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$\mathbb{N}$-COMPACTNESS AND WEIGHTED COMPOSITION MAPS

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ABSTRACT. In this paper we study some conditions on (not necessarily continuous) linear maps T between spaces of real- or complex-valued continuous functions $C(X)$ and $C(Y)$ which allow us to describe them as weighted composition maps. This description depends strongly on the topology in X; namely, it can be given when X is $\mathbb{N}$-compact, but cannot in general if some kind of connectedness on X is assumed. Finally we also give an infimum-preserving version of the Banach-Stone theorem. The results are also proved for spaces of *bounded* continuous functions when $\mathbb{K}$ is a field endowed with a nonarchimedean valuation and it is not locally compact.

1. INTRODUCTION

The classical Banach-Stone theorem states that, for Hausdorff compact spaces X and Y, if T is a surjective linear isometry between the real- or complex-valued spaces of continuous functions $C(X)$ and $C(Y)$, then it is a weighted composition map which derives in a natural way from a homeomorphism. That is, there are a (surjective) homeomorphism $h : Y \to X$ and a function $a \in C(Y)$, $|a(y)| = 1$ for every $y \in Y$, such that $Tf = a \cdot f \circ h$ for every $f \in C(X)$.

On the other hand, a slight modification of the requirements on T derives in the fact that the result is no longer true. For instance, if T is a continuous isomorphism instead of an isometry, we cannot obtain in general a description of T as a weighted composition map.

Of course, in the results mentioned above we consider spaces endowed with a norm. Our aim here is to obtain a similar description of maps without assuming any topology on the spaces of continuous functions. For instance, in general we will not consider the topological spaces to be compact.

Our approach here could be somehow qualified as the opposite to the one taken when studying linear isometries, yet the results we obtain are similar. Namely, an isometry $T : C(X) \to C(Y)$ is a map where the image of each $f \in C(X)$ satisfies $\sup_{y\in Y} |(Tf)(y)| = \sup_{x\in X} |(f)(x)|$. In our case, the maps $T : C(X) \to C(Y)$ we will study are not continuous in principle, and satisfy that there exists $M > 0$ such that

$$\inf_{\substack{y\in Y \\ (Tf)(y)\neq 0}} |(Tf)(y)| \geq M \inf_{\substack{x\in X \\ f(x)\neq 0}} |f(x)| .$$

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As we will see, the behavior of these maps will depend very much on the structure of X. In particular, we obtain that they can be described as weighted composition when the space X is $\mathbb{N}$-compact, but cannot when we assume some connectedness on X, as is shown in the example at the end of this introduction.

The results given in this paper do not apply in general to spaces of real- or complex-valued *bounded* continuous functions. For instance, it is well-known (see [6, Theorems 6.2.10 and 6.2.12]) that if a topological space X is *strongly zerodimensional*, its Stone-Čech compactification βX is zerodimensional (that is, a nonempty T_1-space with a base consisting of open-and-closed sets). Consequently Lemma 2.3 is not true in general in the realm of spaces of bounded functions, making false the rest of the statements for this kind of space.

Nevertheless all these results hold for spaces of *bounded* continuous functions taking values not in $\mathbb{R}$ or $\mathbb{C}$, but in a field $\mathbb{K}$ endowed with a nonarchimedean valuation under which it is not locally compact. Once again, the fact that the field is not locally compact makes a strong difference with respect to the real or complex cases (see [1] and [7]), providing a richer result than obtained when dealing with $\mathbb{R}$ or $\mathbb{C}$.

Unless otherwise stated, the topological space X will be $\mathbb{N}$-compact, that is, homeomorphic to a closed subspace of some power of $\mathbb{N}$, and Y will be completely regular. In the case when $\mathbb{K}$ is a nonarchimedean field (see Situation 2 below and the comment after it), Y will be also zerodimensional. Throughout Sections 2 and 3, we will assume that we are in one of the following two situations:

- **Situation 1.** $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In this case $B(X)$ will denote $C(X)$, that is, the space of all continuous functions $f : X \to \mathbb{K}$. Also $B(Y) = C(Y)$.
- **Situation 2.** $\mathbb{K}$ is a commutative complete field endowed with a nontrivial nonarchimedean valuation, and it is assumed *not* to be locally compact. In this case $B(X)$ will denote $C^*(X)$, that is, the space of all *bounded* continuous functions $f : X \to \mathbb{K}$. Also $B(Y) = C^*(Y)$.

Here we mention the fact that, when X and Y are compact and zerodimensional, the results are valid (with similar proofs) for every commutative complete field endowed with a nontrivial valuation, even if it is locally compact. This will be used in Section 4 when we give a general version of the Banach-Stone theorem for infimum preserving linear maps.

