ON THE NATANZON-TURAEV COMPACTIFICATION
OF THE HURWITZ SPACE

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ABSTRACT. Natanzon and Turaev have constructed by topological methods a compactification of the Hurwitz space, that is, the space of simple branched covers of the two-sphere. Here we show that this compactification is homeomorphic to a compactification mentioned by Diaz and Edidin (in 1996) that was constructed by algebraic methods. Using this we are able to show by example that the Natanzon-Turaev compactification can be singular, that is, not a manifold.

1. INTRODUCTION

The Hurwitz space $H_{n,w}$ is the set of all $n$ sheeted connected coverings of the sphere $S^2$ simply branched over exactly $w$ distinct points and otherwise unbranched. Hurwitz [Hu] showed a natural way to make $H_{n,w}$ into a complex manifold of complex dimension $w$. The space $H_{n,w}$ is not compact. When two or more distinct branch points approach each other the limit of the corresponding covers will not be a cover of the same type. Various compactifications of $H_{n,w}$ and closely related spaces have been constructed and studied by methods of algebraic geometry; see for instance [HM], [DE], and [M]. In [NT] Natanzon and Turaev construct a compactification of $H_{n,w}$ using topological methods. In this paper we show that the compactification constructed in [NT] is homeomorphic to a compactification in [DE].

This allows us to answer some questions brought up in [NT]. Natanzon and Turaev point out that there are no known topological descriptions of the compactifications in algebraic geometry. In view of the homeomorphism we construct, the Natanzon-Turaev compactification is a topological description of one of the compactifications from algebraic geometry. Finally, Natanzon and Turaev asked about the local structure of their compactification, in particular, is it a complex manifold? Methods of algebraic geometry allow one to analyse the local structure of the compactification from [DE] to which the Natanzon-Turaev compactification is homeomorphic. Using this we construct an example to show that the Natanzon-Turaev compactification can be singular, that is, not a complex manifold.
2. The homeomorphism

We start by describing the Natanzon-Turaev compactification and the compactification from [DE] to which it is homeomorphic.

First we describe the Natanzon-Turaev compactification. From the Riemann-Hurwitz formula one deduces that the genus of the cover of $S^2$ in an element of $H^{n,w}$ is $g = \frac{1}{2}(w - 2n + 2)$. Fix a closed oriented connected (though [NT] does not require connected) surface $\Sigma$ of genus $g$. Define $H(\Sigma, n)$ to be the set of equivalence classes of $n$-sheeted simple branched coverings $f : \Sigma \to S^2$, where the equivalence relation is: $f : \Sigma \to S^2$ and $f' : \Sigma \to S^2$ are equivalent if and only if there is a homeomorphism $\alpha : \Sigma \to \Sigma$ such that $f = f'\alpha$. $H(\Sigma, n)$ is the Hurwitz space $H^{n,w}$. Natanzon and Turaev construct a compactification $N(\Sigma, n)$ of $H(\Sigma, n)$. The points of $N(\Sigma, n)$ are equivalence classes of decorated functions, where Natanzon and Turaev define decorated functions and their equivalence as follows. We quote directly from [NT].

**Definition 2.1.** A decorated function (on the surface $\Sigma$) is a triple $(f, E, \{D_e\}_{e \in E})$ where $f : \Sigma \to S^2$ is a simple branched covering, $E$ is a finite subset of $S^2$ disjoint from the set of branch points of $f$, and $\{D_e\}_{e \in E}$ are disjoint closed 2-discs embedded in $S^2$ such that: $e \in \text{Int} D_e$ for all $e \in E$, each $D_e$ contains at least two branch points of $f$, and the circles $\{\partial D_e\}_e$ do not contain branch points of $f$.

An isotopy of a decorated function $(f, E, \{D_e\}_e)$ is a family of decorated functions $(\varphi_t f, E, \{\varphi_t(D_e)\}_e)$ where $\{\varphi_t : S^2 \to S^2\}_{t \in [0, 1]}$ is an isotopy of the identity map $\varphi_0 = \text{id}_{S^2}$ such that for all $t \in [0, 1]$ the homeomorphism $\varphi_t$ preserves (pointwise) $E$ and the branch points of $f$ lying outside $\bigcup_e D_e$.

