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A SOLVABLE GROUP WHOSE CHARACTER DEGREE GRAPH HAS DIAMETER 3

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Abstract. We show that there is a solvable group G so that the character degree graph of G has diameter 3.

Given a finite group G, the set of character degrees of G is the set $\mathrm{cd}(G) = \{\chi(1)|\chi \in \mathrm{Irr}(G)\}$. We define $\rho(G)$ to be the set of all primes that divide degrees in $\mathrm{cd}(G)$. With this set, we associate a graph $\Delta(G)$ called the *character degree graph* of G. The vertex set of $\Delta(G)$ is $\rho(G)$, and there is an edge in $\Delta(G)$ between p and q in $\rho(G)$ if pq divides some degree $a \in \mathrm{cd}(G)$. This graph was first introduced in [8].

In this paper, we are interested in the question of which graphs can occur as $\Delta(G)$ when G is a solvable group. Not all graphs occur in this situation. For example in [10] (or Theorem 18.7 of [9]), Pálfy proved that any set of 3 vertices in $\rho(G)$ spans at least one edge in $\Delta(G)$. This implies that either $\Delta(G)$ has at most two connected components, each of which must be a complete graph, or $\Delta(G)$ has one connected component of diameter at most 3. In this paper, we will be focusing on the diameter of character degree graphs with one connected component. In [8], they proved that the character degree graphs of solvable graphs have diameter at most 3.

The best places to find the basic knowledge about these graphs are [2], [3], and [9]. It is mentioned in [3], [8], and [9] that they know of no examples where the diameter is actually 3. In Theorem 14 (d) of [2], Huppert asked the question: "Is [the diameter of $\Delta(G)$] always at most 2?" We are now able to answer Huppert's question negatively. The purpose of this note is to present a solvable group whose degree graph has a diameter 3.

Theorem. *There exists a solvable group G so that the diameter of $\Delta(G)$ is 3.*

We begin by introducing some notation. Let N be a normal subgroup of G, and $\mathcal{X}$ be a subset of $\mathrm{Irr}(N)$. We define $\mathrm{Irr}(G|\mathcal{X})$ to be the set of characters $\chi \in \mathrm{Irr}(G)$ such that some irreducible constituent of χ_N lies in $\mathcal{X}$. In this spirit, we define $\mathrm{cd}(G|\mathcal{X}) = \{\chi(1)|\chi \in \mathrm{Irr}(G|\mathcal{X})\}$. To be consistent with the notation in [6], we set $\mathrm{Irr}(G|N) = \mathrm{Irr}(G|\mathrm{Irr}(N) \setminus \{1_N\})$ and $\mathrm{cd}(G|N) = \{\chi(1)|\chi \in \mathrm{Irr}(G|N)\}$.

We spend the remainder of this paper constructing a group G and showing that it fulfills the requirements of the Theorem. The construction we use to create our group G can be found in Section 4 of [5]. We start by letting F be the field of order

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2^{15}, and $F\{X\}$ to be the skew polynomial ring with $X\alpha = \alpha^2 X$ for every element $\alpha \in F$. We define R to be the ring obtained as the quotient of $F\{X\}$ by the ideal generated by X^4. Let x be the image in R of X. Write J for the Jacobson radical of R, and note that $P = 1 + J$ is a subgroup of the group of units of R. Also, the elements of P have the form $1 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$ where the $\alpha_i \in F$. The order of P is 2^{45}.

Let C be the multiplicative group $F^\times$. The order of C is $2^{15} - 1 = 7 \cdot 31 \cdot 151$. As in [5], C acts on P. It is not difficult to see that this action is

$$(1 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) \cdot c = 1 + \alpha_1 c\ x + \alpha_2 c^3\ x^2 + \alpha_3 c^7\ x^3,$$

where the $\alpha_i \in F$, $c \in C$, and the multiplication occurs in F. Let T be the semi-direct product of C acting on P. Let $\mathcal{G}$ be the Galois group of F over the prime subfield. Thus, $\mathcal{G}$ is cyclic of order 15. Relying on [5], for any element $\sigma \in \mathcal{G}$, we have

$$(1 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) \cdot \sigma = 1 + {\alpha_1}^\sigma x + {\alpha_2}^\sigma x^2 + {\alpha_3}^\sigma x^3,$$

where the $\alpha_i \in F$, and the action of σ on C is just the action of σ on $F^\times$. It is not difficult to see that $\mathcal{G}$ acts via automorphisms on T where P and C are invariant under this action. Our group G is the semi-direct product of $\mathcal{G}$ acting on T.