For a clopen (that is, closed and open) subset U of X, ξ_U will stand for the characteristic function on U, which is continuous. Given a map f from X into $\mathbb{K}$, the cozero set of f will be the set $c(f) := \{x \in X : f(x) \neq 0\}$. Moreover for any $f \in B(X)$, $f \neq 0$, we denote by $\mathrm{val}\,(f)$ the set $\{|f(x)| : x \in c(f)\}$; as for the function constantly equal to 0, $\mathrm{val}\,(0) := \{0\}$. Finally given a map $T : B(X) \to B(Y)$, we define $Y_0 := \bigcup_{f \in B(X)} c(Tf)$.

Definition 1.1. A map $T : B(X) \to B(Y)$ is said to be weakly separating if $c(Tf) \cap c(Tg) = \emptyset$ whenever $f, g \in B(X)$ and there exist disjoint clopen sets U and V such that $c(f) \subset U$ and $c(g) \subset V$.

Definition 1.2. A linear map $T : B(X) \to B(Y)$ is said to be a weighted composition map if there exist a continuous map $h : Y \to X$ with dense range and $a \in B(Y)$ such that $(Tf)(y) = a(y)f(h(y))$ for every $f \in B(X)$ and every $y \in Y$.

Definition 1.3. A linear map $T : B(X) \to B(Y)$ is said to be a Banach-Stone map if there exist a homeomorphism h from Y onto X and $a : Y \to \mathbb{K}$ continuous, $|a| \equiv 1$, such that $(Tf)(y) = a(y)f(h(y))$ for every $f \in B(X)$ and every $y \in Y$.

Definition 1.4. A linear map $T : B(X) \to B(Y)$ is said to be *infrabounded-below* if there exists $M > 0$ such that $\inf \operatorname{val}(Tf) \geq M \inf \operatorname{val}(f)$ for every $f \in B(X)$.

Also for an infrabounded-below map T, we denote by $\operatorname{infra}(T)$ the supremum of all M satisfying the inequality given in Definition 1.4.

We start by showing that in general not every infrabounded-below map can be represented as a weighted composition, not even when it is bijective and continuous.

Example. Assume that $\mathbb{K} = \mathbb{R}$, and that $X = Y$ is a compact Hausdorff space which contains a connected open set Z with more than one point. Take $x_0 \in Z$. It is clear that $C(X)$ can be expressed as the direct sum $C(X) = A \oplus M_0$, where $A := \{\alpha\xi_X : \alpha \in \mathbb{R}\}$ and $M_0 := \{f \in C(X) : f(x_0) = 0\}$. Consider $g_0 \in C(X)$ such that $1/2 \leq g_0 \leq 1$, $g_0(x_0) = 1/2$, and $g_0 \equiv 1$ outside Z. Now define $T\xi_X := \xi_X$, and $Tf_0 := f_0 g_0$ for every $f_0 \in M_0$, and extend it by linearity to a map $T : C(X) \to C(X)$. It is clear that T is bijective, and it is continuous if we endow $C(X)$ with the sup norm. It is also easy to see that T cannot be described as a weighted composition map. Now let us check that it is infrabounded-below.

Take $f \in C(X)$. Then $f = \alpha\xi_X + f_0$, where $\alpha \in \mathbb{R}$ and $f_0 \in M_0$. Without loss of generality, we assume that $\alpha > 0$. Now notice that, for every $x \notin Z$, $(Tf)(x) = f(x)$. So we concentrate on the behaviour of f in Z.

First, if $f_0(x) \geq 0$ for every $x \in Z$, then it is easy to see that $\alpha = f(x_0) \leq |f(x)|$ and $\alpha = (Tf)(x_0) \leq |(Tf)(x)|$ for every $x \in Z$. As a consequence, $\inf \operatorname{val}(f) = \alpha$ if $c(f) \subset Z$, and $\inf \operatorname{val}(f) = \min\{\alpha, \inf\{|f(x)| : x \notin Z, f(x) \neq 0\}\}$ otherwise. Clearly it coincides with $\inf \operatorname{val}(Tf)$.

On the other hand, if there are points in Z where f_0 takes negative value, then consider $r := \inf_{x \in Z} f_0(x)$. It is clear that, since Z is connected and $f_0(x_0) = 0$, if $r \leq -\alpha$, then there are points in Z where f takes values as close to 0 as we want. Consequently, $\inf \operatorname{val}(f) = 0$ and $\inf \operatorname{val}(Tf) \geq \inf \operatorname{val}(f)$. Finally, if $r > -\alpha$, for every $x \in Z$, we have

$$|(Tf)(x)| = \alpha + f_0(x)g_0(x) \geq \alpha + r = \inf\{|f(x)| : x \in Z\}.$$

As a consequence it is easy to check that for every $f \in C(X)$, $\inf \operatorname{val}(Tf) \geq \inf \operatorname{val}(f)$, and then T is infrabounded-below.

