

## ALMOST POSITIVE CURVATURE ON THE GROMOLL-MEYER 7-SPHERE

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ABSTRACT. D. Gromoll and W. Meyer have represented a certain exotic 7-sphere  $M$  as a biquotient of the compact Lie group  $Sp(2)$ . Thus any invariant normal homogeneous metric on  $Sp(2)$  induces a metric of nonnegative sectional curvature on  $M$ . We show that the simplest such metrics (except the bi-invariant one) induce metrics which have in fact strictly positive curvature outside a subset of  $M$  with measure zero.

There are only very few compact manifolds known which allow metrics of strictly positive sectional curvature. But recently it has been shown ([PW], [Wk]) that much more spaces satisfy a condition which seems to be only slightly weaker: A Riemannian manifold  $M$  is said to have *almost positive curvature* if it has positive curvature on an open subset  $M_0 \subset M$  such that  $M \setminus M_0$  is a set of measure zero.

D. Gromoll and W. Meyer [GM] constructed a metric of nonnegative sectional curvature on the exotic 7-sphere  $M = G/U$  where  $G = Sp(2)$  and

$$U = \{((\begin{smallmatrix} q & \\ & 1 \end{smallmatrix}), (\begin{smallmatrix} q & \\ & q \end{smallmatrix})); q \in Sp(1)\} \subset G \times G.$$

In fact, a subgroup  $U \subset G \times G$  acts on  $G$  by left and right multiplication:  $(u_1, u_2).g := u_1 g u_2^{-1}$ . If this action is free, the orbit space  $G/U$  is a smooth manifold, called a *biquotient*. Any normally homogeneous metric on  $G$  has nonnegative curvature, and if this metric is also  $U$ -invariant, it induces a metric on the orbit space which has also nonnegative curvature by O'Neill's formulas for Riemannian submersions. For the bi-invariant metric and many other normal homogeneous metrics on  $Sp(2)$ , the curvature on  $M = Sp(2)/U$  is even strictly positive near the point  $U.e$  where  $e \in Sp(2)$  is the identity, but this cannot hold on the whole manifold ([E1]). How large is the subset  $M_0 \subset M$  where the curvature is strictly positive? It is known ([W]) that for the bi-invariant metric  $M \setminus M_0$  contains an open subset, so this metric does not have almost positive curvature in the above sense. However the property does hold for the simplest normally homogeneous metrics on  $Sp(2)$  which are not bi-invariant. Using arguments taken from [E1] we will show that  $M \setminus M_0$  is essentially a hypersurface. F. Wilhelm [W] has shown almost positivity for another set of metrics on  $M$ , but his computations are much more involved.

Let  $K = Sp(1) \times Sp(1) \subset Sp(2) = G$ . Then  $G$  is equivariantly diffeomorphic to the homogeneous space  $(G \times K)/K$  where  $K$  sits diagonally in  $G \times K$ . A bi-invariant metric on  $G \times K$  thus induces a normally homogeneous metric on  $G$ . Note

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that  $G/K = \mathbb{H}P^1 = S^4$  is a symmetric space. Such metrics are described in detail in [E2]. They are induced by certain  $Ad(K)$ -invariant inner products on the Lie algebra  $\mathfrak{g}$  and have nonnegative curvature (by O'Neill's formula). Moreover, the 2-planes with curvature zero are those spanned by two orthogonal vectors  $X, Y \in \mathfrak{g}$  with

$$(1) \quad [X, Y] = [X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = [X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = 0$$

where  $X_{\mathfrak{k}}$  and  $X_{\mathfrak{p}}$  are the components of  $X$  with respect to the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ . Since  $G/K$  is a rank-one symmetric space, there are no vanishing commutators in  $\mathfrak{p}$ ; thus we may assume that  $Y$  has no  $\mathfrak{p}$ -component, i.e.  $Y = \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} \in \mathfrak{k}$  where  $y, z$  are imaginary quaternions. Let  $X_{\mathfrak{p}} = \begin{pmatrix} 0 & -\bar{x} \\ x & 0 \end{pmatrix}$  for some nonzero  $x \in \mathbb{H}$ . Then  $[X_{\mathfrak{p}}, Y] = 0$  iff  $zx = xy$  or

$$(2) \quad z = xyx^{-1}.$$

The infinitesimal action of the Lie algebra  $\mathfrak{u}$  of  $U$  on  $G$  is given as follows: For any  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2)$  we have  $V_g := g^{-1}(\mathfrak{u}.g) = \{v_g; v \in \mathbb{R}^3\}$  where  $\mathbb{R}^3 \subset \mathbb{H}$  denotes the set of imaginary quaternions (the Lie algebra of  $Sp(1)$ ) and where

$$(3) \quad v_g = Ad(g^*) \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} \bar{a}va - v & \bar{a}vb \\ \bar{b}va & \bar{b}vb - v \end{pmatrix}.$$

In order to have zero curvature at the point  $U.g \in G/U$  we need to find perpendicular  $X, Y \perp V_g$  satisfying (1), thus spanning a *horizontal zero curvature plane at  $g$* , and in fact this condition is also sufficient (cf. [E1], p. 31, and [GM]).

