PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 3, Pages 643–651
S 0002-9939(01)06159-7
Article electronically published on September 28, 2001

A CHARACTERIZATION OF THE HEREDITARY CATEGORIES DERIVED EQUIVALENT TO SOME CATEGORY OF COHERENT SHEAVES ON A WEIGHTED PROJECTIVE LINE

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(Communicated by Ken Goodearl)

ABSTRACT. Let $\mathcal{H}$ be a connected hereditary abelian category over an algebraically closed field k, with finite dimensional homomorphism and extension spaces. There are two main known types of such categories: those derived equivalent to $\operatorname{mod}\lambda$ for some finite dimensional hereditary k-algebra λ and those derived equivalent to some category $\operatorname{coh}\mathbb{X}$ of coherent sheaves on a weighted projective line $\mathbb{X}$ in the sense of Geigle and Lenzing (1987). The aim of this paper is to give a characterization of the second class in terms of some properties known to hold for these hereditary categories.

INTRODUCTION

Let k be an algebraically closed field and $\mathcal{H}$ a connected abelian k-category. We assume that $\mathcal{H}$ is hereditary, that is the Yoneda $\operatorname{Ext}^2(,)$ vanishes, and we assume that $\mathcal{H}$ has finite dimensional homomorphism and extension spaces.

Such categories are of interest in noncommutative algebraic geometry and have been completely classified under the additional assumptions that $\mathcal{H}$ is noetherian and satisfies Serre duality [RV]. They are also of interest in the representation theory of finite dimensional algebras, now under the additional assumption that $\mathcal{H}$ has a tilting object. Recall that an object T in $\mathcal{H}$ is a tilting object if $\operatorname{Ext}^1(T,T)=0$ and if $\operatorname{Hom}(T,X)=0=\operatorname{Ext}^1(T,X)$ for some object X in $\mathcal{H}$, then X must be 0. The quasitilted algebras introduced in [HRS] are by definition the endomorphism algebras $\operatorname{End}(T)$, and this class of algebras contains the important classes of tilted and canonical algebras.

Note that a noetherian $\mathcal{H}$ with Serre duality does not usually have a tilting object. Examples are given by categories $\operatorname{coh}\mathbb{X}$ of coherent sheaves on a nonsingular projective curve $\mathbb{X}$ different from the projective line $\mathbb{P}^1(k)$. On the other hand $\mathcal{H}$ can have a tilting object without being noetherian. An easy example of this is the opposite category of $\operatorname{coh}\mathbb{P}^1(k)$.

In this paper we assume that $\mathcal{H}$ has a tilting object. It is then a consequence that $\mathcal{H}$ has almost split sequences, and also Serre duality as studied in [RV] (see [HRS]). When $\mathcal{H}$ is not equivalent to $\operatorname{mod}\Lambda$ for a finite dimensional hereditary k-algebra Λ, is is known that there is an equivalence $\tau:\mathcal{H}\to\mathcal{H}$ with the property that if $O\to A\to B\to C\to O$ is an almost split sequence, then $\tau C\simeq A$ [HRS]. Since

Received by the editors January 13, 2000 and, in revised form, September 12, 2000.
1991 *Mathematics Subject Classification.* Primary 16B50, 16E10, 16G70, 18E10, 18E30.

$\mathcal{H}$ has almost split sequences, the familiar concepts of the AR-quiver, and further (standard) stable tubes (which are components of the AR-quiver) from the module theory of finite dimensional algebras carry over to this setting (see [R], [ARS]). For details we refer to section 1.

There are two main known types of such categories $\mathcal{H}$ with tilting objects; those derived equivalent to $\operatorname{mod}\Lambda$ for some finite dimensional hereditary k-algebra Λ and those derived equivalent to some category $\operatorname{coh}\mathbb{X}$ of coherent sheaves on a weighted projective line $\mathbb{X}$, in the sense of [GL]. The first class is characterized by the existence of some indecomposable directing object C [HRe1]. Recall that C is called directing if it does not lie on a cycle of nonzero nonisomorphisms.

The aim of this paper is to give a characterization of the second class in terms of some properties known to hold for these hereditary categories [LP]. The properties are (i) existence of a standard stable tube and (ii) for indecomposable exceptional nondirecting objects E and F in $\mathcal{H}$ we have that $\operatorname{Hom}(E,\tau^{-i}F)=0$ for i large enough. Recall that E is called exceptional if $\operatorname{Ext}^1(E,E)=0$. Note that any summand of a tilting object is exceptional.

