepsilon}$. In fact, assuming that $B_x = B_{x_j}$ for some $j = 1, \dots, N(a,Q)$, equality (2.6) holds. If $B_x \cap I_1 \neq \emptyset$ for some $x \in I_1 \cap B_{\delta,\varepsilon}$ with $B_x \neq B_{x_j}$ for all $j = 1, \dots, N(a,Q)$, then, as will be indicated shortly, the set $(B_x \cap Q) \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}$ contains a ball with radius $(a/\alpha)^{1/n}2^{-i}$ contradicting (2.5). To find such a ball, take $z \in B_x \cap I_1$. Then $B(z, 2(a/\alpha)^{1/n}2^{-i}) \subset Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}$. Since $(a/\alpha)^{1/n}2^{-i} \leq \frac{1}{2}qr_x$, the ball B_x contains a ball with radius $(a/\alpha)^{1/n}2^{-i}$ having z on its boundary such that the centre of the ball belongs to the line going through z and the centre of B_x. Clearly this ball is a subset of $Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}$. This completes the proof of (2.6).

Set

$$I_2 = \Big\{y \in Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k} \mid \operatorname{dist}\Big(y, \partial\Big(Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}\Big)\Big) > 3(\frac{a}{\alpha})^{1/n}2^{-i}\Big\}.$$

Then $I_2 \subset Q \setminus B_{\delta,\varepsilon}$. To see this, assume that there exists $x \in I_2 \cap B_{\delta,\varepsilon}$. From (2.6) we get

$$\operatorname{dist}(x, \partial B_x) > (\frac{a}{\alpha})^{1/n}2^{-i} = 2K(1-2q)2^{-i}.$$

On the other hand $\operatorname{dist}(x, \partial B_x) \leq r_x(1-2q) \leq (K+1/2)(1-2q)2^{-i}$. Hence $I_2 \subset Q \setminus B_{\delta,\varepsilon}$.

Let $B_{x_k} = B(z_k, qr_{x_k})$. Since $\frac{n}{qK}2^{-i}$ is an upper bound for the height of a segment of any ball with radius qr_x having chord with length at most $\sqrt{n}2^{-i}$, the intersections of each of the annuli $B(z_k, qr_x + 3(a/\alpha)^{1/n}2^{-i}) \setminus B_{x_k}$ and Q can be included in a $C_{\beta,\delta}2^{-i}$-plate. Adding parts of the affine $C_{\beta,\delta}$-boundary of Q if necessary concludes the construction of P. Setting

$$I = I_2 \setminus P$$

and

$$E = Q \setminus (P \cup I) \subset Q \cap \bigcup_{k=1}^{N(a,Q)} B_{x_k},$$

equality (2.3) follows since

$$\mu_\delta(Q \cap \bigcup_{k=1}^{N(a,Q)} B_{x_k}) \le N(a)(2K+1)^n C_Q^K \varepsilon$$

by (2.4). □

Let μ be a Radon probability measure on $[0,1]^n$ such that $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. Letting k be a positive integer, set $I = \{1, \dots, 2^{kn}\}$. For all positive integers j, denote by $\mathbf{I}^j$ the set of all j-term sequences of integers belonging to I and by $\mathbf{I}^\infty$ the corresponding set of infinite sequences, that is,

$$\mathbf{I}^j = \{(i_1, \dots, i_j) \mid i_l \in I \text{ for all } l = 1, \dots, j\}$$

and

$$\mathbf{I}^\infty = \{(i_1, i_2, \dots) \mid i_l \in I \text{ for all } l = 1, 2, \dots\}.$$