**Theorem.** *Let  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2)$  with  $a, b \neq 0$ . There exists a zero curvature plane at  $U.g \in G/U$  iff*

$$(*) \quad \det(I - Ad(b^{-1}) - Ad(a^{-1})) = 0.$$

*Proof.* Let  $X, Y \perp V_g$  with (1), spanning a zero curvature plane. Our first claim is that  $X_{\mathfrak{k}}$  and  $Y_{\mathfrak{k}}$  are linearly dependent. In fact, since  $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$ , we may assume  $X_{\mathfrak{k}} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$  and  $Y_{\mathfrak{k}} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$  for  $x, y \in \mathbb{R}^3$ . Thus  $\langle v_g, X \rangle = \langle \bar{a}va - v, x \rangle = \langle v, ax\bar{a} - x \rangle$  and likewise  $\langle v_g, Y \rangle = \langle v, by\bar{b} - y \rangle$ . This vanishes for all  $v \in \mathbb{R}^3$  iff  $ax\bar{a} = x$  and  $by\bar{b} = y$ . If both  $x, y$  are nonzero, we have  $|a|^2 = |b|^2 = 1$  which is impossible since  $|a|^2 + |b|^2 = 1$  (recall that  $g$  is unitary).

Thus we may assume  $X_{\mathfrak{k}} = 0$  and hence by (2)

$$(4) \quad X = \begin{pmatrix} 0 & -\bar{x} \\ x & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y & 0 \\ 0 & xyx^{-1} \end{pmatrix}.$$

Now

$$\langle v_g, X \rangle = 2\langle \bar{b}va, x \rangle = 2\langle v, bx\bar{a} \rangle,$$

and this vanishes if  $bx\bar{a}$  is perpendicular to  $\mathbb{R}^3 \subset \mathbb{H}$ , hence a real number. Thus if  $a \neq 0$ , we get

$$(5) \quad bx = ta$$

for some nonzero  $t \in \mathbb{R}$ . Moreover,  $\langle v_g, Y \rangle = \langle v, ay\bar{a} - y + bxyx^{-1}\bar{b} - xyx^{-1} \rangle$  vanishes for all  $v \in \mathbb{R}^3$  iff

$$(6) \quad ay\bar{a} - y + bxyx^{-1}\bar{b} - xyx^{-1} = 0.$$

By (5) we have  $bxyx^{-1}\bar{b} = |b|^2bxy(bx)^{-1} = |b|^2aya^{-1}$  if also  $b \neq 0$ . Hence

$$(7) \quad ay\bar{a} + bxyx^{-1}\bar{b} = |a|^2aya^{-1} + |b|^2aya^{-1} = aya^{-1} = Ad(a)y.$$

Further (5) implies  $Ad(x) = Ad(b^{-1}a)$ . Therefore  $\langle v_g, Y \rangle = 0$  iff

$$(8) \quad Ad(a)y - Ad(b^{-1})Ad(a)y - y = 0.$$

Thus  $Ad(a)y \neq 0$  is in the kernel of  $I - Ad(b^{-1}) - Ad(a^{-1})$  which implies that the determinant of that matrix vanishes.

Vice versa, if  $\det(I - Ad(b^{-1}) - Ad(a^{-1})) = 0$ , we find a nonzero  $y \in \mathbb{R}^3$  such that  $Ad(a)y$  is in the kernel of this matrix. Now putting  $x = b^{-1}a$  and defining  $X, Y$  by (4), we obtain a horizontal zero curvature plane at  $g$ .  $\square$

*Remarks.* 1. We can determine the horizontal zero curvature planes also in the cases  $a = 0$  or  $b = 0$ , using (6). E.g. if  $b = 0$ , then (6) becomes  $aya^{-1} - y - xyx^{-1} = 0$  which is solvable precisely for those  $a$  such that  $Ad(a)$  turns some vector  $y \in \mathbb{R}^3$  by the angle  $\pi/3$ ; then  $|Ad(a)y - y| = |y|$ , and we find some  $x \in \mathbb{H}$  with  $Ad(x)y = Ad(a)y - y$ . Thus a horizontal zero curvature plane at such  $g$  exists if and only if the (minimal) rotation angle of  $Ad(a)$  is  $\geq \pi/3$ .

2. Note that equation (\*) for  $g$  in the Theorem is invariant under the action of  $U$  and thus determines a hypersurface (possibly with singularities) in  $G/U$ . In fact, if  $u = \left( \begin{pmatrix} q & \\ & 1 \end{pmatrix}, \begin{pmatrix} q \\ & q \end{pmatrix} \right) \in U$ , then

$$u.g = \begin{pmatrix} qaq^{-1} & qbq^{-1} \\ cq^{-1} & dq^{-1} \end{pmatrix}.$$

Thus  $a$  and  $b$  become conjugated by  $q$  which does not change the determinant equation.

#### REFERENCES

- [E1] J.-H. Eschenburg: *Freie isometrische Aktionen auf kompakten Lie-Gruppen mit positiv gekrümmten Orbiträumen*, Schriftenr. Math. Inst. Univ. Münster (2) **32** (1984) MR **86a**:53045
- [E2] J.-H. Eschenburg: *Inhomogeneous spaces of positive curvature*, Diff. Geom. Appl. **2** (1992), 123-132 MR **94j**:53044
- [GM] D. Gromoll and W.T. Meyer: *An exotic sphere with nonnegative sectional curvature*, Ann. of Math. **100** (1974), 401 - 406 MR **51**:11347
- [PW] P. Petersen and F. Wilhelm: *Examples of Riemannian manifolds with positive curvature almost everywhere*, Geom. and Top. **3** (1999), 331 - 367 MR **2000g**:53030
- [W] F. Wilhelm: *An exotic sphere with positive curvature almost everywhere*, Preprint Riverside 1999
- [Wk] B. Wilking: *Manifolds with positive sectional curvature almost everywhere*, preprint

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