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CLOSED GROUPS INDUCED BY FINITARY PERMUTATIONS AND THEIR ACTIONS ON TREES

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Abstract. We describe permutation groups $G \leq Sym(\omega)$ such that G is the closure of the subgroup of all elements with finite support and G can be realized as $Aut(M)$ where M is a saturated structure. We also study isometric actions of such groups on real trees.

0. Introduction

We consider $Sym(\omega)$ as a complete metric space by defining $d(g,h) = \Sigma\{2^{-n} : g(n) \neq h(n)$ or $g^{-1}(n) \neq h^{-1}(n)\}$. It is well-known that a subgroup $G \leq Sym(\omega)$ is closed if G is of the form $Aut(M)$ for an appropriate structure M defined on ω. Below we take a model-theoretic assumption that M is saturated [4].

For $g \in Sym(\omega)$ let $supp(g) = \{x : g(x) \neq x\}$. A permutation is *finitary* if its support is finite. The group of all finitary permutations is denoted by $SF(\omega)$. We study closed permutation groups (ω, G) such that G is the closure of $G \cap SF(\omega)$ and there exists a saturated structure M on ω such that $G = Aut(M)$.

The assumption that M is a countably saturated structure is necessary for model-theoretical methods. Then the action of $Aut(M)$ on ω describes the structure M: for any $n \in \omega$ there is a natural bijection between the family of orbits on M^n and the set of n-types over $\emptyset$. Without that correspondence the class of possible structures becomes intractable. For example $Aut(M)$ has trivial action on ω if M is the standard model of arithmetic $(\omega, +, \cdot)$ as well as in the case when M is the model of unary predicates P_i defined by $P_i(x) \leftrightarrow x = i$.

The saturation assumption above is natural from the algebraic point of view, too. Without that the groups $Aut(M) \cap SF(\omega)$ are quite various: $Sym(\omega)$ and many residually finite, locally finite FC-groups can be so realized. It is worth noting that the latter groups arise as finitary groups with finite orbits [11]. This is impossible when M is saturated. On the other hand by Theorem 3 of [12] (see the proof of Theorem 0.1 below) a transitive finitary group G having a point-stabilizer with an infinite orbit is oligomorphic (for every n the number of orbits of n-tuples is finite). Then by a well-known theorem of Ryll-Nardzewski the closure of G is the automorphism group of a countably categorical structure. Such structures are saturated.

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The main result of the paper is as follows.

Let X be a countably infinite set and F be finite. Let E_F be the equivalence relation on $Y = X \times F$ defined by the *fibres* $\{x\} \times F$. We say that a closed permutation group (Y, H) is a *cell* (E_F*-cell*) if H preserves E_F and induces $Sym(X)$ by the action on the set of E_F-classes. We say that a closed permutation group (Ω, G) is a 2*-dimensional cell* if there exists a partition $\Omega = Y_1 \cup Y_2 \cup ...$ into G-invariant classes such that for any infinite Y_i the group (Y_i, G_i) induced by G is a cell and is also induced by the pointwise stabilizer G_Y of $Y = \bigcup\{Y_j : Y_j \neq Y_i\}$. The following theorem is proved in Section 1 by means of stability theory.

Theorem 0.1. *Let a closed permutation group (ω, G) be equal to the closure of $G \cap SF(\omega)$ and be the automorphism group of a saturated structure on ω. Then (ω, G) is a* 2*-dimensional cell.*

In Section 2 we study actions of 2-dimensional cells on real trees. A short introduction and some motivations are given there. It can be read immediately after Section 0.

1. Groups of finitary permutations

Let M be a countably saturated structure. A tuple $\overline{c} \subset M$ is algebraic over $D \subset M$ if it realizes a formula with parameters from D with finitely many solutions. By $acl(D)$ we denote the algebraic closure of D: the set of all elements algebraic over D. We say that M has *locally finite* algebraic closure if $acl(A) \setminus acl(\emptyset)$ is finite for any finite $A \subset M$. If for all A and B, $acl(A \cup B) = acl(A) \cup acl(B)$, then the algebraic closure is called *disintegrated*.