2. Weighted composition maps and infrabounded-below maps

Lemma 2.1. *Let a linear map $T : B(X) \to B(Y)$ satisfy one of the two following conditions:*

(1) *T is infrabounded-below.*

(2) $\operatorname{val}(Tf) = \{1\}$ *whenever* $\operatorname{val}(f) = \{1\}$, $f \in B(X)$. *In this case also assume that $\mathbb{K}$ is nonarchimedean.*

Then T is weakly separating and injective.

Proof. Since we are giving a unique proof assuming two possible conditions, we will suppose that if T satisfies Condition 2 but it does not satisfy Condition 1, then $\operatorname{infra}(T) = +\infty$.

We will prove the result through the following two claims.

Claim 1. *If U and V are disjoint nonempty clopen subsets of X, then $c(T\xi_U) \cap c(T\xi_V) = \emptyset$.*

Suppose on the contrary that $c(T\xi_U) \cap c(T\xi_V) \neq \emptyset$. Then there exists $y \in Y$ such that $(T\xi_U)(y) = \alpha$ and $(T\xi_V)(y) = \beta$, with $\alpha\beta \neq 0$. Clearly, if we assume Condition 1, $\mathrm{infra}(T) \leq |\alpha|, |\beta|$. Notice also that if we assume Condition 2, then $|\alpha| = 1 = |\beta|$.

Suppose that $|\alpha| \leq |\beta|$. Take $\gamma \in \mathbb{K}$, $|\gamma| \geq 1$, such that

$$0 < |\gamma\alpha + \beta| < \min\{1, \mathrm{infra}(T)\}.$$

Then we have that $\inf \mathrm{val}(\gamma\xi_U + \xi_V) = 1$ and also $\mathrm{val}(\gamma\xi_U + \xi_V) = \{1\}$ if we assume Condition 2. However we have that $|\gamma\alpha + \beta|$ belongs to $\mathrm{val}(T(\gamma\xi_U + \xi_V))$ which is against our hypothesis. Consequently $c(T\xi_U) \cap c(T\xi_V) = \emptyset$.

Claim 2. *Suppose that U is a clopen subset of X, and that $f \in B(X)$ satisfies $c(f) \subset U$. Then $c(Tf) \subset c(T\xi_U)$.*

Although the idea to prove this claim is essentially the same when we deal with archimedean or nonarchimedean fields, there are some basic differences between them. Consequently, we split the proof of this claim into two cases.

Case 1. $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. We know that each complex-valued continuous function is a linear combination of two real-valued continuous functions, namely, its real and its imaginary parts. In the same way, every real-valued continuous function is a linear combination of its positive and its negative parts. Consequently, we can assume that $f \geq 0$. Now suppose that there exists a point $y_0 \in c(Tf) - c(T\xi_U)$. Let $r := |(Tf)(y_0)|$. Next take $M > 0$ such that $M\mathrm{infra}(T) > r$. It is clear that $\inf \mathrm{val}(M\xi_U + f) \geq M$, and consequently $\inf \mathrm{val}(MT\xi_U + Tf) \geq M\mathrm{infra}(T)$. But this contradicts the fact that $|M(T\xi_U)(y_0) + (Tf)(y_0)| = r \in (0, M\mathrm{infra}(T))$. Then the claim is proved in this first case.

Case 2. $\mathbb{K}$ *is a nonarchimedean field.* Take $q \in \mathbb{K}$ such that

$$\sup_{y\in Y} |(Tf)(y)|, \sup_{x\in X} |f(x)| < \min\{1, \mathrm{infra}(T)\} |q|.$$

Then we have that $\mathrm{val}(f + q\xi_U) = \{|q|\}$. So we deduce that

$$\inf \mathrm{val}(T(f + q\xi_U)) \geq \min\{1, \mathrm{infra}(T)\} |q|.$$

Take $y \in c(Tf)$. Since in any case $0 < |(Tf)(y)| < \min\{1, \mathrm{infra}(T)\} |q|$, and consequently $|(Tf + qT\xi_U)(y)| \neq |(Tf)(y)|$, then we have that $|q(T\xi_U)(y)| \neq 0$, which proves that $c(Tf) \subset c(T\xi_U)$ also in this second case.

Now it follows immediately from Claims 1 and 2 that T is weakly separating.