Assume that $\mathcal{H}$ is not equivalent to some $\operatorname{mod}\Lambda$ where Λ is a finite dimensional hereditary k-algebra. When $\mathcal{H}$ has some simple object, then we know that $\mathcal{H}$ is derived equivalent to some category $\operatorname{coh}\mathbb{X}$ [HRe2], but the converse does not hold.

We would like to thank the referee for constructive comments.

1. Standard tubes

In this section we give some results related to standardness of tubes. In addition to being useful in proving our main theorem, these results may also be of independent interest.

Let Λ be a finite dimensional algebra over an algebraically closed field k, and let $\underline{r}$ denote the radical of the category $\operatorname{mod}\Lambda$ of finitely generated Λ-modules (see [R], [ARS]). Recall that a component $\mathcal{C}$ of the AR-quiver is standard if the full subcategory $\operatorname{ind}\mathcal{C}$ of $\operatorname{mod}\Lambda$ whose objects correspond to the vertices of $\mathcal{C}$ is the mesh category of $\mathcal{C}$. The component $\mathcal{C}$ is generalized standard if the restriction of $\underline{r}^{\infty}(\,,\,)$ to $\mathcal{C}$ is zero (see [R], [ARS]). The two concepts are known to be equivalent for stable tubes [L1], [S], but there are examples showing that they do not coincide in general [L2].

Now let $\mathcal{C}$ be a component of the AR-quiver of a hereditary abelian k-category with tilting object. When $\mathcal{C}$ is a stable tube of rank greater than 1, which is standard (or generalized standard), then each object E on the border is clearly exceptional. The notions of standard and generalized standard are defined similarly as in the case of finite dimensional algebras.

Assuming only that a stable tube has some exceptional object at the border, rather than assuming that the tube is standard, we get the following useful information on the perpendicular category $E^{\perp}$. The objects of $E^{\perp}$ are by definition the objects C in $\mathcal{H}$ such that $(E,C)=0=\operatorname{Ext}^1(E,C)$, where (E,C) denotes $\operatorname{Hom}(E,C)$. We say that E is torsionable if E is the factor of a finite direct sum of copies of some tilting object. Note that this is a condition on the objects in $\mathcal{H}$ without reference to a chosen tilting object T. For example all objects in $\operatorname{mod}\Lambda$ for a hereditary finite dimensional algebra Λ are torsionable.

Proposition 1.1. *Let $\mathcal{H}$ be a hereditary abelian k-category with finite dimensional homomorphism and extension spaces, and having a tilting object.*

Assume that E is an indecomposable torsionable exceptional τ-periodic object of infinite length in $\mathcal{H}$, such that if $O \to \tau E \to M \to E \to O$ is an almost split sequence, then M is indecomposable. Then we have the following:

(a) $\operatorname{Hom}(\tau^j E, E) = 0$ *for* $0 < j < t$, *where t is the τ-period of E.*
(b) *M is $\tau_{E^\perp}$-periodic in $E^\perp$.*
(c) *If in addition E is at the border of a stable tube, then the objects on the border are pairwise orthogonal.*

Proof. (a) Let t be the τ-period of E. Then we have $\tau^t E \simeq E$. Consider the almost split sequence $0 \to \tau E \to M \to E \to 0$. We know from [HRe2] that the perpendicular category $E^\perp$ is equivalent to $\operatorname{mod} H$ for some (basic) finite dimensional hereditary k-algebra H. Further M is in $E^\perp$ and $T = H \oplus E$ is a tilting object. The one-point extension algebra $H[M]$ is quasitilted. Since M is indecomposable, it follows that $\operatorname{End}(M) \simeq k$ [HRS].

Let $j = 1$, and apply the functor $(\,, M)$ to the above exact sequence to get the exact sequence $0 \to (E, M) \to (M, M) \to (\tau E, M) \to \operatorname{Ext}^1(E, M)$. Since $M \in E^\perp$ we have $k \simeq (M, M) \simeq (\tau E, M)$. Now apply $(\tau E, \,)$ to the almost split sequence, to get the exact sequence $0 \to (\tau E, \tau E) \to (\tau E, M) \to (\tau E, E) \to \operatorname{Ext}^1(\tau E, \tau E) = 0$. Since $k \simeq (\tau E, \tau E) \simeq (\tau E, M)$, we conclude $(\tau E, E) = 0$.

