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# EPIMORPHISM SEQUENCES BETWEEN HYPERBOLIC 3-MANIFOLD GROUPS

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Abstract. We will show that any infinite sequence of epimorphisms between finitely generated hyperbolic 3-manifold groups eventually consists of isomorphisms.

In this paper, we are interested in sequences $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} \cdots$ of $\pi_1$-surjective maps between geometric 3-manifolds and the problem whether such a sequence has a homotopy equivalence. Our theme here is closely connected with J. Simon's Problem 1.12 (C) and Y. Rong's Problem 3.100 (B) in [5], and related topics are studied in [4], [6], [7], [8], [9], [10], [11], etc. We will consider the case where these 3-manifolds $M_i$ are hyperbolic. If such an $M_i$ has finite volume, then either $M_i$ is a closed manifold or each end of $M_i$ is a $\mathbb{Z} \times \mathbb{Z}$-cusp, and the fundamental group $\pi_1(M_i)$ is finitely generated. Example 3.2 (2) in Reid-Wang-Zhou [7] presented a closed hyperbolic 3-manifold $M$ which admits, for any $n \in \mathbb{N}$, a length $n$ sequence $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} M_n$ of non-homotopy equivalence, $\pi_1$-surjective maps between closed hyperbolic 3-manifolds $M_i$ $(i = 0, 1, \dots, n)$ with $M_0 = M$. On the other hand, Theorem 1 below shows that any infinite sequence of $\pi_1$-surjective maps between such 3-manifolds contains a homotopy equivalence.

**Theorem 1.** *Suppose that*

$$G_0 \xrightarrow{\varphi_0} G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} G_n \xrightarrow{\varphi_n} \cdots$$

*is an infinite sequence of epimorphisms between the fundamental groups $G_n$ $(n = 0, 1, 2, \dots)$ of orientable hyperbolic 3-manifolds (possibly of infinite volume). If $G_0$ is finitely generated, then $\varphi_n$ is isomorphic for all but finitely many $n \in \mathbb{N}$.*

To prove Theorem 1, we use the *character variety* $X(G)$ of representations from a finitely generated group $G$ to $\mathrm{SL}_2(\mathbb{C})$ defined by Culler-Shalen [2, §1]. Fix a generator set $\gamma_1, \dots, \gamma_\nu$ for $G$, and let $\sigma(G) = \{g_1, \dots, g_\mu\}$ be a maximal set such that each $g_j$ has a form $\gamma_{i_1} \cdots \gamma_{i_r} \in G$ for distinct positive integers $i_1, \dots, i_r \le \nu$. By [2, Proposition 1.4.1], the element $\chi_\rho$ of $X(G)$ corresponding to $\rho : G \longrightarrow \mathrm{SL}_2(\mathbb{C})$ can be identified with the point $(\mathrm{tr}(\rho(g_1)), \dots, \mathrm{tr}(\rho(g_\mu))) \in \mathbb{C}^\mu$, where $\mathrm{tr}(\rho(g_j))$ is the trace of the $2 \times 2$ matrix $\rho(g_j)$. Then, $X(G)$ is a closed affine algebraic set in $\mathbb{C}^\mu$. We refer to Hartshorne [3] for the fundamental notation and definitions on algebraic geometry.

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*Proof of Theorem 1.* Let $A$ be the set of infinite sequences $\mathbf{n} = (n_0, n_1, n_2, \dots)$ such that all entries $n_i$ are non-negative integers and at most finitely many of them are non-zero. We equip $A$ with the backward lexicographical order. That is, $\mathbf{n} = (n_0, n_1, n_2, \dots) < \mathbf{m} = (m_0, m_1, m_2, \dots)$ means that $n_{j_0} < m_{j_0}$ for some $j_0 \in \mathbb{N} \cup \{0\}$ and $n_j = m_j$ for all $j \geq j_0 + 1$. Then, $A$ is a well ordered set. For a finitely generated group $G$, let $\alpha(G)$ be the element of $A$ such that the $i$-th term of $\alpha(G)$ is the number of $i$-dimensional irreducible components of $X(G)$.