At this point, we compute the irreducible characters of $G/P \cong C\mathcal{G}$. We know that the subgroup of order 3 in $\mathcal{G}$ centralizes the subgroup of order 31 in C and the subgroup of order 5 in $\mathcal{G}$ centralizes the subgroup of order 7 in C. From this, we see that $\mathrm{cd}(G/P) = \{1, 3, 5, 15\}$. (For those who want the complete count, $\mathrm{cd}(G/P)$ contains fifteen linear characters, ten characters of degree 3, eighteen characters of degree 5, and $2,182$ characters of degree 15.)

Following [5], we define $P^i = 1 + J^i$. Observe that $P^1 = P$ and $P^4 = 1$, and P^i/P^{i+1} is isomorphic to the additive group of F for $i = 1, 2, 3$. It is easy to see that P^i is normal in G for every positive integer i. Observe that CP/P^3 is a Frobenius group with Frobenius kernel P/P^3, and P/P^3 is one of the groups studied in [1] and [11]. (For the remainder of this paper, we will refer to the work in [11].) Using Theorem 4.4 of [11], we see that $\mathrm{Irr}(P/P^3)$ contains $7 \cdot 31 \cdot 151$ nonprincipal linear characters and $2 \cdot 7 \cdot 31 \cdot 151$ characters of degree 2^7. Since under the action of C these characters lie in orbits of size $7 \cdot 31 \cdot 151$, it follows that $\mathrm{Irr}(T/P^3|P/P^3)$ contains 1 character of degree $7 \cdot 31 \cdot 151$ and 2 characters of degree $2^7 \cdot 7 \cdot 31 \cdot 151$. Obviously, these characters are invariant under the action of $\mathcal{G}$, and we obtain 15 characters of degree $7 \cdot 31 \cdot 151$ and 30 characters of degree $2^7 \cdot 7 \cdot 31 \cdot 151$ in $\mathrm{cd}(G/P^3|P/P^3)$. At this time, we have shown $\mathrm{cd}(G/P^3) = \{1, 3, 5, 3 \cdot 5, 7 \cdot 31 \cdot 151, 2^7 \cdot 7 \cdot 31 \cdot 151\}$. It is easy to see that $\mathrm{cd}(G) = \mathrm{cd}(G/P^3) \cup \mathrm{cd}(G|P^3)$. Since we have computed $\mathrm{cd}(G/P^3)$, to determine the structure of $\Delta(G)$ we must find $\mathrm{cd}(G|P^3)$.

Define $B = \{1 + \beta x^2 | \beta \in F\} \subseteq P^2$. Observe that B is a $C\mathcal{G}$-invariant subgroup of P^2 and $P^2 = B \times P^3$. The characters in $\mathrm{Irr}(P^2|P^3)$ have the form $\nu \times \varphi$ where $\nu \in \mathrm{Irr}(B)$ and $1_{P^3} \neq \varphi \in \mathrm{Irr}(P^3)$. To compute $\mathrm{Irr}(P|P^3)$, we use the following facts from [5]. If $s = 1 + \alpha x + i$ and $t = 1 + \beta x^2 + j$ where $i \in J^2$ and $j \in J^3$, then $[s, t] = 1 + (\alpha\beta^2 - \alpha^4\beta)x^3$ (see Corollary 4.2 of [5]). Because of this fact, we define the map $\langle \cdot, \cdot \rangle : F \times F \to F$ by $\langle \alpha, \beta \rangle = \alpha\beta^2 - \alpha^4\beta$. It is not difficult to see that $\langle \cdot, \cdot \rangle$ is a bilinear map. In particular, fixing an element $\alpha \in F$, the map $\langle \alpha, \cdot \rangle : F \to F$ is an additive homomorphism. When $\alpha \neq 0$, the kernel of this homomorphism is $\{0, \alpha^3\}$, and thus, the image $\langle \alpha, F \rangle$ is a hyperplane in F.