Assume we have proved $\operatorname{Hom}(\tau^{j-1} E, E) = 0$ for some $j > 1$. Then since $\operatorname{Ext}^1(E, \tau^j E) = 0$ we know that any nonzero map $g \colon \tau^j E \to E$ is a monomorphism or an epimorphism [HRi]. If g is mono, then since E is periodic we get an infinite descending sequence $\cdots \subsetneq E \subsetneq E \subsetneq E$, contradicting the fact that $\operatorname{End}(E)$ is finite dimensional over k. If g is an epimorphism, there is an infinite chain of proper epimorphisms $E \overset{h}{\twoheadrightarrow} E \overset{h}{\twoheadrightarrow} E \to \cdots \to E \to \cdots$. Then h is not nilpotent, so we have a contradiction. Then we get that $(\tau^j E, E) = 0$. This finishes the proof of (a).

(b) Consider the following part of the AR-quiver of $\mathcal{H}$:

$$\begin{tikzcd}
 & & & & & A \arrow[dr] & & B \arrow[dr] & & & & \\
\tau^3 M \arrow[dr] & & \tau^2 M \arrow[dr] & & \tau M \arrow[ur] \arrow[dr] & & M \arrow[ur] \arrow[dr] & & \tau^{t-1} M \arrow[dr] & & \tau^{t-2} M \arrow[dr] & \\
 & \tau^3 E \arrow[ur] & & \tau^2 E \arrow[ur] & & \tau E \arrow[ur] & & E \arrow[ur] & & \tau^{t-1} E \arrow[ur] & & \tau^{t-2} E
\end{tikzcd}$$

It follows from the above that $\tau^j E$ is in $E^\perp$ for $j \neq 0, 1$. Then $0 \to \tau^{j+1} E \to \tau^j M \to \tau^j E \to 0$ is in $E^\perp$, and is hence almost split in $E^\perp$, for $2 \leq j \leq t-2$. We have the exact sequence $0 \to \tau^2 E \to A \overset{v_1}{\to} M \to 0$ which is in $E^\perp$. Consider $f \colon X \to M$, where X is indecomposable in $E^\perp$, and f is not an isomorphism. We have the almost split sequence $0 \to \tau M \to A \oplus \tau E \overset{(v_1, v_2)}{\longrightarrow} M \to 0$. Hence there is a map $(w_1, w_2) \colon X \to A \oplus \tau E$ such that $(v_1, v_2)(w_1, w_2) = f$. Since $(X, \tau E) \simeq D \operatorname{Ext}^1(E, X) = 0$, where D denotes the duality with respect to the field k, we have that f factors through $v_1 \colon A \to M$. Since $\tau^2 E$ is indecomposable, it follows that the sequence $0 \to \tau^2 E \to A \to M \to 0$ is almost split. Consider the exact sequence $0 \to M \to X \to \tau^{t-1} E \to 0$, which is in $E^\perp$, and the almost split sequence $0 \to M \to X \oplus E \to \tau^{t-1} M \to 0$. Let $f \colon M \to C$, with C indecomposable in $E^\perp$, be a nonisomorphism. Since $(E, C) = 0$ it follows that $g \colon M \to B$ is left almost split in $E^\perp$, and hence $0 \to M \to X \to \tau^{t-1} E \to 0$ is almost split in $E^\perp$. Combining, we see that $\tau_{E^\perp}^{t-1} M \simeq M$, so that M is periodic in $E^\perp$.

(c) This follows directly from (a). $\square$

2. Existence of a standard torsionable tube of rank 1

Assume as before that $\mathcal{H}$ is a connected hereditary abelian k-category with tilting object, and from now on we write "tube" when we mean "stable tube". In this section we assume that $\mathcal{H}$ satisfies the following conditions: (i) There is no simple object in $\mathcal{H}$. (ii) There is some standard tube. (iii) For each indecomposable exceptional torsionable object E the perpendicular category $E^{\perp}$ is equivalent to the module category $\operatorname{mod} H$ for a (basic) wild finite dimensional hereditary k-algebra.

In the derived equivalence class of $\operatorname{coh}\mathbb{X}$, where $\mathbb{X}$ is of tubular type, there are categories $\mathcal{H}$ satisfying (i) and (ii), but not (iii). This is seen by using detailed descriptions in [H] or [LS]. If Λ is the path algebra over k of the quiver

$$\begin{tikzcd} & \bullet \arrow[dl, shift left=2] \arrow[dl] \arrow[dl, shift right=2] & \\ \bullet \arrow[rr, shift left=2] \arrow[rr] \arrow[rr, shift right=2] & & \bullet \arrow[ul, shift left=2] \arrow[ul] \arrow[ul, shift right=2] \end{tikzcd},$$

then one can show that there is some $\mathcal{H}$ derived equivalent to $\operatorname{mod}\Lambda$ which satisfies (i) and (iii), but not (ii).

