PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 4, Pages 1165–1167
S 0002-9939(01)06151-2
Article electronically published on September 19, 2001

ALMOST POSITIVE CURVATURE ON THE GROMOLL-MEYER 7-SPHERE

J.-H. ESCHENBURG

(Communicated by Wolfgang Ziller)

ABSTRACT. D. Gromoll and W. Meyer have represented a certain exotic 7-sphere M as a biquotient of the compact Lie group $Sp(2)$. Thus any invariant normal homogeneous metric on $Sp(2)$ induces a metric of nonnegative sectional curvature on M. We show that the simplest such metrics (except the bi-invariant one) induce metrics which have in fact strictly positive curvature outside a subset of M with measure zero.

There are only very few compact manifolds known which allow metrics of strictly positive sectional curvature. But recently it has been shown ([PW], [Wk]) that much more spaces satisfy a condition which seems to be only slightly weaker: A Riemannian manifold M is said to have *almost positive curvature* if it has positive curvature on an open subset $M_0 \subset M$ such that $M \setminus M_0$ is a set of measure zero.

D. Gromoll and W. Meyer [GM] constructed a metric of nonnegative sectional curvature on the exotic 7-sphere $M = G/U$ where $G = Sp(2)$ and

$$U = \{(\left(\begin{smallmatrix} q & \\ & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} q & \\ & q \end{smallmatrix}\right));\ q \in Sp(1)\} \subset G \times G.$$

In fact, a subgroup $U \subset G \times G$ acts on G by left and right multiplication: $(u_1, u_2).g := u_1 g u_2^{-1}$. If this action is free, the orbit space G/U is a smooth manifold, called a *biquotient*. Any normally homogeneous metric on G has nonnegative curvature, and if this metric is also U-invariant, it induces a metric on the orbit space which has also nonnegative curvature by O'Neill's formulas for Riemannian submersions. For the bi-invariant metric and many other normal homogeneous metrics on $Sp(2)$, the curvature on $M = Sp(2)/U$ is even strictly positive near the point $U.e$ where $e \in Sp(2)$ is the identity, but this cannot hold on the whole manifold ([E1]). How large is the subset $M_0 \subset M$ where the curvature is strictly positive? It is known ([W]) that for the bi-invariant metric $M \setminus M_0$ contains an open subset, so this metric does not have almost positive curvature in the above sense. However the property does hold for the simplest normally homogeneous metrics on $Sp(2)$ which are not bi-invariant. Using arguments taken from [E1] we will show that $M \setminus M_0$ is essentially a hypersurface. F. Wilhelm [W] has shown almost positivity for another set of metrics on M, but his computations are much more involved.

Let $K = Sp(1) \times Sp(1) \subset Sp(2) = G$. Then G is equivariantly diffeomorphic to the homogeneous space $(G \times K)/K$ where K sits diagonally in $G \times K$. A bi-invariant metric on $G \times K$ thus induces a normally homogeneous metric on G. Note

Received by the editors September 28, 2000 and, in revised form, October 23, 2000.
2000 *Mathematics Subject Classification.* Primary 53C20, 53C30; Secondary 57S25, 57R60.
Key words and phrases. Biquotients, exotic 7-sphere, quaternions, zero curvature set.

that $G/K = \mathbb{H}P^1 = S^4$ is a symmetric space. Such metrics are described in detail in [E2]. They are induced by certain $Ad(K)$-invariant inner products on the Lie algebra $\mathfrak{g}$ and have nonnegative curvature (by O'Neill's formula). Moreover, the 2-planes with curvature zero are those spanned by two orthogonal vectors $X, Y \in \mathfrak{g}$ with

$$[X,Y] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = 0 \tag{1}$$

where $X_{\mathfrak{k}}$ and $X_{\mathfrak{p}}$ are the components of X with respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Since G/K is a rank-one symmetric space, there are no vanishing commutators in $\mathfrak{p}$; thus we may assume that Y has no $\mathfrak{p}$-component, i.e. $Y = \left(\begin{smallmatrix} y & 0 \\ 0 & z \end{smallmatrix}\right) \in \mathfrak{k}$ where y, z are imaginary quaternions. Let $X_{\mathfrak{p}} = \left(\begin{smallmatrix} 0 & -\bar{x} \\ x & 0 \end{smallmatrix}\right)$ for some nonzero $x \in \mathbb{H}$. Then $[X_{\mathfrak{p}}, Y] = 0$ iff $zx = xy$ or

$$z = xyx^{-1}. \tag{2}$$

The infinitesimal action of the Lie algebra $\mathfrak{u}$ of U on G is given as follows: For any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sp(2)$ we have $V_g := g^{-1}(\mathfrak{u}.g) = \{v_g;\ v \in \mathbb{R}^3\}$ where $\mathbb{R}^3 \subset \mathbb{H}$ denotes the set of imaginary quaternions (the Lie algebra of $Sp(1)$) and where

$$v_g = Ad(g^*) \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} \bar{a}va - v & \bar{a}vb \\ \bar{b}va & \bar{b}vb - v \end{pmatrix}. \tag{3}$$

In order to have zero curvature at the point $U.g \in G/U$ we need to find perpendicular $X, Y \perp V_g$ satisfying (1), thus spanning a *horizontal zero curvature plane at g*, and in fact this condition is also sufficient (cf. [E1], p. 31, and [GM]).