If $\alpha^7 = 1$, then $(\alpha^4\beta)^2 = \alpha\beta^2$, and $\langle \alpha, \beta\rangle = \alpha^4\beta + (\alpha^4\beta)^2$, for every element $\beta \in F$. Observe that the set $[F, \mathcal{G}] = \{\gamma + \gamma^2 | \gamma \in F\}$ is also a hyperplane of F (in fact, $[F, \mathcal{G}]$ can be seen to be the kernel of the trace map of F), and when $\alpha^7 = 1$, we deduce that $\langle \alpha, F\rangle = [F, \mathcal{G}]$. For any nonzero element $\alpha \in F$, there exist elements $\alpha_0, c \in F$ with $\alpha = \alpha_0 c$ where ${\alpha_0}^7 = 1$ and $c^{31\cdot 151} = 1$. We can write $\beta = \beta_0 c^3$ for some $\beta_0 \in F$, then

$$\langle \alpha, \beta\rangle = \alpha_0 c(\beta_0 c^3)^2 + (\alpha_0 c)^4 \beta_0 c^3 = c^7[{\alpha_0}^4\beta_0 + ({\alpha_0}^4\beta_0)^2].$$

We conclude that $\langle \alpha, F\rangle = c^7([F, \mathcal{G}])$. Since different values of c give different values of α, we see that F contains $31 \cdot 151$ hyperplanes of the form $\langle \alpha, F\rangle$.

Observe that P^3 is isomorphic to F by $1 + \alpha x^3 \mapsto \alpha$. We will view the kernels of characters in $\mathrm{Irr}(P^3)$ as subspaces of F via this isomorphism. Note for every hyperplane in F there is a unique nonprincipal character whose kernel corresponds to the hyperplane. Any action which permutes these hyperplanes must permute the associated characters in the same fashion. We partition the nonprincipal characters of $\mathrm{Irr}(P^3)$ into two sets. Let $\mathcal{A}$ consist of those characters $\varphi \in \mathrm{Irr}(P^3)$ so that $\ker(\varphi)$ corresponds to $\langle \alpha, F\rangle$ for a nonzero element $\alpha \in F$, and write $\mathcal{B}$ for the remaining nonprincipal characters in $\mathrm{Irr}(P^3)$. Because the set of hyperplanes of the form $\langle \alpha, F\rangle$ is stabilized set-wise by $C\mathcal{G}$, both $\mathcal{A}$ and $\mathcal{B}$ are fixed (set-wise) by $C\mathcal{G}$. In particular, $\mathcal{A}$ is a single orbit of size $31 \cdot 151$ under the action of C. Also, $\mathcal{B}$ consists of 6 orbits of size $31 \cdot 151$ under the action of C, and $\mathcal{G}$ permutes these 6 orbits like two 3-cycles.

Given a character $\varphi \in \mathrm{Irr}(P^3)$ and an element $\alpha \in F$, we define the character $\varphi_\alpha \in \mathrm{Irr}(B)$ by $\varphi_\alpha(1 + \beta x^2) = \varphi(1 + \langle \alpha, \beta\rangle x^3)$. Recall that every character in P^2 has the form $\mu \times \varphi$ where $\mu \in \mathrm{Irr}(B)$ and $\varphi \in \mathrm{Irr}(P^3)$. Let $s = 1 + \alpha x + i$ where $i \in J^2$, and note that $s^2 \in P^2$. For any element $t = 1 + \beta x^2 + \gamma x^3 \in P^2$, we have

$$\begin{aligned}(\mu \times \varphi)^s(t) &= (\mu \times \varphi)(t^s) = (\mu \times \varphi)\Big(1 + \beta x^2 + (\gamma + \langle \alpha, \beta\rangle)x^3\Big)\\ &= \mu(1 + \beta x^2)\varphi\Big(1 + (\gamma + \langle \alpha, \beta\rangle)x^3\Big).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mu\varphi_\alpha \times \varphi)(t) &= \mu\varphi_\alpha(1 + \beta x^2)\varphi(1 + \gamma x^3)\\ &= \mu(1 + \beta x^2)\varphi\Big(1 + (\gamma + \langle \alpha, \beta\rangle)x^3\Big).\end{aligned}$$