The aim of this section is to show that under the assumptions (i), (ii) and (iii) any standard tube has rank 1 and is torsionable. We accomplish this through a series of lemmas.

Lemma 2.1. *$\mathcal{H}$ is not derived equivalent to any $\operatorname{mod}\Lambda$ for a finite dimensional hereditary k-algebra Λ, or to any category $\operatorname{coh}\mathbb{X}$ of coherent sheaves on a weighted projective line $\mathbb{X}$.*

Proof. When $\mathcal{H}$ is derived equivalent to $\operatorname{mod}\Lambda$ for a finite dimensional hereditary k-algebra Λ, then Λ cannot be of wild or of finite type, since in these cases there is no tube. When $\mathcal{H}$ is derived equivalent to $\operatorname{mod}\Lambda$ for a tame hereditary algebra Λ, then $\mathcal{H}$ is also derived equivalent to $\operatorname{coh}\mathbb{X}$ for some domestic curve $\mathbb{X}$ [GL]. If $\mathcal{H}$ is derived equivalent to some $\operatorname{coh}\mathbb{X}$, then $\operatorname{coh}\mathbb{X}$ is tubular, since otherwise $\mathcal{H}$ has some simple object (see [H]). But in this case the module categories $E^{\perp}$ are not wild since they contain tubes. $\square$

The following will also be useful.

Lemma 2.2. *Let Y be the direct sum of the indecomposable objects at the border of a tube $\mathcal{C}$ and let E be an exceptional indecomposable torsionable object. Then $(E,Y) \neq 0$ or $(Y,E) \neq 0$.*

Proof. Assume to the contrary that $(E,Y) = 0 = (Y,E) \simeq D\operatorname{Ext}^1(E,Y)$, where D denotes the duality with respect to the field k. Then Y is in $E^{\perp}$, which is equivalent to $\operatorname{mod} H$ for some wild hereditary k-algebra H. Since the whole tube $\mathcal{C}$ lies in $E^{\perp}$, and H is wild by assumption (iii), we have a contradiction. $\square$

Note that $\mathcal{H}$ is a hereditary abelian k-category with tilting object if and only if the opposite category $\mathcal{H}^{op}$ has the same property.

Lemma 2.3. *Let X be at the border of a standard tube $\mathcal{C}$ in $\mathcal{H}$. Then either X is torsionable in $\mathcal{H}$ or X^{op} is torsionable in $\mathcal{H}^{op}$.*

Proof. Assume that X is not torsionable and let E be an indecomposable exceptional torsionable object. Let Y be the direct sum of the nonisomorphic objects in the τ-orbit of X. By possibly replacing $\mathcal{H}$ by $\mathcal{H}^{op}$ we can assume that $\operatorname{Hom}(E,Y) \neq 0$. By possibly replacing E by an object in the same τ-orbit, we can assume that $\mathrm{Hom}(E,X) \neq 0$. We have $E^{\perp} = \mathrm{mod}\, H$ for a finite dimensional (basic) hereditary k-algebra H, and $T = E \oplus H$ is a tilting object [HRe2]. Consider the corresponding torsion exact sequence $0 \to tX \to X \to X/tX \to 0$. Then tX is a nonzero proper subobject of X.

We claim that tX is exceptional. To see this, note first that $\mathrm{Ext}^1(tX,X) \simeq D(\tau^{-1}X, tX) = 0$. For if $\tau^{-1}X \not\simeq X$, then $(\tau^{-1}X, X) = 0$, since X lies in a standard tube, and consequently $(\tau^{-1}X, tX) = 0$. Since the tube $\mathcal{C}$ is standard, we have $\mathrm{End}(X) \simeq k$. If $\tau^{-1}X \simeq X$, then $(X, tX) = 0$ since tX is properly contained in X. We have the exact sequence

$$\cdots \to (tX, X/tX) \to \mathrm{Ext}^1(tX, tX) \to \mathrm{Ext}^1(tX, X).$$

Since $(tX, X/tX) = 0$ and $\mathrm{Ext}^1(tX,X) = 0$, it follows that $\mathrm{Ext}^1(tX,tX) = 0$, so that tX is exceptional.