Finally we are going to see that T is injective. Suppose that $f \in B(X)$, $f \neq 0$. Take a clopen subset A of X such that $a := \inf \mathrm{val}(f\xi_A) > 0$. By hypothesis, there exists $y \in Y$ such that $|(T(f\xi_A))(y)| \geq \min\{1, \mathrm{infra}(T)\}a$. Since T is weakly separating, $c(T(f\xi_A)) \cap c(T(f\xi_{X-A})) = \emptyset$, and we deduce that $|(Tf)(y)| = |(T(f\xi_A))(y)| \geq \min\{1, \mathrm{infra}(T)\}a$. This implies that $Tf \neq 0$. □

As a consequence of the previous lemma, we can apply in our setting all results concerning weakly separating injective maps. In particular, as in [1], if $\beta_0 X$ stands for the Banaschewski compactification of X, we can define a continuous map $h : Y_0 \to \beta_0 X$ which has dense range. Notice that in [1], all results are given in the nonarchimedean context. Exactly the same proofs, with just the natural changes, are valid when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We use $\beta_0 X$ because we need a zerodimensional compactification, and this is not always possible taking βX.

The following lemma, which appears in [1], will be useful in the next results.

Lemma 2.2. *Suppose that $T : B(X) \to B(Y)$ is a weakly separating additive map. If y belongs to Y_0 and $f \in B(X)$ satisfies $(Tf)(y) \neq 0$, then $h(y)$ belongs to the closure of $c(f)$ in $\beta_0 X$.*

Lemma 2.3. *Let $T : B(X) \to B(Y)$ be an infrabounded-below map. Then, for every $y \in Y_0$, $h(y)$ belongs to X.*

Proof. Assume on the contrary that for some $y_0 \in Y_0$, $h(y_0) \notin X$. Then there exists a sequence (U_n) of clopen neighborhoods of $h(y_0)$ in $\beta_0 X$ such that U_{n+1} is strictly contained in U_n for every $n \in \mathbb{N}$, $U_1 = \beta_0 X$, and $X \cap \bigcap_{n=1}^{\infty} U_n = \emptyset$. For each $n \in \mathbb{N}$, define $V_n := U_n - U_{n+1}$.

On the other hand, since $y_0 \in Y_0$, we have that there exists $f_0 \in B(X)$ such that $(Tf_0)(y_0) \neq 0$. Also, by Claim 2 in Lemma 2.1, we have that $\gamma_0 := (T\xi_X)(y_0) \neq 0$.

We separate the archimedean case and the nonarchimedean one.

Case 1. $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. Here $B(X) = C(X)$. Define $g_1 := \sum_{n=1}^{\infty} n\xi_{V_n} \in C(X)$. Suppose that $(Tg_1)(y_0) = \alpha \neq 0$, and take $n_0 \in \mathbb{N}$ such that $|\alpha| \leq n_0 \mathrm{infra}(T)$. It is clear that, if $g_2 := \sum_{n=1}^{n_0} n\xi_{V_n}$, then $h(y_0)$ does not belong to the closure of $c(g_2)$ in $\beta_0 X$, and consequently, by Lemma 2.2, $(Tg_2)(y_0) = 0$. This implies that $(T(g_1 - g_2))(y_0) = \alpha$. Then we are in the situation that

$$\inf \mathrm{val}\,(T(g_1 - g_2)) \leq |\alpha| < (n_0 + 1)\mathrm{infra}(T) = \mathrm{infra}(T) \inf \mathrm{val}\,(g_1 - g_2).$$

Since this cannot happen, we conclude that $(Tg_1)(y_0) = 0$.

But notice that a similar reasoning would lead to the fact that $(T(g_1+\xi_X))(y_0) = (T\sum_{n=1}^{\infty}(n+1)\xi_{V_n})(y_0) = 0$. As a consequence, $(T\xi_X)(y_0) = 0$, which is a contradiction.

Case 2. $\mathbb{K}$ *is a nonarchimedean field.* Here $B(X) = C^*(X)$. Now, since $\mathbb{K}$ is not locally compact, we can take a sequence (α_n) in $\mathbb{K}$ such that $1/2 \leq |\alpha_n| \leq 1$, $|\alpha_n - 1| \geq 1/2$ for each $n \in \mathbb{N}$, and $1 - 1/(n+1) \leq |\alpha_n - \alpha_m|$ for every $n, m \in \mathbb{N}$, $n > m$.

Now define $g_0 := \xi_X$, $g_1 := \sum_{n=1}^{\infty} \alpha_n \xi_{V_n}$, and $g_2 := g_0 - g_1$. It is easy to see that these three functions belong to $C^*(X)$. Now it is clear that, since T is linear, we have $(Tg_1)(y_0) \neq 0$ or $(Tg_2)(y_0) \neq 0$. Also, since $\mathbb{K}$ is nonarchimedean, it is clear that there exist $i \in \{1, 2\}$ with $|\gamma_0| = |(Tg_0)(y_0)| \leq |(Tg_i)(y_0)|$. Since the proof will work in the same way otherwise, we assume that $i = 1$. Then $(Tg_1)(y_0) = \gamma_1$, with $|\gamma_1| \geq |\gamma_0|$. Next take $\delta \in \mathbb{K}$, $|\delta| \leq 1$, such that $0 < |\delta\gamma_1 + \gamma_0| < \mathrm{infra}(T)/2$.