Lemma 1.1. *Let M be a countable saturated structure and $Aut(M)$ be the closure of $Aut(M) \cap SF(\omega)$ in $Sym(\omega)$. Then M has disintegrated locally finite algebraic closure.*

Proof. First we show that the relation $a \in acl(b)$ is an equivalence relation on $M \setminus acl(\emptyset)$. Indeed, let $\phi(x, b)$ have finitely many solutions, a satisfy $\phi(x, b)$ and let D be the $Aut(M/b)$-orbit of a. We may assume that D is the set of solutions $\phi(M, b)$. Suppose b is not algebraic over a. Since M is saturated and the set of finitary automorphisms is dense, every permutation on D, which extends to an automorphism fixing b, extends to a finitary automorphism fixing b. Thus, the $Aut(M/a)$-orbit of b has infinitely many c such that $\phi(M, c) = \phi(M, b)$ (apply the fact that $Aut(M/b)$ induces finitely many permutations on D).

Since a is non-algebraic over $\emptyset$, there is $b' \in M$ of the $Aut(M)$-orbit of b such that a does not satisfy $\phi(x, b')$. A finitary automorphism δ taking b onto b' fixes almost all elements of the $Aut(M/a)$-orbit of b. By the property obtained in the previous paragraph, δ preserves D (containing a). As δ also fixes almost all elements of the $Aut(M/\delta(a))$-orbit of b' it preserves the set $\phi(M, b') \cap \phi(M, b)$ and $\delta(a) \in \phi(M, b') \cap \phi(M, b)$. Since $a \in \phi(M, b) \setminus \phi(M, b')$ we have a contradiction with the fact that $\phi(M, b') \cap \phi(M, b)$ is finite.

We now prove that acl is disintegrated:

$$acl(\{a_1, ..., a_k\}) = \bigcup\{acl(a_i) : i \leq k\}.$$

Suppose all $a_i \in \overline{a} = (a_1, ..., a_k)$ are not algebraic, $a \in acl(\overline{a})$ by a formula $\phi(x, \overline{a})$ with n realizations and a is not algebraic over any a_i. As is shown above, every a_i is not algebraic over a. Thus there is an infinite sequence $\overline{a}_1, \overline{a}_2, ...$ from the

$Aut(M/a)$-orbit of $\overline{a}$ such that $\overline{a}_i \cap \overline{a}_j = \emptyset$ for distinct i, j. Any finitary $\alpha \in Aut(M)$ fixes infinitely many $\overline{a}_i$. This means that infinitely many sets $\phi(M, \overline{a}_i)$ are the same. Thus the $Aut(M)$-orbit of a is contained in some $\phi(M, \overline{a}_i)$. Since the latter is finite, we have a contradiction with the fact that a is non-algebraic.

Now we can show that acl is locally finite. By disintegratedness it suffices to prove that for any $a \in M$, $acl(a) \setminus acl(\emptyset)$ is finite. If the latter does not hold find a and b of the same $Aut(M)$-orbit which are not acl-equivalent and have infinite acl-classes (we can do this because M is saturated). Then there is no finitary automorphism which takes a to b. $\square$

We now need some terminology from *stability theory*. Recall that a theory $Th(M)$ is *ω-stable* if for any countable model N of $Th(M)$ the set $S(N)$ of all complete types over N is countable. A theory $Th(M)$ is *weakly minimal* if for any formula $\psi(x, \overline{y})$ without parameters there are formulas $\phi_i(x, \overline{a}_i), \overline{a}_i \subset M$, $1 \leq i \leq n$, such that $M = \bigcup_{i \leq n} \phi_i(M, \overline{a}_i)$ and for every $\overline{a} \subset M$ and $i \leq n$, either $\psi(M, \overline{a}) \cap \phi_i(M, \overline{a}_i)$ or $\neg\psi(M, \overline{a}) \cap \phi_i(M, \overline{a}_i)$ is finite.