We say that two decorated functions $(f, E, \{D_e\}_e)$ and $(f', E', \{D'_e\}_{e \in E'})$ are equivalent if $E = E'$ and $f'$ may be obtained from $f$ by an isotopy and/or composition with a homeomorphism $\Sigma \to \Sigma$. (The isotopy must also take each $D_e$ to $D'_e$.)

Recall that the set of unordered sets of $k$ not necessarily distinct points in complex projective one-space $\mathbb{CP}^1 = S^2$ is naturally identified with complex projective $k$-space $\mathbb{CP}^k = \mathbb{P}^k$. There is a natural map $q : H(\Sigma, n) = H^{n,w} \to \mathbb{P}^w$ which sends a cover $(f : \Sigma \to S^2) \in H(\Sigma, n)$ to its branch points. This extends to a map (also denoted $q$) $q : N(\Sigma, n) \to \mathbb{P}^w$ which sends a decorated function $(f, E, \{D_e\}_e)$ to the set consisting of the branch points of $f$ outside of $\bigcup_e D_e$ each counted once, plus the points of $E$ each counted with multiplicity equal to the number of branch points of $f$ inside the corresponding $D_e$. As mentioned in [NT] before Lemma 2.2 the extended mapping $q$ is continuous and open.

Now we describe the algebraic compactification mentioned in [DE]. In that article the authors denote the compactification by $\overline{\mathcal{M}}^{k,b}_H$ ($k$ corresponds to $n$ and $b$ to $w$ in $H^{n,w}$). It has been known since Hurwitz that $q : H(\Sigma, n) \to \mathbb{P}^w$ is finite. Therefore the function field of $H(\Sigma, n)$ is a finite extension of the function field of $\mathbb{P}^w$. The compactification $\overline{\mathcal{M}}^{k,b}_H$ is defined to be the normalization of $\mathbb{P}^w$ in the function field of $H(\Sigma, n)$. The map $q : H(\Sigma, n) \to \mathbb{P}^w$ extends to a regular algebraic morphism $\pi : \overline{\mathcal{M}}^{k,b}_H \to \mathbb{P}^w$.

The homeomorphism $g : N(\Sigma, n) \to \overline{\mathcal{M}}^{k,b}_H$ that we claim to exist will be defined to be the identity on $H(\Sigma, n) \subset N(\Sigma, n)$ and $H(\Sigma, n) \subset \overline{\mathcal{M}}^{k,b}_H$. It will also commute with the maps $q$ and $\pi$ to $\mathbb{P}^w$. What is left to define are the values of $g$ on the points of $N(\Sigma, n)$ lying over points of $\mathbb{P}^w$ corresponding to nondistinct points in $\mathbb{P}^1$. 
Denote by \( D \) the points of \( \mathbb{P}^w \) corresponding to nondistinct sets of \( w \) points in \( \mathbb{P}^1 \). Let \( p \in D \). Lemma 4.3 of [DE], which was proved for \( \overline{\mathcal{H}}_{k,b} \) can in the same way be proven for \( \overline{\mathcal{H}}_{k,b} \) to obtain the following.

**Lemma 2.1.** Pick a small connected neighborhood (say any small open ball) \( B \) of \( p \) in \( \mathbb{P}^w \), and pick a point \( r \in B - D \). The fundamental group of \( B - D \) with base point \( r \) acts via monodromy on \( \pi^{-1}(r) \). Define an equivalence relation on \( \pi^{-1}(r) \) by saying that two points are equivalent if and only if they can be taken to each other by the monodromy action. For \( B \) sufficiently small the following are true.

1. Two points of \( \pi^{-1}(r) \) lie in the same monodromy equivalence class iff they lie in the same connected component of \( \pi^{-1}(B - D) \).
2. The closure of each connected component of \( \pi^{-1}(B - D) \) in \( \pi^{-1}(B) \) has exactly one point over \( p \).
3. The closures of the connected components of \( \pi^{-1}(B - D) \) in \( \pi^{-1}(B) \) are all disjoint from each other.

