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ON THE TOTAL CURVATURE OF CONVEX HYPERSURFACES IN HYPERBOLIC SPACES

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Abstract. Let $C_1 \subset C_2 \subset H^n$ be two convex compact subsets of the hyperbolic space H^n with smooth boundary. It is shown that the total curvature of the hypersurface ∂C_2 is larger than the total curvature of ∂C_1.

0. Introduction

Let M^n be an n-dimensional Riemannian manifold and let F be an $(n-1)$-dimensional smooth immersed hypersurface. Denote by $A_q : T_qF \to T_qF$ the shape operator of F at $q \in F$ with respect to a normal field defined in a neighborhood of q and set $K(q) = detA_q$. This is well defined up to sign. When M is the Euclidean space, it is called the Gauss-Kronecker curvature. We adopt the same name for K in general although it is no longer an intrinsic quantity of the hypersurface.

Let H^n denote the hyperbolic space and let $C_0 \subset C_1$ be two convex compact subsets with smooth boundaries. The goal of the paper is to show:

Theorem 1. *With the notations introduced above we have*

$$Vol(S^{n-1}) \leq \int_{\partial C_0} K \leq \int_{\partial C_1} K.$$

Here, K is computed with respect to the outward normal field of ∂C_i, $i = 0, 1$, and $Vol(S^{n-1})$ denotes the Euclidean $(n-1)$-dimensional volume of the unit sphere S^{n-1}.

It is well known that if C is a convex compact subset of the Euclidean space, then

$$\int_F K = Vol(S^{n-1}).$$

In a general Hadamard manifold M^n, as a result of the Gauss-Bonnet theorem, we have for $n = 2$

$$Vol(S^{n-1}) \leq \int_{\partial C_0} K \leq \int_{\partial C_1} K \tag{1}$$

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and for $n = 3$

$$Vol(S^{n-1}) \leq \int_{\partial C_0} K. \tag{2}$$

It seems natural to wonder to what extent the above statements will hold in higher dimensions. Some partial results with respect to (2) were obtained in [1].

Although we have precise results about certain integrals on hypersurfaces due to Chern (the curvature integra [2]), the generalized Gauss-Bonnet-Chern theorem does not seem to help in higher dimension (at least not in an obvious way).

There is another motivation for trying to show that (2) is satisfied for a general nonpositively curved manifold. This is the so-called isoperimetric conjecture (see [3], [4]).

Isoperimetric Conjecture. *Let M^n be a Hadamard manifold and $D \subset M^n$ be a compact domain with smooth boundary. Then it satisfies the Euclidean isoperimetric inequality:*

$$area(\partial D) \geq d_n (vol(D))^{\frac{n-1}{n}},$$

where $d_n = area(S^{n-1})/(vol(B^n))^{\frac{n-1}{n}}$.

This is now settled in dimension 4 by [3] and in dimension 3 by [4]. In fact, the main part of the proof in [4] is to show how (2) implies the isoperimetric inequality. Although it was carried out in dimension 3 only, it is very likely (and is explicitly mentioned in [4]) that it generalizes to higher dimensions. This means that a possible way of proving the isoperimetric conjecture is to establish (2) for a general Hadamard manifold.

1. Construction of a differential form

This is a general construction due to Chern [1] which works on any Riemannian manifold M^n. Our notation follows the notation of the original paper.

Let e_n be a unit normal field defined on some open subset of M^n. At each point extend this to an orthonormal frame $e_1, ..., e_n$ such that e_i is a smooth vector field for $i = 1, ..., n$. At least locally it is certainly possible. We now define the connection forms as

$$\omega^i_j(X) = \langle \nabla_X e_j, e_i \rangle,$$

where $\langle ., . \rangle$ denotes the metric on M^n and X is a vector field. The curvature form is defined as

$$\Omega^i_j(X, Y) = -\langle R(X, Y)e_j, e_i \rangle, \tag{3}$$

where $R(X, Y)$ denotes the curvature tensor defined as: $R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$. Then Cartan's second structural equation reads as

$$\Omega^i_j = d\omega^i_j + \omega^i_k \omega^k_j, \tag{4}$$

where we use the usual summation convention, summing over repeated indices.