Lemma 1.2. *Let M be a countable saturated structure. If $Aut(M)$ is the closure of $Aut(M) \cap SF(\omega)$ in $Sym(\omega)$, then M is an ω-stable weakly minimal structure.*

Proof. To prove that M is ω-stable we use the characterization of ω-stability of Bouscaren and Laskowski from [2] (Corollary 3.12 and Lemma 3.4). By this characterization we should prove that for every algebraically closed $A \subset M$ every *finitely extendible* surjection $f : A \to A$ (for every finite $X \subset A$ the restriction of f on X extends to an automorphism preserving A) extends to an automorphism of M.

We do it as follows. Take a maximal *acl*-independent sequence $B = \{b_0, b_1, ...\}$ in $M \setminus A$. First we show that every elementary permutation g of

$$A_n = acl(A \cup \{b_0, b_1, ..., b_n\})$$

is elementary over b_{n+1} (note that as the algebraic closure is disintegrated, A_n does not contain b_{n+1}). Indeed, suppose g is not elementary over b_{n+1}. Then there is $\overline{a} \in A_n$ such that $(\overline{a}, b_{n+1})$ and $(g(\overline{a}), b_{n+1})$ have distinct types. So, this is true for any realization of $tp(b_{n+1}/(\overline{a}, g(\overline{a})))$. Since the restriction of g on $\overline{a}$ can be extended to a finitary automorphism, the number of these realizations is finite, which contradicts that A_n is algebraically closed.

Let $f : A \to A$ be finitely extendible. Thus f is elementary and by the previous paragraph f is elementary over b_0. Now f can be extended to an elementary permutation f_0 of A_0 fixing b_0. Since f_0 is elementary over b_1, it naturally extends to A_1. Continuing this procedure we get an automorphism of M.

For weak minimality it suffices to prove that the Shelah's degree Deg of M equals 1 (see Theorem 1.17 in [16]). Suppose that there is a sequence $\{\psi(x, \overline{b}_i) : i \in \omega\}$ of non-algebraic formulas with parameters from M such that for some $k \in \omega$ any k formulas from the sequence are inconsistent. Since M is saturated we may assume that all $\overline{b}_i$ realize the same type. Thus for every $i, j \in \omega$ there exists a finitary automorphism α such that $\alpha(\overline{b}_i) = \overline{b}_j$. Thus only finitely many elements of $\phi(M, \overline{b}_i)$ do not satisfy $\phi(x, \overline{b}_j)$. This contradicts k-inconsistency. $\square$

Proof of Theorem 0.1. We use some characterization of finitary permutation groups given by P. M. Neumann in [11] and [12]. A permutation group (Ω, S) is *almost primitive* if there is a maximal S-congruence on Ω with finite classes. Otherwise, (Ω, S) is called *totally imprimitive*. It is a consequence of Theorem 2.4 from [11] that for a transitive totally imprimitive group (Ω, S) of finitary permutations the stabilizer of any element in Ω has no infinite orbits.

Now note that if the group of automorphisms of a countable structure has no infinite orbits, then the structure is not saturated. This shows that if $G = Aut(M)$ where M is saturated, then G has infinite orbits and $G \cap SF(\Omega)$ induces an almost primitive finitary permutation group on any infinite orbit Y_p (p denotes the corresponding non-algebraic 1-type: $Y_p = p(M)$).

By Theorem 3 from [12] any almost primitive transitive group of finitary permutations can be presented as a subgroup of $(D \times \omega, L) = (D, H)wr(\omega, SF(\omega))$ for some permutation group H on a finite D. Moreover, for such a presentation of $G \cap SF(\Omega)$ we have $[L, L] \leq G$. This easily implies that the group induced by G on Y_p induces $(\omega, Sym(\omega))$ for the corresponding presentation $Y_p = D \times \omega$.

Now let $Y_p^* = \bigcup\{acl(a) : a \in Y_p\} \setminus acl(\emptyset)$. By Lemma 1.1 the set Y_p^* can be presented as $Y_p \times F$ for some finite F. Then by the structure of Y_p, the set Y_p^* is a cell under the corresponding group G_p induced by $Aut(M)$.

