

## REAL ANALYTIC SOLUTIONS OF PARABOLIC EQUATIONS WITH TIME-MEASURABLE COEFFICIENTS

JAY KOVATS

(Communicated by David S. Tartakoff)

ABSTRACT. We use Bernstein's technique to show that for any fixed  $t$ , strong solutions  $u(t, x)$  of the uniformly parabolic equation  $Lu := a^{ij}(t)u_{x_i x_j} - u_t = 0$  in  $Q$  are real analytic in  $Q(t) = \{x : (t, x) \in Q\}$ . Here,  $Q \subset \mathbb{R}^{d+1}$  is a bounded domain and the coefficients  $a^{ij}(t)$  are measurable. We also use Bernstein's technique to obtain interior estimates for pure second derivatives of solutions of the fully nonlinear, uniformly parabolic, concave equation  $F(D^2u, t) - u_t = 0$  in  $Q$ , where  $F$  is measurable in  $t$ .

### 0. INTRODUCTION

In [B], Bernstein introduced a method for estimating the maxima of the moduli of derivatives of any order of solutions of linear parabolic equations, under the assumption that the solution and all known functions of the equation are sufficiently smooth. This technique is outlined in §3.11.4 of [La], §4.17 of [LSU], §§5.2–5.4 of [K1], and §8.4 of [K2]. In the simple linear setting  $Lu := a^{ij}(t)u_{x_i x_j} - u_t = 0$ , the equation can be differentiated any number of times in  $x$  and so any derivative (in  $x$ ) of a solution is again a solution. Note that any difference quotient of any order (in  $x$ ) of a solution is also a solution. We assume that the equation is *uniformly parabolic*; i.e., there exist constants  $0 < \lambda \leq \Lambda$  with  $\lambda|\xi|^2 \leq a^{ij}(t)\xi_i \xi_j \leq \Lambda|\xi|^2$ ,  $\forall t$  and  $\forall \xi \in \mathbb{R}^d$ . We assume only that the  $a^{ij}(t)$  are measurable functions of  $t$ . By applying Bernstein's technique, we show that any strong solution of  $Lu = 0$  in  $Q$  is real analytic in  $x$ . This is a generalization of a result which appears in [M], namely if  $u(t, x)$  satisfies  $\Delta u - u_t = 0$  in  $Q$ , then for any fixed  $t$ , the mapping  $x \mapsto u(t, x)$  is analytic. From our result and estimate (1.1) below, it follows that if  $u$  is a solution of  $Lu = 0$  in  $Q$ , then for any fixed  $t$ , any second order derivative  $D^2u(t, \cdot)$  is locally Lipschitz continuous in  $x$  (see Corollary 1.4).

In [B], Brandt uses only the maximum principle applied to finite difference quotients of solutions to prove that second derivatives of strong solutions of  $Lu := a^{ij}(t)u_{x_i x_j} - u_t = f$ , where  $f(t, x)$  is Hölder continuous in  $x$  (with exponent  $\alpha \in (0, 1)$ ) are themselves locally Hölder continuous in  $x$ . In [L], Lieberman uses a Campanato space argument to show that second derivatives are actually locally Hölder continuous in both  $t$  and  $x$ . When  $f \equiv 0$ , the local Hölder continuity of  $D^2u$  in  $(t, x)$  is an immediate consequence of applying the Hölder continuity of *solutions* (see [KS] and §4.2 of [K1]) to solutions  $D^2u$ . Again, these results imply that strong

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Received by the editors October 4, 2000.

1991 *Mathematics Subject Classification*. Primary 35B65, 35K10.

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solutions of  $Lu = 0$  in  $Q$  satisfy a  $C_{\text{loc}}^{2,\alpha}(Q)$  estimate, where  $\alpha \in (0, 1)$ , while our result implies strong solutions satisfy a  $C_{\text{loc}}^{2,1}(Q(t))$  estimate, for any fixed  $t$ .

The fully nonlinear equation  $F(D^2u, t) - u_t = 0$  is *uniformly parabolic* if there exist constants  $0 < \lambda \leq \Lambda$  such that

$$(0.1) \quad \lambda \|N\| \leq F(M + N, t) - F(M, t) \leq \Lambda \|N\|,$$

$\forall M, N \in \mathbb{R}^{d^2}$  with  $N \geq 0$ , where for  $B \in \mathbb{R}^{d^2}$ ,  $\|B\| = \sqrt{\text{tr}(BB^t)} = (\sum_{i,j} b_{ij}^2)^{1/2}$ .