The differential form which is of interest to us is defined as:

$$\Phi = \omega^1_n \omega^2_n ... \omega^{n-1}_n.$$

This is an $(n-1)$-form on M^n which is essentially the same as Chern's form denoted by Φ_0 in [2]. More precisely, Φ_0 is an $(n-1)$-form on the unit tangent bundle and

Φ is its pull back via the map $E_n : M^n \to TM^n$ defined as $E_n(p) = e_n(p)$. As a consequence we have the following important observation:

Fact. *The differential form Φ depends only on the vector field e_n. It does not depend on how e_n is extended to an orthonormal frame $e_1, ..., e_n$.*

From the second structural equation (4) one can derive that

$$d\Phi = \sum \epsilon_{i_1 \dots i_{n-1}} \Omega_n^{i_1} \omega_n^{i_2} \dots \omega_n^{i_{n-1}}, \tag{5}$$

where $\epsilon_{i_1 \dots i_{n-1}}$ is the Kronecker index which is equal to $+1$ or -1 according to whether the permutation $i_1 \dots i_{n-1}$ of the numbers $1, 2, ..., n-1$ is even or odd and the summation is extended over all the indices $i_1 \dots i_{n-1}$ subject to the condition $i_2 < i_3 < ... < i_{n-1}$. It is essentially the same as the n-form Ψ_0 in [2].

2. Convex exhaustion

The other important ingredient is a lemma about convex exhaustion. Although we only need this in the hyperbolic space we state it for Hadamard manifolds. We say that a convex set C with smooth boundary is strictly convex if the second fundamental form of the boundary (with respect to the outward normal) is positive definite everywhere.

Lemma 1. *Let M^n be a Hadamard manifold and $C_0 \subset int(C_1)$ be two compact strictly convex subsets with smooth boundary. Then there exists a continuous function $F : C_1 - int(C_0) \to [0, 1]$ which is smooth in the interior such that: $\nabla F \neq 0$, $\partial C_i = \{p \in C_2 - int(C_1) : F(p) = i\}$ for $i = 0, 1$ and the sublevel sets $C_b = \{p \in C_2 - int(C_1) : F(p) \leq b\}$ for $0 \leq b \leq 1$ are convex.*

Proof. The statement is clear intuitively. Denote by ϱ_i the distance function from the set ∂C_i for $i = 0, 1$.

For $\delta > 0$ we set $N_{2\delta} = \{p \in C_1 : \varrho_1(p) < 2\delta\}$. Since ∂C_1 has a positive definite second fundamental form, we can choose $\delta > 0$ small enough such that: $3\delta < \mathrm{dist}(C_0, \partial C_1)$, the function ϱ_1 is smooth on $N_{2\delta}$ (there are no focal points of ∂C_1 inside $N_{2\delta}$) and

$$-D^2 \varrho_1 > c_1 > 0, \tag{6}$$

when restricted to $\nabla \varrho_1^\perp$ for some positive constant c_1. Here, $D^2 \varrho_1$ denotes the Hessian and we adapted the notation that $-D^2 \varrho_1 > c_1$ on $\nabla \varrho_1^\perp$ if $-D^2 \varrho_1(X, X) > c_1$ for every unit tangent vector $X \in \nabla \varrho_1^\perp$. We observe also that on $N_{2\delta}$ the following inequality holds for the angle between the gradients:

$$\measuredangle(\nabla \varrho_0, -\nabla \varrho_1) < \pi/2 - \alpha, \tag{7}$$

for some $\alpha > 0$ depending on δ.

We are going to construct F in the form

$$F = 1 - e^{-a\varrho_0} f^\epsilon. \tag{8}$$

Here $f = h(\varrho_1)$ is the reparametrized distance function from ∂C_1 and h is a fixed smooth increasing real function $h : [0, \infty) \to [0, 1]$ such that $h(t) = \frac{1}{2\delta} t$ for $0 \leq t \leq \delta$ and $h(t) = 1$ on $[2\delta, \infty)$. For the derivative and the Hessian of f we have

$$df = h' d\varrho_1 \qquad \text{and} \qquad D^2 f = h'' d\varrho_1 \otimes \varrho_1 + h' D^2 \varrho_1.$$

The choice of a, ϵ will be discussed later.

First, we show that the derivative of F is never zero. From the definition we obtain

$$dF = e^{-a\varrho_0} f^{\epsilon}(a d\varrho_0 - \frac{\epsilon}{f} df). \tag{9}$$

Then, taking into consideration that $\nabla f \parallel \nabla \varrho_1$, the statement follows from (7).