Since all Y_p^* form a partition of the non-algebraic part of M (it may happen that $Y_p^* = Y_q^*$ for distinct p and q), the structure M is the union of disjoint infinite cells and finite orbits. If Y is a finite union of finite orbits, then any elementary map $Y \to Y$ extends to an automorphism fixing almost all elements of Y_p^*. Then any type over Y realized by a tuple from Y_p^* has a realization fixed by that automorphism. This implies that the pointwise stabilizer G_Y induces G_p on Y_p^*.

On the other hand as M is weakly minimal any tuple $\bar{c}$ from some infinite Y_q^*'s distinct from Y_p^* is forking-independent from Y_p over Y (chosen above). Then any finite partial elementary map in $Y_p \cup Y$ fixing Y pointwise, extends to one fixing $\bar{c}$. The rest of the proof is clear. □

2. Tree actions

In this section we study isometric actions of $Aut(M)$ on real trees, where $(M, Aut(M))$ is a 2-dimensional cell. The question that we are interested in is whether such a group has a fixed point under an action on an $\mathbb{R}$-tree.

The motivation comes from [14], where actions of some 2-dimensional cells ($Sym(\omega)$ and some profinite groups) on discrete trees are studied. In the case of real trees the question becomes more complicated as the following (typical) example shows.

Let p be a prime number and $\omega = P_1 \supset P_2 \supset ...$ be an infinite sequence of cofinite subsets of ω with infinite intersection. Assume that $|P_i \setminus P_{i+1}| = p^i$ and let R_i be a binary relation defining a single directed cycle on $P_i \setminus P_{i+1}$. Then the structure $(\omega, P_1, P_2, ..., R_1, R_2, ...)$ is saturated and $G = Aut(\omega, P_1, P_2, ..., R_1, R_2, ...)$ is the closure of the subgroup of finitary automorphisms. It is clear that G is the direct sum of $Sym(\bigcap P_i)$ and $\Pi\mathbb{Z}(p^i)$. It is known that both groups (and their sum) satisfy *property FA′*: any element fixes a point under any action of the group on any $\mathbb{Z}$-tree by automorphisms without inversions [1], [14] (a group H acts on a $\mathbb{Z}$-tree *without inversions* if whenever $h \in H$ fixes an edge $[a, b]$ it also fixes the points a, b).

Notice that G has an $\mathbb{R}$-action by isometries where some elements of G do not fix any point. Indeed, $\Pi\mathbb{Z}(p^i)$ is the p-adic completion of $\mathbb{Z}_p \oplus_{k\geq 1} \mathbb{Z}(p^k)$ (Example 2 of Section 40 of [8]). Then $\mathbb{Z}_p$ is a homomorphic image of G. On the other hand, the group of p-adic integers acts on an $\mathbb{R}$-line through an embedding into $\mathbb{R}$ (the former is a subgroup of a 2^ω-dimensional vector space over $\mathbb{Q}$). As a result the group G has a non-trivial action on a real tree (line).

In this section we show how to avoid such examples in the case of closed groups induced by finitary permutations. The main general definitions and results are as follows.

Let G be an infinite group acting on an $\mathbb{R}$-tree T by isometries. For $g \in G$ let T^g be the set of vertices of T fixed by g. An end is *neutral* (resp. attracting; resp. repulsing) for G if for every $g \in G$ there exists a half-line L representing the end such that $g(L) = L$ (resp. $\subset L$; resp. $\supset L$). In [17] J. Tits defines the following properties of an action:

(P1) G has fixed points;

(P2) G has a neutral fixed end and no fixed points;

(P3) G has two (not neutral) fixed ends and no fixed points;

(P4) G has an invariant pair of ends and neither has fixed points nor fixed ends;

(P5) G has no fixed points and has a single end which is not neutral for G.