Under the assumption that  $F(M, t)$  is smooth and concave in  $M \in \mathbb{R}^{d^2}$  and measurable in  $t$ , we can apply Bernstein’s technique to obtain interior estimates on pure second derivatives of solutions of the equation  $F(D^2u, t) - u_t = 0$ . This is possible, since under these assumptions, any derivative of a smooth solution will be a solution to a linear parabolic equation with measurable coefficients, while any pure second derivative will be a subsolution of the same linear equation. This was done by Krylov in [K1], [K3] for more general parabolic equations  $F(D^2u, Du, u, t, x) - u_t = 0$ , using (in [K1]) a variation of Bernstein’s technique called the *monitored maximum method*. This estimate was the key step towards proving the  $W_2^{1,2}$  solvability of fully nonlinear equations when  $F$  is only measurable in  $t$ . (See §§5.4, 6.5 of [K1], §4 of [K3].)

Our notation is standard. (See, for example, [GT], [K2].) We remind the reader that for a point  $z_0 = (t_0, x_0) \in \mathbb{R}^{d+1}$ ,  $Q_\rho(z_0) = (t_0 - \rho^2, t_0) \times B_\rho(x_0)$ , while for a function  $v$ ,  $|v|_{0;Q} := \sup_Q |v|$ . We adopt the convention that repeated indices indicates summation from 1 to  $d$ , so that, for example,  $a^{ij}b^{ji} = \sum_{i,j=1}^d a^{ij}b^{ji} = \text{tr}(AB)$ .

1. LINEAR EQUATIONS WITH TIME-MEASURABLE COEFFICIENTS

**Theorem 1.1.**  $\forall Q_\rho(z_0) \subset\subset Q$  and any multi-index  $\alpha$ , any  $C^\infty$  (in  $x$ ) solution  $u$  of  $Lu := a^{ij}(t)u_{x_i x_j} - u_t = 0$  in  $Q$  satisfies

$$(1.1) \quad |D^\alpha u(z_0)| \leq \left(\frac{N|\alpha|}{\rho}\right)^{|\alpha|} |u|_{0;Q_\rho(z_0)}, \quad \text{where } N = N(d, \lambda, \Lambda).$$

*Proof.* We apply Bernstein’s technique. Take any  $z_0 = (t_0, x_0) \in Q$  and  $\rho > 0$  such that  $Q_\rho(z_0) \subset Q$  and take a function  $\varphi \in C^2(Q_\rho(z_0))$  with  $0 \leq \varphi \leq 1$ ,  $\varphi \equiv 0$  on  $\partial'Q_\rho(z_0)$ ,  $\varphi(z_0) = 1$  and

$$|D\varphi|^2 \leq \frac{N\varphi}{\rho^2}, \quad \|D^2\varphi\| \leq \frac{N}{\rho^2}, \quad |\varphi_t| \leq \frac{N}{\rho^2},$$

where  $N = N(d)$ . Consider, in  $Q_\rho(z_0)$ , the function  $w = \varphi^2 |Du|^2 + C\varphi^2$ , where the constant  $C$  will be determined later. By standard approximation techniques, we may assume  $a^{ij}(t) \in C^\infty$ . Our estimate (1.1) is independent of this smoothness. Then in  $Q_\rho(z_0)$ , since  $u$  and each  $u_{x_i}$  are solutions

$$\begin{aligned} Lw &= 2|Du|^2 a^{kj}(t) \varphi_{x_j} \varphi_{x_k} + 2\varphi |Du|^2 (a^{kj}(t) \varphi_{x_j x_k} - \varphi_t) \\ &\quad + 8\varphi a^{kj}(t) \varphi_{x_j} \sum_i u_{x_i} u_{x_i x_k} + 2\varphi^2 a^{kj}(t) \sum_i u_{x_i x_k} u_{x_i x_j} \\ &\quad + 2\varphi^2 \sum_i u_{x_i} [a^{kj}(t) u_{x_i x_j x_k} - u_{x_i t}] + 2C a^{kj}(t) u_{x_k} u_{x_j} \end{aligned}$$