To show that the sublevel sets are convex we need to show that the Hessian D^2F is positive definite on $\nabla F^{\perp}$. We have

$$\begin{aligned} D^2F =& e^{-a\varrho_0}\big(- a^2 d\varrho_0 \otimes d\varrho_0 + a D^2\varrho_0\big) f^{\epsilon} \\ &+ \epsilon e^{-a\varrho_0} f^{\epsilon-1}\big(\frac{1-\epsilon}{f} df \otimes df - D^2 f\big) \\ &+ \epsilon a e^{-a\varrho_0} f^{\epsilon-1}(d\varrho_0 \otimes df + df \otimes d\varrho_0). \end{aligned} \tag{10}$$

The argument depends on certain estimates of the Hessians $D^2\varrho_i$ for $i = 0, 1$, on various subspaces. It will be useful to keep in mind that ϱ_i (for $i = 0, 1$) are distance functions; therefore the gradients $\nabla\varrho_i$ are eigenvectors of $D^2\varrho_i$ with eigenvalue zero.

First, we consider the region where $f \equiv 1$. On this region $df, D^2 f = 0$ and $\nabla F \parallel \nabla d\varrho_0$. Therefore $D^2F = a f^{\epsilon} e^{-a\varrho_0} D^2\varrho_0$, when it is restricted to vectors in $\nabla F^{\perp}$. But on this subspace $D^2\varrho_0$ is positive definite since it is the distance function from a convex set in a Hadamard manifold.

Next, we consider the region where $1/2 < f < 1$. Since C_0 is strictly convex and compact, we know that $D^2\varrho_0 > c_2 > 0$ for some $c_2 > 0$, when restricted to $(\nabla\varrho_0)^{\perp}$. Therefore, taking (6) and (9) into account, we conclude that $D^2\varrho_0 > c_3 > 0$ for some positive constant c_3, when restricted to $\nabla F^{\perp}$. So, for a small enough a, where the choice of a depends only on c_3, the term $aD^2\varrho_0$ will dominate $a^2 d\varrho_0 \otimes d\varrho_0$. This will remain true throughout the whole region $C_1 - C_0$. The derivatives of h are bounded and so are the terms df and $D^2 f$. Therefore, for a small enough ϵ, the term $aD^2\varrho_0$ will dominate all the other terms as well. The choice of ϵ depends on a and on the bounds for df and $D^2 f$.

At last, we consider the region where $0 < f < 1/2$. It is clear from the definition of the function f that this region is a subset of $N_{2\delta}$. As before, $aD^2\varrho_0$ will dominate $a^2 d\varrho_0 \otimes d\varrho_0$. The term involving $df \otimes df$ is positive semi-definite. As for the rest of the terms, we will show that the term involving $-D^2 f$ will dominate, for a small enough a, the term involving $d\varrho_0 \otimes df + df \otimes d\varrho_0$. Since h is linear on this region, from (6) we obtain $-D^2 f = -h' D^2\varrho_1 > c_1/2\delta > 0$, when restricted to $(\nabla\varrho_1)^{\perp}$. Taking into consideration (7) and (9) we conclude that $-D^2 f > c_4 > 0$ for some sufficiently small $c_4 > 0$, when restricted to $\nabla F^{\perp}$. The constant c_4 depends on c_1, δ and the angle α. So, if the constant $a > 0$ was chosen small enough, where the choice of a depended only on c_2 and c_4, then $-D^2 f$ dominates the term involving $d\varrho_0 \otimes df + df \otimes d\varrho_0$. This concludes the proof of the lemma. □

3. Proof of Theorem 1

With the preparation done in the previous sections, the proof of the theorem is simple. Let us return to the hyperbolic space H^n.

First, we prove the inequality between the two integrals. Assume that $C_0 \subset \mathrm{int}(C_1)$ and both sets are strictly convex. We are going to show that

$$\int_{\partial C_0} K \leq \int_{\partial C_1} K. \tag{11}$$

The general case will follow by a trivial limiting procedure.