For our purposes the point of this lemma is the following obvious corollary.

**Corollary 2.1.** For \( B \) sufficiently small there is a natural bijection between the points of \( \pi^{-1}(p) \) and the connected components of \( \pi^{-1}(B - D) \). The bijection is given by associating to each connected component \( X \) of \( \pi^{-1}(B - D) \) the intersection of the closure of \( X \) in \( \pi^{-1}(B) \) with \( \pi^{-1}(p) \).

Next we see that exactly the same result is true if we replace \( \pi^{-1}(p) \) by \( \mu^{-1}(p) \).

**Proposition 2.1.** For \( B \) as in Lemma 2.7 sufficiently small there is a natural bijection between the points of \( \mu^{-1}(p) \) and the connected components of \( \mu^{-1}(B - D) \). The bijection is given by associating to each connected component \( X \) of \( \mu^{-1}(B - D) \) the intersection of the closure of \( X \) in \( \mu^{-1}(B) \) with \( \mu^{-1}(p) \).

**Proof.** The point \( p \) must consist of \( \ell \) distinct points \( p_1, \ldots, p_\ell \) and \( \ell' \) multiple points \( e_1, \ldots, e_{\ell'} \) of \( \mathbb{P}^1 \). Say the multiplicity of \( e_i \) is \( k_i \).

Step 1. Given any connected component \( X \) of \( \mu^{-1}(B - D) \) we can find an \( (f : \Sigma \to S^2) \in X \) such that all the \( p_i \)'s are branch points of \( f \) and none of the \( e_i \)'s are branch points of \( f \). We can then create a decorated function \( (f, E, \{D_e\}_e) \) representing a point of \( \mu^{-1}(p) \) as follows.

First, the existence of such an \( f \) is clear because \( q(X) = B - D \). Because \( f \in X \), for small \( B \) we have that if we let \( D_{e_i} \) be a small disk around \( e_i \), then \( D_{e_i} \) contains exactly \( k_i \) branch points. Letting \( E = \{ e_1, \ldots, e_{\ell'} \} \), \( (f, E, \{D_e\}_e) \) represents a point of \( \mu^{-1}(p) \).

Step 2. Given any point \( x_0 \) of \( \mu^{-1}(p) \) we can choose a decorated function \( (f, E, \{D_e\}_e) \) representing \( x_0 \) that is obtained in the manner of Step 1, for some connected component of \( \mu^{-1}(B - D) \) possibly depending on \( x_0 \).

It is simply a matter of using an isotopy \( \varphi_i \) to shrink the disks \( D_e \) until they are small enough so that \( \varphi_i f \in \mu^{-1}(B - D) \).

Step 3. Two decorated functions \( (f, E, \{D_e\}_e) \) and \( (f', E, \{D'_e\}_e) \) obtained as in steps 1 and 2 are equivalent iff \( f \) and \( f' \) lie in the same connected component of \( \mu^{-1}(B - D) \).

By applying isotopies we can assume the disks \( D_e = D'_e \) for all \( e \in E \). If \( f \) and \( f' \) lie in the same connected component \( X \), then a path in \( X \) connecting them can be used to create the desired equivalence as in [NT], proof of Theorem 3.9.
Given \( f \) and \( f' \) correspond to the branch points of \( X \) inside disks centered at \( P \) the resulting path in \( D \) will be an open neighborhood of \( Q \). For this it is sufficient to show that for each \( Q \) of connected components of \( q \) open neighborhoods would lead to two connected components, a contradiction.

From steps 1–3 we see that the number of points in \( q^{-1}(p) \) equals the number of connected components of \( q^{-1}(B - D) \). To complete the proof it is enough to show that no connected component of \( q^{-1}(B - D) \) can have more than one point of \( q^{-1}(p) \) in its closure. Assume to the contrary that some connected component had at least two points of \( q^{-1}(p) \) in its closure. From [NT], Lemma 2.2, we know that \( N(\Sigma, n) \) is Hausdorff. We can find disjoint open neighborhoods of these two points. After possibly shrinking \( B \) the intersection of \( q^{-1}(B - D) \) with these two disjoint open neighborhoods would lead to two connected components, a contradiction. \( \square \)

We can now define the map \( g \) on points of \( N(\Sigma, n) \) lying over points of \( D \subset \mathbb{P}^w \). Given \( p \in D \) and \( x \in q^{-1}(p) \), for a sufficiently small open ball \( B \) around \( p \), \( x \) will be the only point of \( q^{-1}(p) \) lying in the closure of some connected component \( X \) of \( q^{-1}(B - D) \). Similarly, the closure of \( X \) in \( \pi^{-1}(B) \) contains a unique point \( y \) of \( \pi^{-1}(p) \). Define \( g(x) = y \).