$$\begin{aligned}
 &+ 2Cu (a^{kj}(t)u_{x_j x_k} - u_t) \\
 &= 2|Du|^2 a^{kj}(t)\varphi_{x_j} \varphi_{x_k} + 2\varphi|Du|^2 (a^{kj}(t)\varphi_{x_j x_k} - \varphi_t) \\
 &\quad + 8\varphi a^{kj}(t)\varphi_{x_j} \sum_i u_{x_i} u_{x_i x_k} + 2\varphi^2 a^{kj}(t) \sum_i u_{x_i x_k} u_{x_i x_j} + 2Ca^{kj}(t)u_{x_k} u_{x_j} \\
 &\geq 2|Du|^2 \lambda |D\varphi|^2 + 2\varphi|Du|^2 (a^{kj}(t)\varphi_{x_j x_k} - \varphi_t) \\
 &\quad - 8\varphi\Lambda|D\varphi| \cdot |Du| \cdot \|D^2u\| + 2\varphi^2 \lambda \|D^2u\|^2 + 2C\lambda|Du|^2 \\
 &\geq 2|Du|^2 \lambda |D\varphi|^2 + 2\varphi|Du|^2 (a^{kj}(t)\varphi_{x_j x_k} - \varphi_t + C\lambda) \\
 &\quad - 8\varphi\Lambda|D\varphi| \cdot |Du| \cdot \|D^2u\| + 2\varphi^2 \lambda \|D^2u\|^2
 \end{aligned}$$

since  $0 \leq \varphi \leq 1$  and  $\sum_i a^{kj}(t)u_{x_i x_k} u_{x_i x_j} \geq \sum_i \lambda |D(u_{x_i})|^2 = \lambda \sum_{i,k} u_{x_i x_k}^2 = \lambda \|D^2u\|^2$ .

In the second to the last inequality we have used Cauchy's inequality and the fact that  $|a^{kj}x_k y_j| \leq \sqrt{a^{kj}x_k x_j} \sqrt{a^{kj}y_k y_j}$  for any  $x, y \in \mathbb{R}^d$ . First choose  $C$  so large that  $|a^{ij}(t)\varphi_{x_i x_j} - \varphi_t| \leq \frac{C\lambda}{2}$ . Observe that  $|a^{ij}\varphi_{x_i x_j} - \varphi_t| \leq \|A\| \cdot \|D^2\varphi\| + |\varphi_t| \leq \frac{\Lambda d^{1/2}N}{\rho^2} + \frac{N}{\rho^2} = \frac{N}{\rho^2}(\Lambda d^{1/2} + 1)$ . So  $C \geq \frac{2N}{\lambda\rho^2}(\Lambda d^{1/2} + 1)$  will do. By Young's inequality  $\forall \varepsilon > 0$  we have

$$\begin{aligned}
 Lw &\geq 2|Du|^2 \lambda |D\varphi|^2 + \varphi|Du|^2 C\lambda - 8\Lambda \left( \varepsilon|Du|^2 |D\varphi|^2 + \frac{\varphi^2 \|D^2u\|^2}{\varepsilon} \right) \\
 &\quad + 2\varphi^2 \lambda \|D^2u\|^2 \\
 &\geq |Du|^2 |D\varphi|^2 \left( 2\lambda - 8\Lambda\varepsilon + \frac{C\lambda\rho^2}{N} \right) + \varphi^2 \|D^2u\|^2 \left( 2\lambda - \frac{8\Lambda}{\varepsilon} \right) \\
 &\geq |Du|^2 |D\varphi|^2 \left( \frac{C\lambda\rho^2}{N} - 8\Lambda\varepsilon \right) + \varphi^2 \|D^2u\|^2 \left( 2\lambda - \frac{8\Lambda}{\varepsilon} \right)
 \end{aligned}$$

Now choose  $\varepsilon$  st  $\frac{C\lambda\rho^2}{N} - 8\Lambda\varepsilon = \frac{C\lambda\rho^2}{2N}$ . This gives

$$Lw \geq |Du|^2 |D\varphi|^2 \frac{C\lambda\rho^2}{2N} + \varphi^2 \|D^2u\|^2 \left( 2\lambda - \frac{128\Lambda^2 N}{C\lambda\rho^2} \right).$$