This shows $(\mu \times \varphi)^s = \mu\varphi_\alpha \times \varphi$. Notice that the stabilizer of $\mu \times \varphi$ depends only on φ. We see that s stabilizes $\mu \times \varphi$ if and only if $\varphi_\alpha = 1_B$. This occurs when the kernel of φ has the form $\langle \alpha, F\rangle$, and by definition $\varphi \in \mathcal{A}$.

When $\varphi \in \mathcal{B}$, we showed in the previous paragraph that P^2 is the stabilizer of $\mu \times \varphi$ for every character $\mu \in \mathrm{Irr}(B)$, and $(\mu \times \varphi)^P$ is irreducible. In particular, φ is fully ramified with respect to P/P^3. We deduce that $\mathrm{Irr}(P|\mathcal{B})$ consists of $6 \cdot 31 \cdot 151$ characters of degree 2^{15}. Recall that C partitions $\mathcal{B}$ into 6 orbits of size $31 \cdot 151$. We deduce that $\mathrm{Irr}(T|\mathcal{B})$ consists of 42 characters of degree $2^{15} \cdot 31 \cdot 151$. Also, our earlier comments on the action of $\mathcal{G}$ on $\mathcal{B}$ imply that 3 divides the size of every orbit for $\mathcal{G}$ on $\mathrm{Irr}(T|\mathcal{B})$. Since the subgroup of order 5 in $\mathcal{G}$ centralizes the subgroup of order 7 in C, we conclude that $\mathrm{Irr}(G|\mathcal{B})$ consists of 70 characters of degree $2^{15} \cdot 3 \cdot 31 \cdot 151$. In particular, $\mathrm{cd}(G|\mathcal{B}) = \{2^{15} \cdot 3 \cdot 31 \cdot 151\}$.

We saw that C acts transitively on $\mathcal{A}$, so $\mathrm{Irr}(G|\mathcal{A}) = \mathrm{Irr}(G|\varphi)$ where φ is the character in $\mathrm{Irr}(P^3)$ whose kernel is $[F, \mathcal{G}]$. To simplify our computations, we partition $\mathrm{Irr}(P^2|\varphi)$ into two sets. Write $\mathcal{A}_1$ for the P-orbit in $\mathrm{Irr}(P^2|\varphi)$ that contains $1_B \times \varphi$ and $\mathcal{A}_2$ for the remaining characters in $\mathrm{Irr}(P^2|\varphi)$. It is easy to see that $\mathrm{Irr}(G|\mathcal{A}) = \mathrm{Irr}(G|\varphi) = \mathrm{Irr}(G|\mathcal{A}_1) \cup \mathrm{Irr}(G|\mathcal{A}_2)$.

Take S to be the stabilizer in G of $1_B \times \varphi$. Define Q to be the set of elements $1 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$ where the elements α_i all lie in the subfield of order 8 in F. Observe that Q is a subgroup of P and $S = QP^2$. Let C_7 be the subgroup of order 7 in C, and observe that Q is invariant under the action of C_7. Furthermore, S/P^2 is irreducible under the action of C_7. Also, the character $1_B \times \varphi$ is C_7-invariant. Using Problem 6.12 of [4], we know that either $1_B \times \varphi$ extends to S or $1_B \times \varphi$ is fully ramified with respect to S/P^2. Because $|S : P^2| = 8$ is not a square, we conclude that $1_B \times \varphi$ extends to S. Using Glauberman's lemma, we see that φ has a unique C_7-invariant extension $\hat{\varphi} \in \mathrm{Irr}(S)$. Now, $\hat{\varphi}^P$ is a C_7-invariant irreducible character of P whose degree is 2^{12}. The remaining 7 characters in $\mathrm{Irr}(S|1_B \times \varphi)$ form an orbit under the action of C_7. They induce irreducibly to P in a C_7-orbit of characters whose degrees are 2^{12}. In $\mathrm{Irr}(T|\mathcal{A}_1)$, this yields seven characters of degree $2^{12} \cdot 31 \cdot 151$ and one character of degree $2^{12} \cdot 7 \cdot 31 \cdot 151$. Since $1_B \times \varphi$ is $\mathcal{G}$-invariant, the character of degree $2^{12} \cdot 7 \cdot 31 \cdot 151$ extends to 15 characters of this degree in $\mathrm{Irr}(G)$. It is not difficult to see that $\mathcal{G}$ stabilizes one of the characters of degree $2^{12} \cdot 31 \cdot 151$ and permutes the others in two orbits of size 3. This yields 15 characters in $\mathrm{Irr}(G)$ of degree $2^{12} \cdot 31 \cdot 151$ and 10 characters of degree $2^{12} \cdot 3 \cdot 31 \cdot 151$. We conclude that