Since we now know that X has some nonzero torsionable exceptional subobject, we can choose some indecomposable exceptional torsionable subobject F of X. Then $F^{\perp} = \mathrm{mod}\, H'$ for some finite dimensional hereditary k-algebra H', and $T' = F \oplus H'$ is a tilting object in $\mathcal{H}$ [HRe2].

Let P be indecomposable projective in $F^{\perp}$. Since it follows from Lemma 2.1 that $\mathcal{H}$ is not derived equivalent to $\mathrm{mod}\, H_1$ for any finite dimensional hereditary k-algebra H_1, we know that the one-point extension $H'[\mathrm{M}]$ is quasitilted nontilted, where M is given by the almost split sequence $0 \to \tau F \to M \to F \to 0$. Hence M is a sincere H'-module (see [H]), so that $\mathrm{Hom}(P, M) \neq 0$.

Now let $f\colon F' \to X$ be a minimal right $\mathrm{add}F$-approximation. Since X is assumed not to be torsionable, $f\colon F' \to X$ is not surjective. Consider the exact sequences $0 \to \mathrm{Ker}\, f \to F' \to \mathrm{Imf} \to 0$ and $0 \to \mathrm{Imf} \to X \to \mathrm{coker}\, f \to 0$. Since $\mathrm{Ext}^1(F,F') = 0$ and $\mathrm{Ext}^1(F,F') \to \mathrm{Ext}^1(F, \mathrm{Imf})$ is an epimorphism, it follows that $\mathrm{Ext}^1(F, \mathrm{Imf})$ is 0. Applying the functor $(F, \,)$ to the second exact sequence gives the exact sequence

$$0 \to (F, \mathrm{Imf}) \to (F, X) \to (F, \mathrm{coker}\, f) \to \mathrm{Ext}^1(F, X) \to \mathrm{Ext}^1(F, \mathrm{coker}\, f) \to 0.$$

Since $f'\colon F' \to \mathrm{Imf}$ is a minimal right $\mathrm{add}F$-approximation, and $\mathrm{End}(F) \simeq k$, because F is exceptional [HRe2], it follows that $(F, \mathrm{Imf}) \to (F, X)$ is an isomorphism. Since $\mathrm{Ext}^1(F, \mathrm{Imf}) = 0$, we then have $(F, \mathrm{coker}\, f) = 0$. It follows that $\mathrm{Ext}^1(F,X) \simeq D\,\mathrm{Hom}(\tau^{-1}X, F) = 0$ since $\mathrm{Hom}(\tau^{-1}X, X) = 0$. Hence we get $\mathrm{Ext}^1(F,X) = 0$, so that $\mathrm{Ext}^1(F, \mathrm{coker}\, f) = 0$. This shows that $\mathrm{coker}\, f$ is in $F^{\perp}$.

Since X is not torsionable, τX is also not torsionable, so that $\mathrm{Ext}^1(F \oplus H', \tau X) \neq 0$. We have $\mathrm{Ext}^1(F, \tau X) \simeq D(X, F) = 0$ since $F \subset X$ is a proper inclusion and $\mathrm{End}(F) \cong k$. Hence we have $\mathrm{Ext}^1(H', \tau X) \simeq D(X, H') \neq 0$, so that there is some indecomposable projective H'-module Q such that $(X, Q) \neq 0$. Since $(F', Q) = 0$, the exact sequence $F' \to X \to \mathrm{coker}\, f \to 0$ gives that $(\mathrm{coker}\, f, Q) \neq 0$. Since $\mathrm{coker}\, f$ and Q are in $\mathrm{mod}\, H'$, there is some indecomposable projective H'-module P' which is a summand of $\mathrm{coker}\, f$. Since $\mathrm{Ext}^1(P', F') = 0$, we have $\mathrm{Ext}^1(P', \mathrm{Imf}) = 0$, so that the split inclusion $P' \to \mathrm{coker}\, f$ factors through $X \to \mathrm{coker}\, f$. Then P' would be a summand of X, which is a contradiction.

This shows that X is torsionable. $\square$

We can now see that our tube must have rank 1.

Lemma 2.4. *Any standard tube has rank* 1.

Proof. Assume that there is some standard tube of rank greater than 1, and let F be at the border of this tube. Since the tube is standard, F is exceptional, and it follows from Lemma 2.3 that we can assume that F is torsionable. Since we have assumed that $\mathcal{H}$ has no simple object, $F^{\perp} = \operatorname{mod} H$ would contain a periodic H-module by Proposition 1.1, which is a contradiction to H being wild. This finishes the proof. □

3. The main result

The aim of this section is to prove our desired characterization of the hereditary k-categories derived equivalent to some category $\operatorname{coh}\mathbb{X}$.