Divide $[0,1]^n$ into 2^{kn} dyadic subcubes, enumerate them and denote them by D_j, $j = 1, \dots, 2^{kn}$. Define $f : [0,1]^n \to [0,1]^n$ by setting $f(x) = 2^k x \bmod 1$. For each $x \in [0,1]^n$ we define a sequence $\mathbf{i}^x = (i_1, i_2, \dots) \in \mathbf{I}^\infty$ such that $f^{l-1}(x) \in D_{i_l}$ for all $l = 1, 2, \dots$. Note that for all $x = (x_1, \dots, x_n)$ the sequence $\mathbf{i}^x$ is unique unless x_j is a dyadic rational for some $j = 1, \dots, n$. (If $f^l(x)_i$ is a dyadic rational for some l and i, one can choose between indices corresponding to "left" and "right" cube. If one chooses left, then $f^{l+1}(x)_i = 1$ and otherwise $f^{l+1}(x)_i = 0$.) For a positive integer l and $\mathbf{i} = (i_1, i_2, \dots) \in \mathbf{I}^\infty$ let $\mathbf{i}|_l = (i_1, \dots, i_l) \in \mathbf{I}^l$ be the sequence of the first l digits of $\mathbf{i}$ and for all $j = 1, \dots, 2^{kn}$ let $\mathrm{n}_j(\mathbf{i}|_l)$ be the number of j's in $\mathbf{i}|_l$.

We can attach a sequence (P_l^μ) of probability measures on I to μ such that for all $j = 1, \dots, 2^{kn}$ $\mathrm{P}_l^\mu(\{j\})$ gives the probability that the lth digit (in the above representation) of a random number (with respect to μ) in $[0,1]^n$ equals j, that is,

$$\mathrm{P}_l^\mu(\{j\}) = \sum_{\substack{(i_1,\dots,i_l)\in\mathbf{I}^l \\ i_l = j}} \mu(D_{i_1,\dots,i_l})$$

where $D_{i_1,\dots,i_l}$ is the closed dyadic subcube of $[0,1]^n$ of side-length 2^{-kl} consisting of points whose expansion begins with $(i_1, \dots, i_l)$. The measures P_l^μ are well-defined since $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. We use the notation P^μ for the product measure $\prod_{l=1}^\infty \mathrm{P}_l^\mu$ on the code space $\mathbf{I}^\infty$.

2.7. Proposition. *Let μ be a Radon probability measure on $[0,1]^n$ such that $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. Let $p \le 2^{-kn}$ and $L \in I$. Assume that $\limsup_{l\to\infty} \frac{1}{l} \sum_{i=1}^l \mathrm{P}_i^\mu(\{j\}) \le p$ for all $j = 1, \dots, L$. Then*

$$\dim_{\mathrm{p}} \mu \le -\frac{1}{\log 2^k}(Lp \log p + (1-Lp)\log(\frac{1-Lp}{2^{kn}-L})) =: \alpha(p, L).$$

Proof. The strong law of large numbers [Fe, X.7.1] gives for all $j = 1, \dots, L$ that

$$\limsup_{l\to\infty} \frac{1}{l}\, \mathrm{n}_j(\mathbf{i}|_l) \le p$$

for P^μ-almost all $\mathbf{i} \in \mathbf{I}^\infty$. Defining

$$E_{p,L} = \{x \in [0,1]^n \mid \limsup_{l\to\infty} \frac{1}{l}\, \mathrm{n}_j(\mathbf{i}^x|_l) \le p \text{ for all } j = 1,\dots,L\},$$

this implies that $\mu(E_{p,L}) = 1$. Since $E_{p,L}$ is a Borel set it is enough to prove that $\dim_{\mathrm{p}}(E_{p,L}) \le \alpha(p,L)$.

Let ρ be a probability measure on I such that $\rho(\{j\}) = p$ for all $j = 1,\dots,L$ and $\rho(\{j\}) = (1-Lp)/(2^{kn}-L)$ for all $j = L+1,\dots,2^{kn}$. Let ν be the image of the infinite product of the measures ρ under the map $\pi : \mathbf{I}^\infty \to [0,1]^n$. Note that since $p \le 2^{-kn}$ we have

$$-u\log p - (1-u)\log\big(\frac{1-Lp}{2^{kn}-L}\big) \le \alpha(p,L)\log 2^k$$

for all $u \le Lp$. Let $x \in E_{p,L}$. The equality

$$\log\nu(D_{kl}(x)) = \log p \sum_{j=1}^{L} \mathrm{n}_j(\mathbf{i}^x|_l) + \log\big(\frac{1-Lp}{2^{kn}-L}\big) \sum_{j=L+1}^{2^{kn}} \mathrm{n}_j(\mathbf{i}^x|_l)$$