**Proposition 2.2.** The map \( g \) is a homeomorphism.

**Proof.** It is clear that \( g \) is bijective. We show that \( g \) is continuous. The proof that \( g^{-1} \) is continuous is similar.

Let \( U \subset \overline{SH}_{k,b} \) be an open set. We wish to show that \( g^{-1}(U) \subset N(\Sigma, n) \) is open. For this it is sufficient to show that for each \( x \in g^{-1}(U) \) we can find an open set \( V_x \) of \( N(\Sigma, n) \) with \( x \in V_x \subset g^{-1}(U) \). Clearly this can be done for \( x \in H(\Sigma, n) \) so we assume \( x \in N(\Sigma, n) - H(\Sigma, n) \). Set \( y = g(x) \). For a sufficiently small ball \( B \) around \( \pi(y) \) the connected component \( Y \) of \( \pi^{-1}(B) \) containing \( y \) will be an open neighborhood of \( y \) contained in \( U \). The connected component \( X \) of \( q^{-1}(B) \) containing \( x \) will be an open neighborhood of \( x \). We wish to show \( g(X) \subset Y \), so that \( X \subset g^{-1}(U) \). Clearly \( g(X \cap H(\Sigma, n)) = Y \cap H(\Sigma, n) \), in fact \( X \cap H(\Sigma, n) = Y \cap H(\Sigma, n) \). Pick any \( x_0 \in X - H(\Sigma, n) \). To compute \( g(x_0) \) we find a sufficiently small ball \( B_0 \) around \( q(x_0) \), we can assume \( B_0 \subset B \), so that Corollary 2.1 and Proposition 2.1 apply. Say \( X_0 \) is the connected component of \( q^{-1}(B_0 - D) \) with \( x_0 \) in its closure. Then \( g(x_0) \) will be the point over \( q(x_0) \) in the closure of \( X_0 \) in \( \pi^{-1}(B_0) \). But \( B_0 \subset B \) says \( X_0 \subset X \cap H(\Sigma, n) = Y \cap H(\Sigma, n) \), so \( g(x_0) \in Y \). \( \square \)

As pointed out in [DE], section 4.4, \( \overline{SH}_{k,b} \) is a projective variety and it is certainly normal. In view of the homeomorphism \( g \) we could define a complex structure on
that singularities. N codimension at least 2. As we shall see in the next section $N(\Sigma, n)$ can have singularities.

3. A singular example

We shall study $N(\Sigma, n)$ when $\Sigma = S^2 = \mathbb{P}^1$ and $n = 3$. Thus we are studying degree three covers of $S^2$ simply branched at four points. We have the map $q : N(S^2, 3) \to \mathbb{P}^4$. We will show that over points of $\mathbb{P}^4$ corresponding to two distinct points of $\mathbb{P}^1$ each taken with multiplicity 2, $N(S^2, 3)$ has two points—one nonsingular and one singular.