Let us see that there exists at most one $n \in \mathbb{N}$ with $|\delta\alpha_n - 1| < 1/2$. This is clear if $|\delta| < 1$. So suppose that $|\delta| = 1$ and that we can take $n_1, n_2 \in \mathbb{N}$, $n_1 \neq n_2$, such that the distances from $\delta\alpha_{n_1}$ and $\delta\alpha_{n_2}$ to 1 are strictly less than $1/2$. As a consequence, $|\alpha_{n_1} - \alpha_{n_2}| = |\delta\alpha_{n_1} - \delta\alpha_{n_2}| < 1/2$, contradicting the way we have chosen the sequence (α_n). If it exists, let $n_0 \in \mathbb{N}$ such that $|\delta\alpha_{n_0} - 1| < 1/2$. It is clear that $h(y_0)$ does not belong to the closure of V_{n_0} in $\beta_0 X$, and consequently, if we define $k_0 := g_0 - g_0\xi_{V_{n_0}}$, and $k_1 := g_1 - g_1\xi_{V_{n_0}}$, then by Lemma 2.2 $(Tk_0)(y_0) = \gamma_0$, and $(Tk_1)(y_0) = \gamma_1$.

We deduce that if $x \in V_{n_0}$, then $(k_0 - \delta k_1)(x) = 0$. On the other hand, if $x \in X - V_{n_0}$, then there exists $n \in \mathbb{N} - \{n_0\}$ such that $x \in V_n$, and in this way,

$$\begin{aligned} |(k_0 - \delta k_1)(x)| &= |g_0(x) - \delta g_1(x)| \\ &= |g_0(x)|\,|1 - \delta\alpha_n| \\ &\geq 1/2. \end{aligned}$$

We conclude that $\inf \operatorname{val}(k_0 + \delta k_1) \geq 1/2$, and by hypothesis, that

$$\inf \operatorname{val}(Tk_0 + \delta Tk_1) \geq \operatorname{infra}(T)/2.$$

But notice that $|(Tk_0)(y_0) + \delta(Tk_1)(y_0)| = |\gamma_0 + \delta\gamma_1|$ belongs to $(0, \operatorname{infra}(T)/2)$. This contradiction implies that $h(y_0)$ belongs to X.

Consequently the lemma is proved. □

Lemma 2.4. *Let $T : B(X) \to B(Y)$ be an infrabounded-below map. If $y \in Y_0$ and $f \in B(X)$ satisfy $f(h(y)) = 0$, then $(Tf)(y) = 0$.*

Proof. Consider $y_0 \in Y_0$ and $f_0 \in B(X)$ such that $f_0(h(y_0)) = 0$. We are going to prove that $(Tf_0)(y_0) = 0$. Suppose that this is not the case, but $(Tf_0)(y_0) = \alpha \neq 0$.

Once again, at this point the archimedean and the nonarchimedean cases deserve separate attention.

Case 1. $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. First notice that, by Claim 2 in Lemma 2.1, we have $c(Tf_0) \subset c(T\xi_X)$. Also, multiplying by a constant if necessary, we can assume that $\alpha = (T\xi_X)(y_0)/2$. Now take an open neighborhood U of $h(y_0)$ such that $|f_0(x)| < M$ for every $x \in U$, where $M > 0$ satisfies $(1 - M)\operatorname{infra}(T) > |(T\xi_X)(y_0)/2|$. Consider $g_0 \in C(X)$ such that $g_0 \equiv 1$ on a neighborhood of $h(y_0)$, $0 \leq g_0 \leq 1$, and $g_0(X - U) \equiv 0$. It is clear that, since T is weakly separating and $f_0 = f_0 g_0 + f_0(1 - g_0)$, then by Lemma 2.2, $(Tf_0 g_0)(y_0) = \alpha$. It is also clear that $\inf \operatorname{val}(\xi_X - f_0 g_0) \geq 1 - M$, and consequently

$$\inf \operatorname{val}(T\xi_X - Tf_0 g_0) \geq (1 - M)\operatorname{infra}(T),$$

which is in contradiction with the fact that

$$|(T\xi_X - Tf_0 g_0)(y_0)| = |\alpha| < (1 - M)\operatorname{infra}(T).$$

Case 2. $\mathbb{K}$ *is a nonarchimedean field.* As above, we have $c(Tf_0) \subset c(T\xi_X)$. In the same way, multiplying by a constant if necessary, we can assume that $0 < |\alpha - (T\xi_X)(y_0)| < \operatorname{infra}(T)$. Now take the set $U_0 := \{x \in X : |f_0(x)| < 1\}$, which is clopen. By Lemma 2.2, we see that $(T(f_0 \xi_{X - U_0}))(y_0) = 0$, and consequently $(T(f_0 \xi_{U_0}))(y_0) = \alpha$. We easily check that $\operatorname{val}(f_0 \xi_{U_0} + \xi_X) = \{1\}$, and consequently $\inf \operatorname{val}(T(f_0 \xi_{U_0} + \xi_X)) \geq \operatorname{infra}(T)$. But, on the other hand, we know that $|(T(f_0 \xi_{U_0} + \xi_X))(y_0)| \in (0, \operatorname{infra}(T))$, which is a contradiction.