gives

$$\liminf_{l\to\infty} \frac{1}{l}\log\big(\frac{\nu(D_{kl}(x))}{2^{-klt}}\big) \ge -\log 2^k\alpha(p,L) + t\log 2^k$$

where $D_{kl}(x)$ is the dyadic cube of side-length 2^{-kl} containing x. Thus if $t > \alpha(p,L)$, then $\liminf_{l\to\infty} \frac{\nu(D_{kl}(x))}{2^{-klt}} = \infty$, implying

$$\limsup_{l\to\infty} \frac{\log(\nu(D_{kl}(x)))}{\log 2^{-kl}} \le t.$$

By [Fa, Proposition 2.3 (d)] we get $\dim_{\mathrm{p}}(E_{p,L}) \le \alpha(p,L)$. □

Let k and i be positive integers. Dyadic cubes in $\mathcal{D}_{ki}$ form *a brood* if they belong to the same dyadic cube belonging to $\mathcal{D}_{k(i-1)}$. Note that each brood consists of 2^{kn} dyadic cubes. Given a measure μ on $\mathbb{R}^n$, order the cubes of every brood such that $\mu(D_j) \le \mu(D_{j+1})$ for all $j = 1,\dots,2^{kn}$. Let $\mathcal{D}^j_{ki}(\mu)$ be the set of the jth cubes of all broods.

2.8. Theorem. *Let μ be a Radon probability measure on $[0,1]^n$ such that* $\mathrm{por}(\mu) \ge \frac{1}{2}(1-\beta)$ *for $0 \le \beta \le \frac{1}{18}$ and $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. For all $0 < \delta < \frac{1}{18}$ let μ_δ be as in Lemma 2.1. Then there is an integer L with $2^{k(\beta,\delta)n} \ge L \ge 2^{k(\beta,\delta)n} - c2^{k(\beta,\delta)(n-1)}$ where c is a constant depending only on n and $k(\beta,\delta) \to \infty$ as $\delta \to 0$ and $\beta \to 0$ such that the following inequality is valid: for all $j = 1,\dots,L$ we have*

$$\limsup_{l\to\infty} \frac{1}{l} \sum_{i=1}^{l} \sum_{D\in\mathcal{D}^j_{ki}(\mu_\delta)} \mu_\delta(D) \le \delta\mu_\delta([0,1]^n).$$

Proof. Let k_0 be the largest integer such that $\beta + \delta \le 2^{-2k_0}$. Set $K = 2^{k_0}$. Let $0 < \varepsilon < \varepsilon_\delta$ and let $i \ge i_0$ where ε_δ and i_0 are as in Theorem 2.2. Let k be the largest integer such that $2^{-k} > 4(6+4n)2^{-k_0}$. Then $2^{-k} > 4C_{\beta,\delta}$ where $C_{\beta,\delta}$ is as in Theorem 2.2. Consider $Q \in \mathcal{D}_{ki}$. Let $Q = E_Q \cup P_Q \cup I_Q$ be the partition of Q given in Theorem 2.2. Take any $x \in P_Q$. Then $D_{k(i+1)}(x)$ and its neighbouring cubes in $\mathcal{D}_{k(i+1)}$ cover a part of P_Q such that the $\mathcal{L}^{n-1}$-measure of the covered part of both inner and outer boundary of P_Q is at least $2^{-(n-1)}2^{-k(i+1)(n-1)}$. By

the convexity of P_Q the $\mathcal{L}^{n-1}$-measure of the outer boundary of P_Q is less than $2n2^{-ki(n-1)}$. Hence we need at most $3^{2n}2n2^{n-1}2^{k(n-1)}$ cubes from $\mathcal{D}_{k(i+1)}$ to cover P_Q. Thus there are $L \geq 2^{kn} - 3^{2n}2n2^{n-1}2^{k(n-1)}$ cubes in $\mathcal{D}_{k(i+1)}$ which belong to $E_Q \cup I_Q$. Clearly $\mu_\delta(D_j) \leq \mu_\delta(E_Q \cup I_Q)$ for all $j = 1, \ldots, L$. Theorem 2.2 and Lemma 2.1 give