We first give two preliminary results. For these we assume in addition to our assumptions on hereditary k-categories in section 2 that we have some torsionable standard tube of rank one, not just some standard tube.

Lemma 3.1. *Under the above assumptions, let X be an indecomposable object at the border of some torsionable standard tube of rank* 1*.*

Then there is some indecomposable exceptional object E with $\operatorname{Hom}(E, X) \neq 0$ *and X not in* $\operatorname{Fac} E$*, where the objects of* $\operatorname{Fac} E$ *are the factors of finite direct sums of copies of E.*

Proof. Assume to the contrary that whenever $\operatorname{Hom}(E, X) \neq 0$ for some indecomposable exceptional object E, then X is in $\operatorname{Fac} E$. Since $\operatorname{End}(E) \simeq k$ [HRe2], this implies $\operatorname{Hom}(X, E) = 0$, and hence $l(E) = \dim_k \operatorname{Hom}(E, X) - \dim_k \operatorname{Hom}(X, E) > 0$. We choose E such that $l(E) = a > 0$ is smallest possible.

Let $f \colon E^t \to X$ be a minimal right add$E$-approximation, which is necessarily surjective. The perpendicular category $E^{\perp}$ is equivalent to $\operatorname{mod} H$ for some finite dimensional (basic) hereditary k-algebra H, and $T = E \oplus H$ is a tilting object [HRe2]. Consider the exact sequence $0 \to K \to E^t \xrightarrow{f} X \to 0$. Applying the functor $(E, \ )$ to this exact sequence gives rise to the exact sequence $0 \to (E, K) \to (E, E^t) \to (E, X) \to \operatorname{Ext}^1(E, K) \to \operatorname{Ext}^1(E, E^t) \to \operatorname{Ext}^1(E, X) \to 0$. Since $\operatorname{End}(E) \simeq k$, it follows that K is in $E^{\perp}$. Hence the exact sequence $0 \to K \to E^t \xrightarrow{f} X \to 0$ is in $\operatorname{Fac} T$, so that $0 \to (T, K) \to (T, E^t) \to (T, X) \to 0$ is exact. Now $(T, E) = (k, M, \mathrm{id})$ as a $\left(\begin{smallmatrix} k & 0 \\ M & H \end{smallmatrix}\right)$-module, and $(T, X) = (\operatorname{Hom}(E, X), \operatorname{Hom}(H, X), f)$, where $\operatorname{Hom}(E, X) = k^t$ for some t, $\operatorname{Hom}(H, X) = I$ is in $\operatorname{mod} H$ and $f \colon M^t \to I$ is an H-homomorphism. The map $f \colon M^t \to I$ is surjective since $\mathrm{id} \colon M \to M$ is surjective. The tilting object T gives rise to a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{H}$, where $\mathcal{T} = \operatorname{Fac} T$. We have a split torsion pair $(\mathcal{R}', \mathcal{L}')$ in $\operatorname{mod} H[M]$, and $(T, \ ) \colon \mathcal{T} \to \mathcal{L}'$ is an equivalence of categories (see [HRS]). Then the almost split sequence $0 \to X \to Y \to X \to 0$ in $\mathcal{H}$ lies in $\mathcal{T}$, and hence $0 \to (T, X) \to (T, Y) \to (T, X) \to 0$ is an exact sequence in $\operatorname{mod} H[M]$, which lies in $\mathcal{L}'$. It is then almost split, and hence $D\operatorname{Tr}(T, X) \simeq (T, X)$, where Tr denotes the transpose.

We now show that the H-module I is injective. Consider the minimal projective $H[M]$-resolution $0 \to (0, K, 0) \to (k^t, M^t, \mathrm{id}) \to (k^t, I, f) \to 0$. Using that $D\operatorname{Tr}(T, X) \simeq (T, X)$, we get a minimal injective resolution $0 \to (k^t, I, f) \to (k^r, I(K/\underline{r}K), g) \to (k^t, 0, 0) \to 0$, where $I(K/\underline{r}K)$ denotes the injective envelope of $K/\underline{r}K$. Hence I is isomorphic to $I(K/\underline{r}K)$, and is hence an injective H-module.