Let $\varphi : G \longrightarrow H$ be an epimorphism between finitely generated groups. Then, the induced map $\varphi^* : X(H) \longrightarrow X(G)$ is an injective regular map. Here, $\varphi^*$ being *regular* means that it is the restriction to $X(H)$ of a polynomial map between the affine complex spaces containing $X(H)$ and $X(G)$. In fact, by identifying $\mathrm{tr}(\rho(\varphi(g_j)))$ with $\mathrm{tr}((\varphi^*\rho)(g_j))$ for $\rho \in X(H)$ and $g_j \in \sigma(G)$, one can consider that $X(H)$ is an algebraic subset of $X(G)$ in $\mathbb{C}^\mu$. If an irreducible component $C$ of $X(H)$ is contained in an irreducible component $D$ of $X(G)$, then $\dim(C) \leq \dim(D)$. Moreover, if $\dim(C) = \dim(D)$, then we have $C = D$. This implies that $\alpha(G) \geq \alpha(H)$, and the equality $\alpha(G) = \alpha(H)$ holds if and only if $X(G) = X(H)$.

Here, we return to an infinite sequence

$$G_0 \xrightarrow{\varphi_0} G_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} G_n \xrightarrow{\varphi_n} \cdots$$

given in the statement of Theorem 1. Since $A$ is well ordered, there exists $n_0 \in \mathbb{N}$ with $\alpha(G_{n_0}) = \alpha(G_n)$ for all $n \geq n_0$. Since $X(G_n) = X(G_{n+1})$ if $n \geq n_0$, there exists $\chi_\rho \in X(G_{n+1})$ with $\varphi_n^*\chi_\rho = \chi_{\widetilde{\eta}_n}$, where $\widetilde{\eta}_n : G_n \longrightarrow \mathrm{SL}_2(\mathbb{C})$ is a lift of a holonomy $\eta_n : G_n \longrightarrow \mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$ of a hyperbolic 3-manifold $M_n$ with $\pi_1(M_n) = G_n$. Since $\widetilde{\eta}_n = \rho \circ \varphi_n$ is a faithful representation, $\varphi_n$ is monic and hence isomorphic. This completes the proof. □

A knot $K$ in $S^3$ is called *hyperbolic* if the complement $S^3 - K$ admits a hyperbolic structure of finite volume. Note that, for $G = \pi_1(S^3 - K)$, $X(G)$ has no 0-dimensional irreducible component. Moreover, it is very often the case that the dimension of $X(G)$ is one. For example, if an exterior $E(K)$ for $K$ has no closed essential surface, then $\dim X(G) = 1$ (see Cooper et al. [1, §2.4, Proposition]). Theorem 2 below is a knot-group version of Theorem 1, where epimorphisms are not necessarily assumed to be peripheral preserving.

**Theorem 2.** *Let $K_0$ be a hyperbolic knot in $S^3$ with $\dim X(G_0) = 1$ and $n(K_0) \in \mathbb{N}$ the number of irreducible components of $X(G_0)$ for $G_0 = \pi_1(S^3 - K_0)$. Suppose that*

$$G_0 \xrightarrow{\varphi_0} G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} G_n$$

*is a finite sequence of epimorphisms between the fundamental groups $G_i$ of the complements of hyperbolic knots $K_i$ in $S^3$ $(i = 1, 2, \dots, n)$ starting from $G_0$. If the length $n$ of the sequence is not less than $n(K_0)$, then at least one of these epimorphisms $\varphi_j$ is an isomorphism.*

*Proof.* Let $a_1(G_i)$ be the number of (1-dimensional) irreducible components of $X(G_i)$. Suppose that any $\varphi_i$ $(i = 0, 1, \dots, n-1)$ are not isomorphic. Then, by the argument similar to that in Theorem 1, we have $a_1(G_i) > a_1(G_{i+1})$. Since $a_1(G_0) = n(K_0)$ and $a_1(G_n) \geq 1$, the inequality $n < n(K_0)$ holds. □

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