$$\mathrm{cd}(G|\mathcal{A}_1) = \{2^{12} \cdot 31 \cdot 151, 2^{12} \cdot 3 \cdot 31 \cdot 151, 2^{12} \cdot 7 \cdot 31 \cdot 151\}.$$

The final case to consider consists of characters in $\mathcal{A}_2$. Let σ be an element of order 5 in $\mathcal{G}$. It is easy to see that Q is the centralizer in P of σ. From Fitting's theorem, we have $B = (Q \cap B) \times [B, \sigma]$. Fix $0 \neq \alpha \in F$, and consider an element β in the subfield of order 8 in F. We will show that $1 + \beta x^2$ lies in $\ker(\varphi_\alpha)$. When $\beta = 0$, this is obviously the case, so we assume that $\beta \neq 0$. This implies $\beta^7 = 1$. Observe that

$$\langle \alpha, \beta \rangle = \alpha\beta^2 + \alpha^4\beta = (\alpha\beta^2 + (\alpha\beta^2)^2) + (\alpha^2\beta^4 + (\alpha^2\beta^4)^2) \in [F, \mathcal{G}] = \ker(\varphi),$$

and this proves the claim. We have 2^{12} characters of the form φ_α. This implies that $\mathrm{Irr}(B/(Q \cap B)) = \{\varphi_\alpha | \alpha \in F\}$. Using the earlier direct product, we see that the orbits of $\mathrm{Irr}(P^2|\varphi)$ under the action of P are identified with the cosets of $\mathrm{Irr}([B, \sigma])$ as a subgroup of $\mathrm{Irr}(B)$. Since C_7 acts transitively on the nonidentity elements of $Q \cap B \cong B/[B, \sigma]$, we deduce that $\mathcal{A}_2$ is a single orbit under the action of PC_7, and $\mathrm{Irr}(G|\mathcal{A}_2) = \mathrm{Irr}(G|\mu \times \varphi)$ where $\mu \times \varphi$ is any character in $\mathcal{A}_2$. Fix the character $\mu \in \mathrm{Irr}(B)$ so that $\mu \times \varphi \in \mathcal{A}_2$ and the kernel of μ contains $[B, \sigma]$.

Define $Q^i = Q \cap P^i$ for $i = 1, 2, 3, 4$. Note that S is the stabilizer of $\mu \times \varphi$ in P, and $S/([B, \sigma] \times P^3) \cong Q/Q^3$. Observe that Q/Q^3 satisfies the hypotheses of the groups studied in [11]. Using those results, we see that $\mathrm{Irr}(Q/Q^3|Q^2/Q^3)$ consists of 14 characters of degree 2. Note that these characters form 2 orbits under the action of C_7. Thus, $\mathrm{Irr}(S|\mu \times \varphi)$ contains 14 characters of degree 2. These induce irreducibly to 14 characters of degree 2^{13} in $\mathrm{Irr}(P)$. We already noted that C_7 acts faithfully on these characters. It is easy to see that C_7 in C also acts faithfully on $\mathrm{Irr}(P|\mu \times \varphi)$, and $\mathrm{Irr}(T|\mathcal{A}_2)$ consists of 2 characters of degree $2^{13} \cdot 7 \cdot 31 \cdot 151$. In