Finally, choose  $C$  so large that  $2\lambda - \frac{128\Lambda^2 N}{C\lambda\rho^2} \geq 0$ . Observe that for

$$C = \max \left\{ \frac{2N}{\lambda\rho^2}(\Lambda d^{1/2} + 1), \frac{64\Lambda^2 N}{\lambda^2 \rho^2} \right\},$$

we have  $Lw \geq 0$  in  $Q_\rho(z_0)$ . Hence by the maximum principle  $\sup_{Q_\rho(z_0)} w = \sup_{\partial' Q_\rho(z_0)} w$  and so

$$|Du(z_0)|^2 \leq C \sup_{Q_\rho(z_0)} u^2 = \frac{N_1(\lambda, \Lambda, d)}{\rho^2} \sup_{Q_\rho(z_0)} u^2,$$

which gives (1.1) for  $|\alpha| = 1$ . So assume our claim is true for  $|\alpha| = k$ . We verify it holds for  $|\alpha| = k + 1$ . Fix  $z_0 \in Q$  and  $\rho > 0$  such that  $Q_\rho(z_0) \subset Q$ . Then  $Q_{\frac{\rho}{k+1}}(z_0) \subset Q_\rho(z_0) \subset Q$  and  $\alpha = \beta + e_i$ , for some  $i = 1, \dots, d$ , where  $|\beta| = k$ . Moreover for  $z \in Q_{\frac{\rho}{k+1}}(z_0)$ , we have  $Q_{\frac{k\rho}{k+1}}(z) \subset Q_\rho(z_0)$  and hence by the inductive

hypothesis

$$\begin{aligned} |D^\alpha u(z_0)| &= |(D^\beta u(z_0))_{x_i}| \leq \frac{N}{\left(\frac{\rho}{k+1}\right)} |D^\beta u|_{0;Q_{\frac{\rho}{k+1}}(z_0)} \\ &\leq \frac{N}{\left(\frac{\rho}{k+1}\right)} \cdot \left(\frac{Nk}{\left(\frac{k\rho}{k+1}\right)}\right)^k |u|_{0;Q_\rho(z_0)} \\ &= \left(\frac{N(k+1)}{\rho}\right)^{k+1} |u|_{0;Q_\rho(z_0)}. \end{aligned}$$

□

**Lemma 1.2.** Any strong solution  $u = u(t, x)$  of  $Lu := a^{ij}(t)u_{x_i x_j} - u_t = 0$  in  $Q$  is  $C^\infty$  in  $x$ .

*Proof.* For our solution  $u$ , consider  $u^\epsilon$ , the mollifier (in  $x$ ) of  $u$ . For any fixed  $t$  and any  $x$

$$\begin{aligned} (u^\epsilon)_t(t, x) &= (u_t)^\epsilon(t, x) = (a^{ij}(t)u_{x_i x_j})^\epsilon(t, x) \\ &= a^{ij}(t)(u_{x_i x_j})^\epsilon(t, x) = a^{ij}(t)(u^\epsilon)_{x_i x_j}(t, x), \end{aligned}$$

and hence  $u^\epsilon$  is a  $C^\infty$  (in  $x$ ) solution of  $Lu^\epsilon = 0$  and hence by Theorem 1, for any  $\alpha$

$$(1.2) \quad |D^\alpha u^\epsilon(z_0)| \leq \left(\frac{N|\alpha|}{\rho}\right)^{|\alpha|} |u^\epsilon|_{0;Q_\rho(z_0)} \leq \left(\frac{N|\alpha|}{\rho}\right)^{|\alpha|} |u|_{0;Q}.$$

That is, for any  $\alpha$ , the sequence  $\{D^\alpha u^\epsilon\}$  is uniformly bounded and equicontinuous. By the Arzela-Ascoli theorem, it has a subsequence  $\{D^\alpha u^\epsilon\}$  (by relabelling if necessary) converging uniformly on compacta of  $Q$  to some continuous function  $v = v_\alpha$ . Hence  $\forall \xi \in C_0^{|\alpha|}(Q(t))$ , where  $Q(t) = \{x : (t, x) \in Q\}$ , integration by parts gives

$$\begin{aligned} (-1)^{|\alpha|} \int_{Q(t)} u(t, x) D^\alpha \xi(x) dx &= \lim_{\epsilon \rightarrow 0} (-1)^{|\alpha|} \int_{Q(t)} u^\epsilon(t, x) D^\alpha \xi(x) dx \\ &= \lim_{\epsilon \rightarrow 0} \int_{Q(t)} D^\alpha (u^\epsilon)(t, x) \xi(x) dx = \int_{Q(t)} v(t, x) \xi(x) dx, \end{aligned}$$