Let F_i be the property that G satisfies (P1) $\vee ... \vee$ (Pi) for any action on a tree. Proposition 2.3.4 of [17] claims that if $(m,n) = (1,1), (2,2), (2,4)$, or $(5,5)$ and $N \leq H \leq G$, where N is normal and $G/N \models F_m$, then given a G-action on a tree with $H \models$ (Pn), the group G satisfies (Pn). If G acts on a tree such that every element of G fixes a point, then any two of them have a common fixed point (Proposition 1.2 of [6]) and by Sections 1.6 from [17] G satisfies (P1) $\vee$ (P2) (see also [15], Exercise 2, p. 66).

Note that actions of G on $\mathbb{Z}$-trees by automorphisms without inversions form a subclass of real tree actions because every action on a $\mathbb{Z}$-tree T uniquely defines an action on an $\mathbb{R}$-tree naturally extending T. The group G is called an *FA-group* if G satisfies (P1) for any action without inversions on a $\mathbb{Z}$-tree. It follows from Section 6.1 of [15] that any uncountable group G satisfying F_2 has property FA if it has *uncountable cofinality*: G cannot be presented as the union of a countable chain of proper subgroups.

Let (Ω, G) be a 2-dimensional cell and (Y_{i_k}, G_{i_k}), $k \in J$, be the sequence of all infinite cells in the corresponding partition $\Omega = Y_1 \cup Y_2 \cup ....$ Let $\omega = \bigcup_{k\in J} X_k$, where X_k are pairwise disjoint countably infinite sets. We assume that each cell Y_{i_k} is of the form $X_k \times D_k$, where D_k is finite and G_{i_k} induces $Sym(X_k)$. As a result we obtain a surjective homomorphism ρ from (Ω, G) onto the automorphism group of the structure $(\omega, X_k)_{k\in J}$ (of unary predicates). We call *ρ the cover homomorphism.*

Theorem 2.1. *Let a closed permutation group (Ω, G) be a 2-dimensional cell and let $\rho : G \to Aut(\omega, X_i)_{i\in J}$ be the cover homomorphism. Then for any isometric action of G on an $\mathbb{R}$-tree if any $h \in Ker\rho$ fixes a point, then so does any $g \in G$. Moreover, G has property FA if $Ker\rho$ is not the union of a countable chain of proper subgroups.*

We start with two preliminary statements. The first one will be proved in Section 3.

Proposition 2.2. *Let $X_i, i \in I$, be a partition of ω into infinite sets and $M = (\omega, X_i)_{i\in I}$ be the corresponding structure of unary predicates. Then for any n the space $(Aut(M))^n$ has a comeagre orbit under the action of $Aut(M)$ by conjugation. The group $Aut(M)$ has uncountable cofinality.*

Lemma 2.3. *Let $X_i, i \in I$, be a partition of ω into infinite sets, $G_i = Sym(X_i)$ and (ω, G) be the permutation group $\Pi_I G_i$ under the natural action on $\bigcup_{i \in I} X_i$. Then any $g \in G$ has a fixed point under any G-action on an $\mathbb{R}$-tree. Moreover G has property FA.*

Proof. Let G act on an $\mathbb{R}$-tree T. Let $g \in G$. We start with the case when g has no non-trivial finite cycles (then we can use some arguments from the proof of Theorem 6.1 from [14]). We may assume that for any X_i the element g does not induce a unique infinite cycle on X_i (otherwise we consider g^2; it fixes T-points if and only if g does). Then $g = h_1 h_2$, where h_1 and h_2 are disjoint permutations, for any X_i both h_1 and h_2 have infinitely many fixed points in X_i and do not have finite non-trivial cycles. Then each of them is expressed as $f_1 f_2$, where f_1 and f_2 have infinitely many infinite orbits on every X_i and for every infinite f_i-orbit Y the permutation f_{3-i} fixes Y pointwise or acts as f_i^{-1}. By Proposition 2.1 of [6] it is enough to show that any f_i has a fixed T-point. We obtain this following the idea from the proof of Lemma 6.5 of [14]: let $\Omega = supp(f_i)$; then $f_i \in S < G$, such that (Ω, S) is isomorphic to the left regular action of $SL(3, \mathbb{Z})$ on $|I|$ copies of $SL(3, \mathbb{Z})$. By the $\mathbb{R}$-version of Theorem 16 of [15] (see [6], p. 682), f_i fixes a point from T.