Let F be the smooth function of Lemma 1 and define the unit vector field e_n by $e_n = \nabla F/|\nabla F|$. This is defined on $\mathrm{int}(C_1) - C_0$ but it extends continuously to the boundary. To the vector field e_n we construct the form Φ as in the previous section and by Stokes's theorem we have

$$\int_{\partial C_0 \cup \partial C_1} \Phi = \int_{C_1 - C_0} d\Phi. \tag{12}$$

To evaluate the integrals in (12) we are going to compute the forms Φ and $d\Phi$. Let $q \in C_1 - \mathrm{int}(C_0)$ be an arbitrarily chosen point.

Since Φ depends only on the vector field e_n, we can express Φ in a special frame. Let us choose the frame $e_1, ..., e_n$ such that at the point $q \in C_1 - \mathrm{int}(C_0)$ the vectors $e_1, ..., e_{n-1}$ are the principal directions for the hypersurface $\{F = F(q)\}$. For the other points of the hypersurface the vectors $e_1, ..., e_{n-1}$ may no longer be principal directions. Then, from the definition of the ω_n^i's we have

$$\omega_n^i(e_j) = \delta_j^i \lambda_j, \qquad \text{for} \quad 1 \le i, j \le n-1 \tag{13}$$

at $q \in C_1 - \mathrm{int}(C_0)$, where λ_j denotes the principal curvature at q of the hypersurface $\{F = F(q)\}$ in the direction of e_j and δ_j^i is the Kronecker symbol. Therefore

$$\Phi(e_1, ..., e_{n-1}) = \lambda_1 \cdot ... \cdot \lambda_{n-1} = K$$

at $q \in C_1 - \mathrm{int}(C_0)$, where K denotes the Gauss-Kronecker curvature of the hypersurface $\{F = F(q)\}$ with respect to the normal field e_n. Since $q \in C_1 - \mathrm{int}(C_0)$ was chosen arbitrarily, the left-hand side of (12) reads as follows:

$$\int_{\partial C_0 \cup \partial C_1} \Phi = \epsilon_{n12...n-1} \left(\int_{\partial C_1} K - \int_{\partial C_0} K \right) = (-1)^{n-1} \left(\int_{\partial C_1} K - \int_{\partial C_0} K \right).$$

The curvature tensor has the form $R(X,Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y$; therefore

$$\Omega_n^i(e_j, e_k) = 0$$

if $\{j, k\} \neq \{i, n\}$ as sets. From this and (13) we have

$$\Omega_n^{i_1} \omega_n^{i_2} ... \omega_n^{i_{n-1}}(e_1, ..., e_n) = \epsilon_{i_1 n i_2 ... i_{n-1}} K_{i_1 n} \lambda_{i_2} \cdot ... \cdot \lambda_{i_{n-1}}$$

at $q \in C_1 - \mathrm{int}(C_0)$, where the indices satisfy the condition $i_2 < i_3 < ... < i_{n-1}$ and $K_{in} \equiv -1$ denotes the sectional curvature of the two-plane determined by e_i, e_n and λ_i is the principal curvature at the point $q \in C_1 - \mathrm{int}(C_0)$ in the direction of e_i. Then (12) reads as follows:

$$\int_{\partial C_1} K - \int_{\partial C_0} K = \int_{C_1 - C_0} \sum -K_{i_1 n} \lambda_{i_2} \cdot ... \cdot \lambda_{i_{n-1}},$$

where the summation is extended over all the indices $i_1 ... i_{n-1}$ subject to the condition $i_2 < i_3 < ... < i_{n-1}$.

Since all sublevel sets are convex, all the principal curvatures are positive. Therefore, the integral on the right-hand side is positive. This completes the proof of the theorem, when $C_0 \subset \mathrm{int}(C_1)$ and both sets are strictly convex. The general case follows by slightly "blowing up" the sets; that is, instead of C_0 we consider an η-neighborhood C_η and instead of C_1 we take a 2η-neighborhood $C_{1+2\eta}$. These are now strictly convex sets satisfying the conditions set forth at the beginning of the proof. Then letting η go to 0 will yield the general case.

All that remains is to prove the inequality

$$Vol(S^{n-1}) \leq \int_{\partial C_0} K.$$

This is a simple consequence of (11). Choose a ball B_ϵ inside C_0. Applying (11), we obtain

$$\int_{\partial B_\epsilon} K < \int_{\partial C_0} K.$$

Letting ϵ go to 0 will yield the desired inequality. This completes the proof of the theorem.

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