that is,  $D^\alpha u = v$  in the weak sense. But (1.2) and the fact that  $\{D^\alpha u^\epsilon\} \rightarrow v$  implies  $v \in L^\infty_{loc}(Q)$ , i.e.  $u(t, \cdot) \in W^{|\alpha|, \infty}_{loc}(Q(t)) = C^{|\alpha|-1, 1}(Q(t))$ . Since  $\alpha$  is arbitrary,  $u(t, \cdot) \in C^\infty(Q(t))$ . □

**Theorem 1.3.** If  $Q \subset \mathbb{R}^{d+1}$  is a bounded domain, any solution  $u(t, x)$  of  $Lu := a^{ij}(t)u_{x_i x_j} - u_t = 0$  in  $Q$  is real analytic in  $x$ .

*Proof.* By Lemma 1.2, we know  $u$  is  $C^\infty$  in  $x$ . Fix any point  $z_0 = (t_0, x_0) \in Q$ . We will show that  $u$  can be represented as a power series (in  $x$ ) in a neighborhood of  $z_0$ . So set  $\rho = \frac{1}{4} \text{dist}(z_0, \partial'Q)$ . Observe that for  $z \in Q_\rho(z_0)$ ,  $Q_\rho(z) \subset Q_{2\rho}(z_0) \subset Q$  and hence for any such  $z$ , by Theorem 1 we have  $|D^\alpha u(z)| \leq \left(\frac{N|\alpha|}{\rho}\right)^{|\alpha|} |u|_{0;Q_\rho(z)} \leq$

$\left(\frac{N|\alpha|}{\rho}\right)^{|\alpha|} |u|_{0;Q_{2\rho}(z_0)} := \left(\frac{N|\alpha|}{\rho}\right)^{|\alpha|} M$ . By Taylor’s theorem, for any  $k \geq 1$  we have

$$u(t, x) = \sum_{|\alpha| \leq k-1} \frac{D^\alpha u(t_0, x_0)}{\alpha!} (x - x_0)^\alpha + \sum_{|\alpha|=k} \frac{D^\alpha u(t_0, \xi)}{\alpha!} (x - x_0)^\alpha,$$

where  $\xi = \gamma x + (1 - \gamma)x_0$  for some  $\gamma \in (0, 1)$ . Since  $(t_0, \xi) \in Q_\rho(z_0)$ , we have  $|D^\alpha u(t_0, \xi)| \leq \left(\frac{N|\alpha|}{\rho}\right)^{|\alpha|} M$ . Observe that the second sum on the right goes to zero as  $k \rightarrow \infty$  for  $|x - x_0| \leq \frac{\rho}{2Nde}$ , since by a strong version of Stirling’s formula,  $k^k < \frac{e^k k!}{\sqrt{2\pi}}$ , we have

$$\begin{aligned} \sum_{|\alpha|=k} \frac{|D^\alpha u(t_0, \xi)|}{\alpha!} |x - x_0|^{|\alpha|} &\leq \sum_{|\alpha|=k} \frac{1}{\alpha!} \left(\frac{N|\alpha|}{\rho}\right)^{|\alpha|} M \left(\frac{\rho}{2Nde}\right)^{|\alpha|} \\ &= \frac{M}{(2de)^k} \sum_{|\alpha|=k} \frac{|\alpha|^{|\alpha|}}{\alpha!} \\ &< \frac{M}{(2de)^k} \sum_{|\alpha|=k} \frac{|\alpha|! e^{|\alpha|}}{\sqrt{2\pi} \alpha!} \\ &= \frac{M}{(2d)^k \sqrt{2\pi}} \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} \\ &= \frac{M}{(2d)^k \sqrt{2\pi}} \cdot d^k \rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

and so  $u(t, x) = \sum_\alpha \frac{D^\alpha u(t_0, x_0)}{\alpha!} (x - x_0)^\alpha$ . □