Consequently, in both cases we conclude that $(Tf_0)(y_0) = 0$. □

Lemma 2.5. *Let $T : B(X) \to B(Y)$ be an infrabounded-below map. Then $c(T\xi_X)$ is clopen.*

Proof. The proof is immediate once we notice that $c(T\xi_X) = \{y \in Y : |(T\xi_X)(y)| \geq \operatorname{infra}(T)/2\} = \{y \in Y : |(T\xi_X)(y)| > \operatorname{infra}(T)/2\}$. □

Theorem 2.6. *Let $T : B(X) \to B(Y)$ be an infrabounded-below map. Then it is a weighted composition map.*

Proof. Define $a := T\xi_X \in B(Y)$. Let $y_0 \in Y_0$. By Lemma 2.3, we know that $h(y_0) \in X$. Now take $f \in B(X)$, and suppose that $f(h(y_0)) \neq 0$. Clearly, if we define $k \in B(X)$ as $k(x) := f(x)/f(h(y_0))$ for $x \in X$, we have $k(h(y_0)) = \xi_X(h(y_0))$ and by Lemma 2.4, $(Tk)(y_0) = a(y_0)$, that is, $(Tf)(y_0) = a(y_0)f(h(y_0))$. On the other hand, also by Lemma 2.4, it is clear that if $f(h(y_0)) = 0$, then $(Tf)(y_0) = 0$ and $(Tf)(y_0) = a(y_0)f(h(y_0))$ holds too. But this can be stated not only for points $y \in Y_0$. In particular, if x_0 is any point in X by defining $h(y) := x_0$ for every $y \in Y - Y_0$, we also have $(Tf)(y) = a(y)f(h(y))$ for every $y \in Y$ and every $f \in B(X)$, because $a \equiv 0$ on $Y - Y_0$.

On the other hand, it is easy to see, by using Claim 2 of Lemma 2.1, that $Y_0 = c(T\xi_X)$, and by Lemma 2.5, it is clopen. This implies that the new h defined on the whole Y is continuous.

Finally the fact that h has dense range in X was mentioned after Lemma 2.1. □

3. The surjective case

In this section we also assume that the space Y is $\mathbb{N}$-compact. On the other hand, recall that a bijective additive map $T : B(X) \to B(Y)$ is said to be separating if $(Tf)(Tg) \equiv 0$ whenever $fg \equiv 0$ $(f, g \in B(X))$, and that it is said to be biseparating if both T and T^{-1} are separating.

Theorem 3.1. *Let $T : B(X) \to B(Y)$ be a surjective and infrabounded-below map. Then it is a weighted composition map, where h is a (surjective) homeomorphism and $|a(y)| \geq \mathrm{infra}(T)$ for every $y \in Y$.*

Proof. By Theorem 2.6, there exist $a \in B(Y)$ and a continuous map $h : Y \to X$ of dense range such that $(Tf)(y) = a(y)f(h(y))$ for every $y \in Y$ and every $f \in B(X)$. Also, as we saw in the proof of Theorem 2.6, $c(a) = Y_0$, and consequently, in our case, $c(a) = Y$, that is, $a(y) \neq 0$ for every $y \in Y$. On the other hand, since $a = T\xi_X$ and T is infrabounded-below, we deduce that $|a(y)| \geq \mathrm{infra}(T)$ for every $y \in Y$.

Next, T is surjective, and then, by Lemma 2.1, it is bijective. Also it is easy to see that T is separating (and linear). Finally, taking into account that X is zerodimensional, when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we know by [3, Theorem 17] that T is biseparating. Also, since both X and Y are realcompact, we conclude by [2, Proposition 3] that h is a surjective homeomorphism. When $\mathbb{K}$ is nonarchimedean, the same result follows from [1, Theorems 3.1 and 3.2]. □

Theorem 3.2. *Let $T : B(X) \to B(Y)$ be surjective and linear. Then the following statements are equivalent:*

(1) *T is a Banach-Stone map.*

(2) $\inf \mathrm{val}\,(f) = \inf \mathrm{val}\,(Tf)$ *for every $f \in B(X)$.*

Also, if $\mathbb{K}$ is nonarchimedean, both statements are equivalent to

(3) *If $f \in B(X)$, then* $\mathrm{val}\,(Tf) = \{1\}$ *whenever* $\mathrm{val}\,(f) = \{1\}$.