Theorem. *Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Sp(2)$ with $a, b \neq 0$. There exists a zero curvature plane at $U.g \in G/U$ iff*

$$\det(I - Ad(b^{-1}) - Ad(a^{-1})) = 0. \tag{$*$}$$

Proof. Let $X, Y \perp V_g$ with (1), spanning a zero curvature plane. Our first claim is that $X_{\mathfrak{k}}$ and $Y_{\mathfrak{k}}$ are linearly dependent. In fact, since $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$, we may assume $X_{\mathfrak{k}} = \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $Y_{\mathfrak{k}} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & y \end{smallmatrix}\right)$ for $x, y \in \mathbb{R}^3$. Thus $\langle v_g, X\rangle = \langle \bar{a}va - v, x\rangle = \langle v, ax\bar{a} - x\rangle$ and likewise $\langle v_g, Y\rangle = \langle v, by\bar{b} - y\rangle$. This vanishes for all $v \in \mathbb{R}^3$ iff $ax\bar{a} = x$ and $by\bar{b} = y$. If both x, y are nonzero, we have $|a|^2 = |b|^2 = 1$ which is impossible since $|a|^2 + |b|^2 = 1$ (recall that g is unitary).

Thus we may assume $X_{\mathfrak{k}} = 0$ and hence by (2)

$$X = \begin{pmatrix} 0 & -\bar{x} \\ x & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y & 0 \\ 0 & xyx^{-1} \end{pmatrix}. \tag{4}$$

Now

$$\langle v_g, X\rangle = 2\langle \bar{b}va, x\rangle = 2\langle v, bx\bar{a}\rangle,$$

and this vanishes if $bx\bar{a}$ is perpendicular to $\mathbb{R}^3 \subset \mathbb{H}$, hence a real number. Thus if $a \neq 0$, we get

$$bx = ta \tag{5}$$

for some nonzero $t \in \mathbb{R}$. Moreover, $\langle v_g, Y\rangle = \langle v, ay\bar{a} - y + bxyx^{-1}\bar{b} - xyx^{-1}\rangle$ vanishes for all $v \in \mathbb{R}^3$ iff

$$ay\bar{a} - y + bxyx^{-1}\bar{b} - xyx^{-1} = 0. \tag{6}$$

By (5) we have $bxyx^{-1}\bar{b} = |b|^2 bxy(bx)^{-1} = |b|^2 aya^{-1}$ if also $b \neq 0$. Hence

$$ay\bar{a} + bxyx^{-1}\bar{b} = |a|^2 aya^{-1} + |b|^2 aya^{-1} = aya^{-1} = Ad(a)y. \tag{7}$$

Further (5) implies $Ad(x) = Ad(b^{-1}a)$. Therefore $\langle v_g, Y\rangle = 0$ iff

$$Ad(a)y - Ad(b^{-1})Ad(a)y - y = 0. \tag{8}$$

Thus $Ad(a)y \neq 0$ is in the kernel of $I - Ad(b^{-1}) - Ad(a^{-1})$ which implies that the determinant of that matrix vanishes.

Vice versa, if $\det(I - Ad(b^{-1}) - Ad(a^{-1})) = 0$, we find a nonzero $y \in \mathbb{R}^3$ such that $Ad(a)y$ is in the kernel of this matrix. Now putting $x = b^{-1}a$ and defining X, Y by (4), we obtain a horizontal zero curvature plane at g. $\square$

Remarks. 1. We can determine the horizontal zero curvature planes also in the cases $a = 0$ or $b = 0$, using (6). E.g. if $b = 0$, then (6) becomes $aya^{-1} - y - xyx^{-1} = 0$ which is solvable precisely for those a such that $Ad(a)$ turns some vector $y \in \mathbb{R}^3$ by the angle $\pi/3$; then $|Ad(a)y - y| = |y|$, and we find some $x \in \mathbb{H}$ with $Ad(x)y = Ad(a)y - y$. Thus a horizontal zero curvature plane at such g exists if and only if the (minimal) rotation angle of $Ad(a)$ is $\geq \pi/3$.

2. Note that equation $(*)$ for g in the Theorem is invariant under the action of U and thus determines a hypersurface (possibly with singularities) in G/U. In fact, if $u = \left(\left(\begin{smallmatrix} q & \\ & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} q & \\ & q \end{smallmatrix}\right)\right) \in U$, then

$$u.g = \begin{pmatrix} qaq^{-1} & qbq^{-1} \\ cq^{-1} & dq^{-1} \end{pmatrix}.$$

Thus a and b become conjugated by q which does not change the determinant equation.

References

[E1] J.-H. Eschenburg: *Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen*, Schriftenr. Math. Inst. Univ. Münster (2) **32** (1984) MR **86a:**53045

[E2] J.-H. Eschenburg: *Inhomogeneous spaces of positive curvature*, Diff. Geom. Appl. **2** (1992), 123-132 MR **94j:**53044

[GM] D. Gromoll and W.T. Meyer: *An exotic sphere with nonnegative sectional curvature*, Ann. of Math. **100** (1974), 401 - 406 MR **51:**11347

[PW] P. Petersen and F. Wilhelm: *Examples of Riemannian manifolds with positive curvature almost everywhere*, Geom. and Top. **3** (1999), 331 - 367 MR **2000g:**53030

[W] F. Wilhelm: *An exotic sphere with positive curvature almost everywhere*, Preprint Riverside 1999

[Wk] B. Wilking: *Manifolds with positive sectional curvature almost everywhere*, preprint

Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Germany

E-mail address: `eschenburg@math.uni-augsburg.de`