**Corollary 1.4.** *If  $Q \subset \mathbb{R}^{d+1}$  is a bounded domain, any solution  $u(t, x)$  of  $Lu := a^{ij}(t)u_{x_i x_j} - u_t = 0$  in  $Q$  has locally Lipschitz second derivatives (in  $x$ ). That is, for any fixed  $t$ ,  $u(t, \cdot) \in C_{loc}^{2,1}(Q(t))$  and for any  $(t, x), (t, y) \in Q$  and  $i, j = 1, \dots, d$ , we have*

$$(1.3) \quad |u_{x_i x_j}(t, x) - u_{x_i x_j}(t, y)| \leq \frac{N_1}{\rho^3} \cdot |u|_{0;Q} \cdot |x - y|,$$

where  $N_1 = N_1(d, \lambda, \Lambda)$  and  $\rho = \frac{1}{4} \min\{d_{z_x}, d_{z_y}\}$ , where  $d_{z_x} = \text{dist}((t, x), \partial'Q)$ .

*Proof.* Fix  $(t, x), (t, y) \in Q$ . Denote these points  $z_x, z_y$ , respectively and set  $\rho = \frac{1}{4} \min\{d_{z_x}, d_{z_y}\}$ . Then either  $z_y \in Q_\rho(z_x)$  or  $z_y \notin Q_\rho(z_x)$ . If  $z_y \in Q_\rho(z_x)$ , then  $z_\xi = (t, \xi) \in Q_\rho(z_x)$ , where  $\xi = \gamma x + (1 - \gamma)y$  and  $\gamma \in (0, 1)$ . Hence  $Q_\rho(z_\xi) \subset Q_{2\rho}(z_x) \subset Q$  and so by the mean-value theorem and Theorem 1.1, we get

$$\begin{aligned} |u_{x_i x_j}(t, x) - u_{x_i x_j}(t, y)| &\leq |D(u_{x_i x_j})(t, \xi)| \cdot |x - y| \\ &\leq \left(\frac{N3}{\rho}\right)^3 |u|_{0;Q_\rho(z_\xi)} \cdot |x - y| \leq \left(\frac{N3}{\rho}\right)^3 |u|_{0;Q} \cdot |x - y|. \end{aligned}$$

On the other hand, if  $z_y \notin Q_\rho(z_x)$ , i.e.,  $|x - y| \geq \rho$ , then since  $z_x \in Q_{d_{z_x}}(z_x) \subset Q$ , Theorem 1.1 gives

$$\begin{aligned} \frac{|u_{x_i x_j}(t, x) - u_{x_i x_j}(t, y)|}{|x - y|} &\leq \frac{|u_{x_i x_j}(t, x)| + |u_{x_i x_j}(t, y)|}{\rho} \\ &\leq \frac{1}{\rho} \left[ \left( \frac{N2}{d_{z_x}} \right)^2 |u|_{0; Q_{d_{z_x}}(z_x)} + \left( \frac{N2}{d_{z_y}} \right)^2 |u|_{0; Q_{d_{z_y}}(z_y)} \right] \\ &\leq \frac{1}{\rho} \left[ \left( \frac{N2}{4\rho} \right)^2 |u|_{0; Q} + \left( \frac{N2}{4\rho} \right)^2 |u|_{0; Q} \right] = \frac{N^2}{2\rho^3} |u|_{0; Q} \end{aligned}$$

so in either case we get (1.3). □

### 2. SIMPLE FULLY NONLINEAR EQUATIONS WITH TIME-MEASURABLE COEFFICIENTS

In what follows we assume that the nonlinear operator  $F(M, t)$  is smooth and concave in  $M \in \mathbb{R}^{d^2}$  and measurable in  $t$ . Since  $F$  is differentiable in  $M$ , our uniform parabolicity condition (0.1) now reads

$$(2.0) \quad \lambda|\xi|^2 \leq F_{ij}(M, t)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \text{where } F_{ij} = \frac{\partial F}{\partial m_{ij}},$$

while the concavity of  $F$  in  $M$  implies  $N^t \cdot D^2F(M, t) \cdot N \leq 0, \forall N \in \mathbb{R}^{d^2}$ , i.e.  $F_{ij,kl}(M, t)N_{ij}N_{kl} \leq 0$ , where  $F_{ij,kl} = \frac{\partial^2 F}{\partial m_{ij}\partial m_{kl}}$ .