Proof. (1) ⇒ (2) and (1) ⇒ (3) are easy.

As for (2) ⇒ (1), by Theorem 3.1, we know that $Tf = a \cdot f \circ h$ for every $f \in B(X)$, where h is a (surjective) homeomorphism and $|a(y)| \geq \mathrm{infra}(T) = 1$ for every $y \in Y$.

We are going to see that $|a(y)| = 1$ for every $y \in Y$. Suppose that there is $y_0 \in Y$ such that $|a(y_0)| > 1$. Then take $\epsilon > 0$ and a clopen neighborhood U of y_0 such that $|a(y)| > 1 + \epsilon$ for every $y \in U$. Define $f_0 := T^{-1}\xi_U$. By applying the hypothesis it is clear that there exists a point $y_1 \in U$ such that $|f_0(h(y_1))| \geq 1$. Now

$$1 = |\xi_U(y_1)| = |(Tf_0)(y_1)| = |a(y_0)f_0(h(y_1))| \geq 1 + \epsilon,$$

which is absurd. We conclude that $|a(y)| = 1$ for every $y \in Y$ and we are done.

Finally we are going to see (3) ⇒ (1). First, we have, by Lemma 2.1, that T is weakly separating and bijective. In [1, Theorem 3.1], it is proved that, when $\mathbb{K}$ is nonarchimedean (even if it is locally compact), every bijective weakly separating linear map is biseparating. Now, using [1, Theorem 3.2], there exist $a \in B(Y)$ and a (surjective) homeomorphism $h : Y \to X$ such that, for every $f \in B(X)$ and every $y \in Y$, $(Tf)(y) = a(y)f(h(y))$. Also $a = T\xi_X$, and consequently $v(a) = \{1\}$. Now, since $Y_0 = c(a) = Y$, we conclude that T is a Banach-Stone map. □

Remarks. 1. Recall that in classical results concerning the Banach-Stone theorem, (local) compactness on the topological spaces X and Y is assumed. Notice that this is not the case in Theorem 3.2, where we do not call for (local) compactness on the underlying spaces.

2. In general, we have to assume $\mathbb{N}$-compactness of our spaces X and Y. For instance, consider a pseudocompact not $\mathbb{N}$-compact space X (e.g., the set of ordinal numbers less than the first uncountable ordinal number, [7, 5.12]) and let Y be its $\mathbb{N}$-compactification, which is compact (see for instance [8, page 44]). It is easy to see that the canonical isomorphism $T : C(X) \to C(Y)$, sending each $f \in C(X)$ into its continuous extension to Y, cannot be described as a weighted composition map.

3. In the proof of Theorem 3.2 ((3) ⇒ (1)), when $\mathbb{K}$ is not locally compact, and X and Y are compact, the result also holds applying [3, Corollary 21] instead of [1, Theorem 3.2]. Exactly the same comment applies to Theorem 3.1.

4. Banach-Stone maps and preservation of the infimum

Unlike the rest of the paper, in this section we assume that we are in one of the following situations:

- **Situation 1.** $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In this case X and Y will denote Hausdorff compact spaces (not necessarily zerodimensional) and $B(X) = C(X)$, $B(Y) = C(Y)$.
- **Situation 2.** $\mathbb{K}$ is a commutative complete field endowed with a nontrivial nonarchimedean valuation. If $\mathbb{K}$ is not locally compact, then X and Y will denote $\mathbb{N}$-compact spaces, and $B(X) = C^*(X)$, $B(Y) = C^*(Y)$. If $\mathbb{K}$ is locally compact, then X and Y will denote compact and zerodimensional spaces, and $B(X) = C(X)$, $B(Y) = C(Y)$.

Now $B(X)$ and $B(Y)$ become Banach spaces when endowed with the sup norm $\|\cdot\|_\infty$.

At this point, it is worth mentioning that when we replace the real or complex field for a nonarchimedean one, the Banach-Stone theorem does not hold. Given a zerodimensional compact space X, when we study linear isometries from the space $C(X)$ of $\mathbb{K}$-valued continuous functions onto itself, not only may they not be weighted composition maps ([5, 4]), but the set of linear surjective isometries which are not of this kind is dense in the space of all linear isometries ([4]).

Theorem 4.1. *Let $T : B(X) \to B(Y)$ be surjective and linear. Then the following statements are equivalent:*

(1) *T is a Banach-Stone map.*

(2) $\inf_{x\in X} |f(x)| = \inf_{y\in Y} |(Tf)(y)|$ *for every $f \in B(X)$.*

Proof. (1) ⇒ (2) is easy.