Since $f \colon M^t \to I$ is surjective, we see that $\operatorname{Hom}_{H[M]}((T, X), Z) = 0$ for all Z in $\operatorname{mod} H$. Note that $\operatorname{mod} H$ is a subcategory of both $\mathcal{H}$ and $\operatorname{mod} H[M]$, and with such an identification $(T, \ )$ acts as the identity on $\operatorname{mod} H \subset \mathcal{H}$. Hence we also have

$\mathrm{Hom}_{\mathcal{H}}(X,Z) = 0$ for Z in $\mathrm{mod}\, H$. This shows that $l(Z) = \dim_k \mathrm{Hom}(Z,X) - \dim_k \mathrm{Hom}(X,Z) \geq 0$. Let Z be in $\mathrm{mod}\, H$, and let S be a simple submodule of Z. Then S is exceptional and also torsionable since it is in $\mathrm{Fac}\, H \subset \mathrm{Fac}\, T$. Hence we have $l(S) > 0$ by Lemma 2.2, and consequently $l(Z) > 0$. In particular $l(S) \geq a$ for each simple submodule S of M. Since $l(E) = a = l(\tau E)$, we have $l(M) = 2a$, and hence $\dim_k \mathrm{soc}\, M \leq 2$, where $\mathrm{soc}\, M$ denotes the socle of M. Since by assumption $\mathcal{H}$ is not derived equivalent to any $\mathrm{mod}\, \Lambda$ for a finite dimensional hereditary k-algebra Λ, then $H[M]$ is not tilted. Hence we have $\dim_k(\mathrm{soc}\, M) = 2$, since M is indecomposable [H]. But then we have $l(M/\mathrm{soc}\, M) = 0$, which is a contradiction. This finishes the proof of the lemma. □

Lemma 3.2. *Under the above assumptions on the hereditary k-category $\mathcal{H}$, it follows that $\mathcal{H}$ does not satisfy condition $(*)$: If E and F are indecomposable exceptional nondirecting objects, there is some integer n such that* $\mathrm{Hom}(E, \tau^{-i}F) = 0$ *for $i > n$.*

Proof. Let X be at the border of a torsionable standard tube of rank 1. By Lemma 3.1 we can choose some indecomposable exceptional object E such that $\mathrm{Hom}(E,X) \neq 0$ and X is not in $\mathrm{Fac}\, E$.

Consider the exact sequence $0 \to \mathrm{Ker}\, f \to E^t \xrightarrow{f} X \to \mathrm{coker}\, f \to 0$, where $f\colon E^t \to X$ is a minimal right add E-approximation. We know that $f \neq 0$ and $\mathrm{coker}\, f \neq 0$. We have the associated exact sequences $0 \to \mathrm{Imf} \to X \to \mathrm{coker}\, f \to 0$ and $0 \to \mathrm{Ker}\, f \to E^t \to \mathrm{Imf} \to 0$.

We first want to show that $\mathrm{coker}\, f$ is an exceptional object in $E^\perp$. As before we have $(E, \mathrm{coker}\, f) = 0 = \mathrm{Ext}^1(E, \mathrm{coker}\, f)$, so that $\mathrm{coker}\, f$ is in $E^\perp$. We have the exact sequence $(X, \mathrm{coker}\, f) \to (\mathrm{Imf}, \mathrm{coker}\, f) \to \mathrm{Ext}^1(\mathrm{coker}\, f, \mathrm{coker}\, f) \to \mathrm{Ext}^1(X, \mathrm{coker}\, f) \to \mathrm{Ext}^1(\mathrm{Imf}, \mathrm{coker}\, f) \to 0$. Since $(E^t, \mathrm{coker}\, f) = 0$, we have $(\mathrm{Imf}, \mathrm{coker}\, f) = 0$. Since $\tau X \simeq X$, we have $\mathrm{Ext}^1(X, \mathrm{coker}\, f) \simeq D(\mathrm{coker}\, f, X)$. If $(\mathrm{coker}\, f, X) \neq 0$, then the composition $X \to \mathrm{coker}\, f \to X$ is a nonzero map which is not an isomorphism. But $\mathrm{End}(X) \simeq k$ since X lies at the border of a standard tube, and hence we have a contradiction. This shows that $\mathrm{Ext}^1(X, \mathrm{coker}\, f) = 0$, and hence $\mathrm{Ext}^1(\mathrm{coker}\, f, \mathrm{coker}\, f) = 0$.