**Lemma 2.1.** *Let  $u$  be a smooth solution of the equation  $F(D^2u, t) - u_t = 0$  in  $Q$ , where  $F(0, \cdot) = 0$ . Then there exists a linear operator  $L := a^{ij}(t, x)\frac{\partial^2}{\partial x_i\partial x_j} - \frac{\partial}{\partial t}$ , with  $\lambda|\xi|^2 \leq a^{ij}(t, x)\xi_i\xi_j \leq \Lambda|\xi|^2$ , such that  $Lu \leq 0$  and  $\forall |e| = 1, Lu_e = 0$  and  $Lu_{ee} \geq 0$  in  $Q$ .*

*Proof.* Differentiating the equation  $u_t(t, x) = F(D^2u(t, x), t)$  with respect to any vector  $|e| = 1$  gives

$$(u_e)_t(t, x) = F_{ij}(D^2u(t, x), t)(u_e)_{x_i x_j} := a^{ij}(t, x)(u_e)_{x_i x_j}.$$

Thus  $Lu_e = 0$  in  $Q$ , where  $a^{ij}(t, x)$  satisfies (2.0). Differentiating the equation again with respect to  $e$  (omitting the obvious argument  $(D^2u(t, x), t)$ ) and using the fact that  $F$  is concave gives

$$(u_{ee})_t(t, x) = F_{ij,kl}(u_e)_{x_k x_l}(u_e)_{x_i x_j} + F_{ij}(u_{ee})_{x_i x_j} \leq F_{ij}(u_{ee})_{x_i x_j},$$

which gives  $Lu_{ee} \geq 0$  in  $Q$ . Finally, to conclude  $Lu \leq 0$ , observe that since  $F$  is concave, so is  $g(r) = F((1-r)D^2u(t, x), t)$ . Hence  $\forall r$  we have  $g(r) \leq g'(0)r + g(0)$ . In particular for  $r = 1, g(1) \leq g'(0) + g(0)$ , which from the equation and  $F(0, t) = 0$  gives  $0 \leq -a^{ij}(t, x)u_{x_i x_j} + u_t$ . □

For fixed  $\rho > 0$ , we define  $Q^\rho := \{z_0 \in Q : Q_\rho(z_0) \subset Q\}$ . We now prove the interior estimates for pure second derivatives of solutions of  $F(D^2u, t) - u_t = 0$  in  $Q$ , where  $F(0, \cdot) = 0$ . For a proof of this for fully nonlinear constant coefficient elliptic equations, see §9.1 of [CC]. In §§5.2–5.4 of [K1], as well as §4 in [K3], Krylov proves this result for more general parabolic equations  $F(D^2u, Du, u, t, x) - u_t = 0$ ,

under additional assumptions on  $F$ , using a variation of Bernstein’s technique he calls the *monitored maximum method*. This interior estimate for  $u_{ee}$  led to interior estimates for  $\|u_t\|_{L^2}, \|D^2u\|_{L^2}$ , which in turn led to the  $W_2^{1,2}$  solvability of fully nonlinear equations when  $F$  is only measurable in  $t$ . The proof of this was based on the  $C^{2,\alpha}$  solvability for functions  $F$  which are smooth in  $t$ . (See §§5.4, 6.5 of [K1].)

**Theorem 2.2.** *Let  $u$  be a  $C^4$  solution of the concave equation  $F(D^2u, t) - u_t = 0$  in  $Q$ , where  $F \in C^2(\mathbb{R}^{d^2})$  and  $F(0, \cdot) = 0$ . Then there is a constant  $N = N(d, \lambda, \Lambda)$  such that  $\forall e \in \mathbb{R}^d$  with  $|e| = 1$*

$$(2.1) \quad |u_{ee}^+|_{0;Q^\rho} \leq \frac{N}{\rho^2} |u|_{0;Q}.$$

*Proof.* For fixed  $\rho > 0$ , recall  $Q^\rho := \{z_0 \in Q : Q_\rho(z_0) \subset Q\}$ . So take any  $z_0 = (t_0, x_0) \in Q^\rho$  and take a function  $\varphi \in C^2(Q_\rho(z_0))$  with  $0 \leq \varphi \leq 1, \varphi \equiv 0$  on  $\partial'Q_\rho(z_0), \varphi(z_0) = 1$  and

$$|D\varphi|^2 \leq \frac{N\varphi}{\rho^2}, \quad \|D^2\varphi\| \leq \frac{N}{\rho^2}, \quad |\varphi_t| \leq \frac{N}{\rho^2},$$

where  $N = N(d)$ . As in Theorem 1.1, for a constant  $C$  to be determined later, consider, in  $Q_\rho(z_0)$ , the function