Consider the case when g does not have infinite cycles. Express $g = g_1 g_2$ as a product of disjoint permutations such that both g_1 and g_2 have infinitely many fixed points in any X_i. Each of them can be presented as $f_1 f_2$, where for every X_i both f_1 and f_2 have infinitely many orbits of any finite length and if Y is an f_i-orbit, then the permutation f_{3-i} fixes Y pointwise or acts as f_i^{-1}. To apply Proposition 2.1 of [6] we need to show that any f_i has a fixed T-point. If f_i does not fix any T-point, then f_i acts on some T-line by translation (see Section 3.1 in [17]). This defines a non-trivial homomorphism from the centralizer $C_G(f_i)$ into $\mathbb{R}$. It is easy to see that $C_G(f_i)$ is the direct product of all groups of the following form: $(C_n wr Sym(\omega))^I$ (C_n denotes a cyclic group consisting of n elements). Straightforward computations show that such groups have trivial abelianization. This is a contradiction.

We now consider the general case. Express $g = g_1 g_2$ as a product of disjoint permutations such that g_1 has no non-trivial finite cycles and g_2 has no infinite cycles. By the above cases and Proposition 2.1 of [6] it fixes points on T.

To prove FA it suffices to verify that G cannot be expressed as the union of an increasing ω-chain of of proper subgroups. This follows from Proposition 2.2. $\square$

D. Macpherson and S. Thomas have proved that a Polish group having a comeagre conjugacy class has the property that any element fixes a point under any action on a $\mathbb{Z}$-tree. Their arguments work for $\mathbb{R}$-trees and can be applied in the proof above.

Proof of Theorem 2.1. Let (Ω, G) be a 2-dimensional cell and ρ be the cover homomorphism. By Lemma 2.3 the group $\rho(G)$ satisfies F_2 and has property FA. If $K := ker\rho$ acts on an $\mathbb{R}$-tree and any of its elements fixes a point, then applying Proposition 2.3.4 of [17] we see that any $g \in G$ fixes a point.

The kernel K satisfies (FA) if and only if its cofinality is uncountable. On the other hand for any action of G on a $\mathbb{Z}$-tree T, the group $\rho(G)$ acts on the tree T^K of K-fixed points. By Lemma 2.3 we obtain that if K has uncountable cofinality, then G fixes some points of T^K. $\square$

It is worth noting that even in the case when all fibres and finite cell of a 2-dimensional cell are of bounded size (this holds if the structure is ω-categorical), the kernel K can be of countable cofinality (Theorem 1.3 of [14]) and without FA. On the other hand this assumption implies that K is periodic. Then every $h \in K$ fixes a point under any action on an $\mathbb{R}$-tree (otherwise h acts by translation on an $\mathbb{R}$-line [17]). By Theorem 2.1 every $g \in G$ fixes a point under any action on an $\mathbb{R}$-tree. As we noted above this implies $G \models F_2$. So we have the following corollary.

Corollary 2.4. *Let a closed permutation group (Ω, G) be a 2-dimensional cell and $\rho : G \to Aut(\omega, X_i)_{i \in J}$ be the cover homomorphism. If the 2-cell decomposition of G involves finite cells and fibres of bounded size, then $G \models F_2$.*

3. Generic automorphisms

Let M be a countable first-order structure. An automorphism $\alpha \in Aut(M)$ is *generic* if its conjugacy class is comeagre. J. Truss has shown in [18] that if the set $\mathbb{P}$ of all finite partial maps extendible to automorphisms contains a cofinal subset $\mathbb{P}'$ closed under conjugacy and having the amalgamation property and the joint embedding property, then there is a generic automorphism. In fact the arguments work for the space $(Aut(M))^n$ as follows.