Let C be an indecomposable summand of $\mathrm{coker}\, f$. Then C is exceptional by what we just proved, and C is torsionable since X is torsionable. Assume first that C lies in a tube. We then claim that there is an indecomposable exceptional torsionable object at the border of this tube: Clearly all $\tau^i C$ are also exceptional and torsionable. If C is not at the border, we have an almost split sequence $0 \to \tau C \to A \oplus B \to C \to 0$, where A and B are indecomposable and the induced map $A \to C$ is a monomorphism. If $\mathrm{Ext}^1(A,A) \neq 0$, then $\mathrm{Hom}(\tau^{-1}A, A) \neq 0$. Hence the composition $C \to \tau^{-1}A \to A \to C$ gives a nonzero map which is not an isomorphism, since $C \to \tau^{-1}A$ is an epimorphism. Continuing this way we obtain that all objects at the border are exceptional. We replace C by an object at the border which is a factor of C. Hence we still have that the new object is torsionable. Since C automatically has infinite length, $C^\perp$ is equivalent to $\mathrm{mod}\, H$ for some hereditary finite dimensional wild k-algebra H. If $0 \to \tau C \to F \to C \to 0$ is almost split, it follows from Proposition 1.1 that F is a periodic H-module, which gives a contradiction.

We can now assume that C has infinite τ-period. Since $\tau X \simeq X$, we have an epimorphism $X \to \tau^i C$ for each i, and hence a nonzero map $E \to \tau^i C$ for each

i. The modules E and C are both indecomposable exceptional. Since by Lemma 2.1 $\mathcal{H}$ is not derived equivalent to $\operatorname{mod} \Lambda$ for some finite dimensional hereditary k-algebra Λ, we know that E and C are not directing [HRe1]. This shows that condition $(*)$ does not hold, and the proof is finished. $\square$

We are now in the position to prove our main result.

Theorem 3.3. *The following are equivalent for a hereditary abelian k-category $\mathcal{H}$ with finite dimensional homomorphism and extension spaces, and having a tilting object*:

(a) *$\mathcal{H}$ is derived equivalent to some* $\operatorname{coh} \mathbb{X}$.
(b) (i) *There exists a standard stable tube.*
(ii) *For indecomposable exceptional nondirecting objects E and F with infinite τ-orbit there is some integer n such that* $\operatorname{Hom}(E, \tau^{-i}F) = 0$ *for $i > n$.*

Proof. (a) $\Rightarrow$ (b). Properties (i) and (ii) are known to hold for the hereditary categories derived equivalent to some $\operatorname{coh} \mathbb{X}$ [LP].

(b) $\Rightarrow$ (a). Assume that (i) and (ii) are satisfied. If $\mathcal{H}$ is derived equivalent to some $\operatorname{mod} \Lambda$ for Λ a finite dimensional hereditary k-algebra Λ, then Λ must be tame since $\mathcal{H}$ has a tube (see [R]). Hence $\mathcal{H}$ is derived equivalent to some $\operatorname{coh} \mathbb{X}$. If $\mathcal{H}$ has a simple object, we are then done by [HRe2].

Assume now that $\mathcal{H}$ has no simple object. Let E be an indecomposable exceptional torsionable object. Then $E^{\perp} = \operatorname{mod} H$ for a finite dimensional hereditary k-algebra H. If H is of finite type, we know that the quasitilted algebra $H[M]$ is tilted, where M is the middle term of the almost split sequence with right hand term E [HRS]. Then $\mathcal{H}$ is equivalent to $\operatorname{mod} \Lambda$ for a finite dimensional hereditary k-algebra Λ. Since $\mathcal{H}$ has a tube by assumption, Λ must be tame, and hence $\mathcal{H}$ is derived equivalent to $\operatorname{coh} \mathbb{X}$ for some domestic curve $\mathbb{X}$ [GL]. If H is tame, then since $H[M]$ is quasitilted, M is either preprojective, preinjective or simple regular. In the first two cases, $H[M]$ is tilted [HRS]. Using the same argument as above, it follows that $\mathcal{H}$ is derived equivalent to $\operatorname{coh} \mathbb{X}$ for some domestic curve $\mathbb{X}$. If M is simple regular, then $H[M]$ is either tilted from a tame hereditary algebra [R] or $H[M]$ can be obtained as the endomorphism ring of a tilting object for some $\operatorname{coh} \mathbb{X}$ [LM]. In both cases we have that $\mathcal{H}$ is derived equivalent to some $\operatorname{coh} \mathbb{X}$. If H is wild, the standard assumptions from section 2 are satisfied. By the results of section 2 also the standard assumptions from section 3 are valid. Then we get a contradiction to (ii), and we are done. $\square$

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