Next we prove (2) ⇒ (1). Suppose that $\|f\|_\infty = 1$. Then, for every $\alpha \in \mathbb{K}$, $|\alpha| < 1$, $\inf_{x\in X} |(\xi_X - \alpha f)(x)| > 0$, and consequently $\inf_{y\in Y} |(T\xi_X - \alpha Tf)(y)| > 0$. This implies that, for every $y \in Y$, $|\alpha|\,|(Tf)(y)| \neq |(T\xi_X)(y)|$. Since this happens for every $\alpha \in \mathbb{K}$, $|\alpha| < 1$, we conclude that $|(Tf)(y)| \leq |(T\xi_X)(y)|$ for every $y \in Y$.

On the other hand, since $|(T\xi_X)(y)| \geq 1$ for every $y \in Y$, we can define a new surjective linear map $S : B(X) \to B(Y)$ as $Sf := Tf/T\xi_X$, $f \in B(X)$. Now, from what we saw above, we have that $\|Sf\|_\infty \leq 1$ whenever $\|f\|_\infty = 1$. Let us see that S is an isometry, that is, that $\|Sf\|_\infty = 1$ for such an f. If this is not the case, there exists $M \in (0,1)$ such that $|(Sf)(y)| \leq M$ for every $y \in Y$. Then $|(Tf)(y)| \leq M\,|(T\xi_X)(y)|$ for every $y \in Y$. It is easy to see that, whenever $\alpha \in \mathbb{K}$, $|\alpha| < 1/M$, then for every $y \in Y$, $|(T\xi_X)(y) - \alpha(Tf)(y)| \geq |(T\xi_X)(y)| - |\alpha|\,|(Tf)(y)| \geq 1 - |\alpha|\,M$. Consequently, in this case $\inf_{y\in Y} |(T\xi_X - \alpha Tf)(y)| > 0$, that is, $\inf_{x\in X} |(\xi_X - \alpha f)(x)| > 0$. But notice that this is not true, because $\|f\|_\infty = 1$. We deduce that S is an isometry.

Now suppose that $\mathbb{K}$ is nonarchimedean. We are going to prove that $\operatorname{val}(Sf) = \{1\}$ whenever $\operatorname{val}(f) = \{1\}$, $f \in B(X)$. Suppose that there exists a point $y_0 \in Y$ such that $r := |(Sf)(y_0)| \in (0,1)$. Then $|(Tf)(y_0)| = r\,|(T\xi_X)(y_0)|$, and we can take $\alpha \in \mathbb{K}$, $|\alpha| = r$, such that $\alpha(T\xi_X)(y_0) = (Tf)(y_0)$. Then it is clear that $\inf_{y\in Y} |(\alpha T\xi_X - Tf)(y)| = 0$, but it is easy to see that $\inf_{x\in X} |(\alpha\xi_X + f)(x)| \geq r$, which contradicts our hypothesis. As a consequence, $\operatorname{val}(Sf) \subset [1, +\infty)$ and, since S is an isometry, we conclude that $\operatorname{val}(Sf) = \{1\}$.

So applying the Banach-Stone theorem in the case $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and applying Theorem 3.2 (see also Remark 3 after it) when $\mathbb{K}$ is nonarchimedean, we conclude that S is a Banach-Stone map, that is, there exists a homeomorphism h from Y onto X and a function $a \in B(Y)$, $|a| \equiv 1$, such that $(Sf)(y) = a(y)f(h(y))$ for every $f \in B(X)$ and every $y \in Y$. This implies that, for every $y \in Y$ and every $f \in B(X)$, $(Tf)(y) = (T\xi_X)(y)a(y)f(h(y))$. Also, we know that $|(T\xi_X)(y)| \geq 1$ for every $y \in Y$. Consequently, if we want to prove that T is a Banach-Stone map, it is enough to see that $|(T\xi_X)(y)| \leq 1$ for every $y \in Y$.

Notice first that, for every $g \in B(Y)$, we have $T^{-1}g = g \circ h^{-1}/(T\xi_X \cdot a) \circ h^{-1}$. In particular, if we take $g := \xi_Y \in B(Y)$, this implies that $T^{-1}\xi_Y = 1/(T\xi_X \cdot a) \circ h^{-1}$. Then it is clear that if $\|T\xi_X\|_\infty > 1$, then $\inf_{x\in X} \left|(T^{-1}\xi_Y)(x)\right| < \inf_{y\in Y} |\xi_Y(y)|$, which goes against our hypothesis. Consequently $T\xi_X \equiv 1$ and we are done.

We conclude that T is a Banach-Stone map. □

Remark. In all the results given in this paper the assumption of ℕ-compactness on the space X plays a relevant role. Also, by the example given before Section 2, we conclude that we cannot expect to describe any infrabounded-below map as a weighted composition whenever X contains an open connected subset with more than one point. We end the paper with the following more general question.

Problem. Assuming that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, characterize all completely regular spaces X such that every infrabounded-below map defined on $C(X)$ can be described as a weighted composition map.

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