Let $\mathbb{P}_n$ be the set of all n-tuples $\overline{\alpha}$ of finite maps in M which have extensions to tuples from $(Aut(M))^n$. We say that $\mathbb{P}'_n \subseteq \mathbb{P}_n$ is *closed under conjugacy* if for any $\overline{\alpha} \in \mathbb{P}'_n$ and $\gamma \in Aut(M)$ the tuple of maps $(\alpha_i)^\gamma$ $(1 \leq i \leq n)$ from $\gamma^{-1}(Dom(\alpha_i))$ onto $\gamma^{-1}(Range(\alpha_i))$ belongs to $\mathbb{P}'_n$. The set $\mathbb{P}_n$ has the natural ordering (by extension); *cofinality* is defined with respect to that order. We say that $\mathbb{P}'_n$ has *the joint embedding property* (*the amalgamation property*) if for any $\overline{\alpha}, \overline{\beta} \in \mathbb{P}'_n$ (extending $\overline{\gamma} \in \mathbb{P}'{}_n$) there exists $\delta \in Aut(M)$ (fixing $\cup(Dom(\gamma_i) \cup Range(\gamma_i))$ pointwise) such that $\overline{\alpha} \cup \overline{\beta}^\delta \in \mathbb{P}'_n$. The argument of Theorem 2.1 in [18] proves the following proposition.

Proposition 3.1. *If $\mathbb{P}_n$ has a cofinal subset $\mathbb{P}'_n$ closed under conjugacy and having JEP and AP, then $(Aut(M))^n$ has a generic tuple $\overline{\gamma}$ (the set of all conjugates of $\overline{\gamma}$ is comeagre in the space $(Aut(M))^n$).*

An $Aut(M)$-invariant set Γ of finite subsets of M is an *amalgamation base* [9] if:

-any tuple $\overline{\alpha}$ of finite maps extendible to a tuple of automorphisms has an extension to a tuple of automorphisms inducing a tuple of permutations on some $A \in \Gamma$ with $Dom(\overline{\alpha}) \subseteq A$;

- for any $A, B, C \in \Gamma$ with $A \subseteq B, A \subseteq C$ there exists $\gamma \in Aut(M/A)$ such that $C \cup \gamma(B) \in \Gamma$ and whenever $\alpha, \beta \in Aut(M)$ induce permutations of $\gamma(B)$ and C which agree on A then there exists $\delta \in Aut(M)$ extending both $\alpha|\gamma(B)$ and $\beta|C$.

Theorems 2.9 and 5.3 of [9] prove that if M has an amalgamation base as above (in [9] that definition is slightly wider), then M has the *small index property*: any subgroup of $Aut(M)$ of index less than continuum is open. Moreover, it is easy to see that the family of all tuples of partial automorphisms with the same domain (from Γ) which induce permutations on the domain, satisfies the assumptions of Proposition 3.1. So, $(Aut(M))^n$ has generic tuples.

Proof of Proposition 2.2. To prove the first statement of the theorem consider the set Prm_n of all pairs $(D, \overline{\alpha})$ where D is a finite subset of M and $\overline{\alpha}$ is an n-tuple of permutations of D preserving the predicates X_i. The proof that Prm_n works as $\mathbb{P}'_n$ in Proposition 3.1 is straightforward. Note that the set of all finite subsets form an amalgamation base.

To prove the second statement we use the argument of Theorem 1.3 of [3]. Let $G := Aut(M) = \bigcup_{i\in\omega} G_i$, where $G_1 < G_2 < ...$ is a chain of proper subgroups. Notice that no G_m is open. Suppose, to the contrary, that some G_m is open. Then there is a finite X such that the pointwise stabilizer G_X is a subgroup of G_m. Take X' of the same type in M such that $X \cap X' = \emptyset$. By increasing m if necessary we may assume that G_m contains all automorphisms mapping X onto X'. Then for any X'' of the same G_X-orbit with X' we have $G_{X''} \leq G_m$.

Let $\alpha \in G$. Let β be a finitary automorphism of M such that $\beta|_X = \alpha|_X$. Then $\beta^{-1}\alpha \in G_X$ and there exists X'' as above such that $\beta \in G_{X''}$. Then $\alpha \in G_m$. We see $G = G_m$.

The proof can now be completed as in the proof of Theorem 6.1 of [9]. □

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