$$\begin{aligned}\limsup_{l\to\infty} \frac{1}{l} \sum_{i=1}^{l} \sum_{D\in\mathcal{D}^j_{ki}(\mu_\delta)} \mu_\delta(D) &\leq \limsup_{l\to\infty} \frac{1}{l} \sum_{i=1}^{l} \sum_{Q\in\mathcal{D}_{ki}} \mu_\delta(E_Q \cup I_Q) \\ &\leq \limsup_{l\to\infty} \frac{1}{l} \sum_{i=1}^{l} ((2K+1)^n N\varepsilon + \delta)\mu_\delta([0,1]^n) \xrightarrow[\varepsilon\to 0]{} \delta\mu_\delta([0,1]^n).\end{aligned}$$

Since $k_0 \to \infty$ as δ and β tend to zero we may let k tend to infinity when $\delta \to 0$ and $\beta \to 0$. □

2.9. Corollary. *Assume that μ is a Radon measure on $\mathbb{R}^n$. If $0 < \beta \leq 1$ such that $\mathrm{por}(\mu) \geq \frac{1}{2}(1-\beta)$, then $\dim_{\mathrm{p}}(\mu) \leq d(\beta)$ where $d(\beta) \to n-1$ as $\beta \to 0$.*

Proof. The claim follows from Theorem 2.8 and from the obvious generalization of [JJ, Lemma 3.3] (see [JJ, Corollary 3.4]) with $d(\beta) = \lim_{\delta\to 0} \alpha(\delta, L)$ where L is as in Theorem 2.8. Note that by the choices of k and k_0 in Theorem 2.8 we have $C_n(\beta+\delta)^{-1/2} \leq 2^k \leq \widetilde{C}_n(\beta+\delta)^{-1/2}$ for constants C_n and $\widetilde{C}_n$ depending only on n. By the lower and upper bounds given in Theorem 2.8 for L we obtain that $d(\beta) \to n-1$ when $\beta \to 0$. □

2.10. *Remark.* After finishing this paper we obtained the preprint [BS] from D. B. Beliaev and S. K. Smirnov where similar dimension results have been proved using different methods.

Acknowledgements

We acknowledge the financial support of the Academy of Finland (projects 46208 and 38955).

References

[BS] D. B. Beliaev and S. K. Smirnov, *On dimension of porous measures*, preprint.

[EJJ] J.-P. Eckmann, E. Järvenpää, and M. Järvenpää, *Porosities and dimensions of measures*, Nonlinearity **13** (2000), 1–18. CMP 2000:07

[Fa] K. J. Falconer, *Techniques in Fractal Geometry*, John Wiley & Sons, Chichester, 1997. MR **99f:**28013

[Fe] W. Feller, *An Introduction to Probability Theory and Its Applications*, John Wiley & Sons, New York, 1950. MR **12:**424a

[JJ] E. Järvenpää and M. Järvenpää, *Porous measures on the real line have packing dimension close to zero*, preprint (http://www.math.jyu.fi/research/papers.html, number 212).

[JJM] E. Järvenpää, M. Järvenpää, and R. Daniel Mauldin, *Deterministic and random aspects of porosities*, preprint (http://www.math.jyu.fi/research/papers.html, number 221).

[M1] P. Mattila, *Distribution of sets and measures along planes*, J. London Math. Soc. (2) **38** (1988), 125–132. MR **89f:**28019

[M2] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and rectifiability*, Cambridge University Press, Cambridge, 1995. MR **96h:**28006

[MM] M. E. Mera and M. Morán, *A zero-one half law for porosity of measures*, preprint.

[S] A. Salli, *On the Minkowski dimension of strongly porous fractal sets in* $\mathbb{R}^n$, Proc. London Math. Soc. (3) **62** (1991), 353–372. MR **91m:**28008

Department of Mathematics, P.O. Box 35, University of Jyväskylä, FIN-40351 Jyväskylä, Finland

E-mail address: esaj@math.jyu.fi

Department of Mathematics, P.O. Box 35, University of Jyväskylä, FIN-40351 Jyväskylä, Finland

E-mail address: amj@math.jyu.fi