Let $D \subset \mathbb{P}^4$ be the discriminant locus consisting of nondistinct points and fix $O \in D$ where $O$ corresponds to two distinct points each with multiplicity 2. Locally near $O$, $D$ consists of two smooth branches crossing transversally. Each branch corresponds to allowing one of the two multiplicity 2 points to become two distinct points. Pick a point $P \in \mathbb{P}^4 - D$ near $O$. By standard techniques from Hurwitz space theory (see [F], proof of Proposition 1.5, [A], proof of Theorem 2.7, or [DE], section 4.2 shortly before Lemma 4.2) the fiber of $q$ over $P$ corresponds to equivalence classes of ordered 4-tuples of simple transpositions $[\sigma_1, \sigma_2, \sigma_3, \sigma_4]$, $\sigma_i \in S_3$ (the symmetric group on three letters), such that the product $\sigma_1 \sigma_2 \sigma_3 \sigma_4 = (1)$ and $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ generates a transitive subgroup of $S_3$, where the equivalence relation is $[\sigma_1, \sigma_2, \sigma_3, \sigma_4]$ is equivalent to $[\tau_1, \tau_2, \tau_3, \tau_4]$ if there exists an $\alpha \in S_3$ such that $\sigma_i = \alpha \tau_i \alpha^{-1}$, $i = 1, \ldots, 4$. Each $\sigma_i$ represents the ramification point over one of the four points of $\mathbb{P}^1$ represented by $P$. One computes that over $P$ there are four points which we may represent as: $[(12), (12), (23), (23)]$, $[(12), (23), (12), (13)]$, $[(12), (23), (13), (23)]$, and $[(12), (23), (23), (12)]$.

We may assume that we have set things up so that one branch of $D$ near $O$ corresponds to the first two points becoming one double point and the other branch represents the last two points becoming one double point. Again from standard Hurwitz space techniques (see [F], proof of Proposition 1.5, [A], proof of Theorem 2.7, or [DE], end of section 4.2) we see that the monodromy action on the inverse image of $P$ generated by a loop based at $P$ going around branch 1 is generated by $[\sigma_1, \sigma_2, \sigma_3, \sigma_2] \mapsto [\sigma_2, \sigma_2^{-1} \sigma_1 \sigma_2, \sigma_3, \sigma_4]$ and around branch 2 it is $[\sigma_1, \sigma_2, \sigma_3, \sigma_4] \mapsto [\sigma_1, \sigma_2, \sigma_4, \sigma_4^{-1} \sigma_3 \sigma_4]$. One computes that in both cases $[(12), (12), (23), (23)]$ does not move but that the other three points are permuted cyclically. Remember that after applying the monodromy transformation you might need to conjugate by an appropriate element of $S_3$ to get the ordered 4-tuple to be one of the four we have chosen to represent the fiber.

Thus over a small neighborhood of $O$ in $\mathbb{P}^w$, $N(S^2, 3)$ has two components. One is a single sheet mapping isomorphically onto the small neighborhood. This gives the nonsingular point of $q^{-1}(O)$. The other component consists of three sheets all coming together and ramifying to order 3 over each branch of $D$. We now concentrate on that component; call it $X$.

Choose local coordinates $u, v, x, y$ on $\mathbb{P}^4$ near $O$ so that $D$ has local equation $xy = 0$. The ramification to order 3 along both $x = 0$ and $y = 0$ says that in $X$, $xy$ has a cube root. In $\mathbb{C}^5$ with coordinates $u, v, x, y, z$ take the hypersurface $X' = \{xy = z^3\}$. $X'$ maps to $\mathbb{P}^4$ by $(u, v, x, y, z) \mapsto (u, v, x, y)$ (locally near $O$ of course). One easily computes that the singularities of $X'$ are $x = y = z = 0$. 
$X'$ is normal because it is a hypersurface with singularities in codimension greater than 1; see [Ha], Proposition II.8.23. $X'$ also has the appropriate monodromy along $xy = 0$. By uniqueness of normalization $X'$ near $(0, \ldots, 0)$ is isomorphic to $X$ near $q^{-1}(O) \cap X$. Thus $X$ is singular.

Even if we get a loop backwards in the monodromy the only other possibility is $z^3 = x^2 y$ which also has a singular normalization.

As a final remark we note that since $\overline{SH}_{k,b}$ is normal any nonsingular variety $Z$ finite over $\mathbb{P}^w$ compactifying $H^{n,w}$ would be isomorphic to $\overline{SH}_{k,b}$. Since a nonsingular variety is normal such a $Z$ would have to be the normalization of $\mathbb{P}^w$ in the function field of $H^{n,w}$, hence equal to $\overline{SH}_{k,b}$. Thus we cannot make $N(S^2,3)$ nonsingular by finding a different complex structure to put on it.

References


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