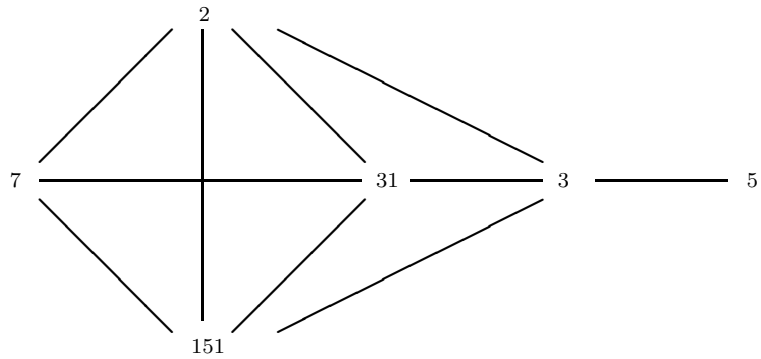

FIGURE 1.

fact, these are the only characters of this degree that lie in $\mathrm{Irr}(T|\mathcal{A})$. Therefore, they must be invariant under the action of $\mathcal{G}$, and the only characters in $\mathrm{Irr}(G|\mathcal{A}_2)$ are 30 characters of this degree. In particular, $\mathrm{cd}(G|\mathcal{A}_2) = \{2^{13} \cdot 7 \cdot 31 \cdot 151\}$.

We note that $\mathrm{cd}(G) = \mathrm{cd}(G/P^3) \cup \mathrm{cd}(G|\mathcal{A}_1) \cup \mathrm{cd}(G|\mathcal{A}_2) \cup \mathrm{cd}(G|\mathcal{B})$, and thus, we have shown that

$$\begin{aligned} \mathrm{cd}(G) \quad = \quad & \{1,\ 3,\ 5,\ 3 \cdot 5,\ 7 \cdot 31 \cdot 151,\ 2^7 \cdot 7 \cdot 31 \cdot 151,\ 2^{12} \cdot 31 \cdot 151, \\ & 2^{12} \cdot 3 \cdot 31 \cdot 151,\ 2^{12} \cdot 7 \cdot 31 \cdot 151,\ 2^{13} \cdot 7 \cdot 31 \cdot 151,\ 2^{15} \cdot 3 \cdot 31 \cdot 151\}. \end{aligned}$$

Using this set, we determine that $\Delta(G)$ is the graph in Figure 1. It is clear that this graph has diameter 3.

Our group G is quite big. (The order of G is on the order of 10^{19}.) It is reasonable to ask whether there is a smaller example. Our intuition is that there is no smaller example, but at this time we do not have a proof of this. We do know that $|\rho(G)|$ is as small as possible. When the diameter of $\Delta(G)$ is 3, it follows that $\rho(G)$ has at least 4 vertices. In the paper [12], Zhang proved that $\Delta(G)$ cannot be a graph with 4 vertices and diameter 3. In our preprint [7], we show that $\Delta(G)$ cannot be a graph with 5 vertices and diameter 3. Therefore, in any example, $\rho(G)$ must have size at least 6, and 6 is the cardinality of $\rho(G)$ in our example.

In conclusion, we would like to consider the possibility of finding other solvable groups whose character degree graphs have diameter 3. We believe that other such groups can be found among those described in Section 4 of [5]. The construction of these groups is similar to one used to construct G following the Main Theorem. We conjecture that if one uses a field of order p^{qr} where p is a prime power and $q < r$ are distinct primes that do not divide $(p^{qr} - 1)/(p - 1)$ and we factor out the ideal generated by X^{q+1}, then the resulting groups will have character degree graphs of diameter 3. Notice that when $p = 2$ and $q = 3$ a proof similar to the one outlined in this paper should yield a similar result. In fact, our proof can probably be generalized to prove the conjecture for all cases when $q = 3$, but difficulties arise in our proof when q is changed from 3. It seems likely to us that solvable groups whose degree graphs have diameter 3 must be somewhat rare. If this is the case, it should be possible to classify these groups.

I conclude by thanking Don White and John McVey for reading a preprint of this result, and for their helpful comments in writing this article.

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