$$v = \varphi^2 (u_{ee}^+)^2 + C u_e^2.$$

In  $Q_\rho(z_0) \cap \{u_{ee} \geq 0\}$ , for  $L := a^{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t}$ , we have, since  $Lu_e = 0, Lu_{ee} \geq 0$  by Lemma 2.1

$$\begin{aligned} Lv &= 2u_{ee}^2 a^{ij} \varphi_{x_i} \varphi_{x_j} + 2\varphi u_{ee}^2 (a^{ij} \varphi_{x_i x_j} - \varphi_t) + 8\varphi u_{ee} a^{ij} \varphi_{x_i} u_{ee x_j} \\ &\quad + 2\varphi^2 a^{ij} u_{ee x_j} u_{ee x_i} + 2\varphi^2 u_{ee} (a^{ij} u_{ee x_i x_j} - u_{eet}) \\ &\quad + 2C a^{ij} u_{e x_j} u_{e x_i} + 2C u_e (a^{ij} u_{e x_i x_j} - u_{et}) \\ &\geq 2u_{ee}^2 a^{ij} \varphi_{x_i} \varphi_{x_j} + 2\varphi u_{ee}^2 (a^{ij} \varphi_{x_i x_j} - \varphi_t) + 8\varphi u_{ee} a^{ij} \varphi_{x_i} u_{ee x_j} \\ &\quad + 2\varphi^2 a^{ij} u_{ee x_j} u_{ee x_i} + 2C a^{ij} u_{e x_j} u_{e x_i} \\ &\geq 2u_{ee}^2 \lambda |D\varphi|^2 + 2\varphi u_{ee}^2 (a^{ij} \varphi_{x_i x_j} - \varphi_t) - 8\Lambda \varphi u_{ee} |D\varphi| \cdot |Du_{ee}| \\ &\quad + 2\varphi^2 \lambda |Du_{ee}|^2 + 2C \lambda |Du_e|^2 \\ &\geq 2u_{ee}^2 \lambda |D\varphi|^2 + 2\varphi u_{ee}^2 (a^{ij} \varphi_{x_i x_j} - \varphi_t + C\lambda) - 8\Lambda \varphi u_{ee} |D\varphi| \cdot |Du_{ee}| \\ &\quad + 2\varphi^2 \lambda |Du_{ee}|^2, \end{aligned}$$

since  $0 \leq \varphi \leq 1$  and  $u_{ee} = Du_e \cdot e \leq |Du_e|$ . First choose  $C$  so large that  $|a^{ij} \varphi_{x_i x_j} - \varphi_t| \leq \frac{C\lambda}{2}$ . Observe that  $|a^{ij} \varphi_{x_i x_j} - \varphi_t| \leq \|A\| \cdot \|D^2\varphi\| + |\varphi_t| \leq \frac{\Lambda d^{1/2} N}{\rho^2} + \frac{N}{\rho^2} = \frac{N}{\rho^2} (\Lambda d^{1/2} + 1)$ . So  $C \geq \frac{2N}{\lambda \rho^2} (\Lambda d^{1/2} + 1)$  will do. By Young’s inequality  $\forall \varepsilon > 0$  we

have

$$\begin{aligned}
Lv &\geq 2u_{ee}^2\lambda|D\varphi|^2 + \varphi u_{ee}^2 C\lambda - 8\Lambda \left( \varepsilon u_{ee}^2 |D\varphi|^2 + \frac{\varphi^2 |Du_{ee}|^2}{\varepsilon} \right) + 2\varphi^2\lambda |Du_{ee}|^2 \\
&= u_{ee}^2 |D\varphi|^2 (2\lambda - 8\Lambda\varepsilon) + \varphi u_{ee}^2 C\lambda + \varphi^2 |Du_{ee}|^2 \left( 2\lambda - \frac{8\Lambda}{\varepsilon} \right) \\
&\geq u_{ee}^2 |D\varphi|^2 (2\lambda - 8\Lambda\varepsilon) + \frac{u_{ee}^2 C\lambda |D\varphi|^2 \rho^2}{N} + \varphi^2 |Du_{ee}|^2 \left( 2\lambda - \frac{8\Lambda}{\varepsilon} \right) \\
&= u_{ee}^2 |D\varphi|^2 \left( 2\lambda - 8\Lambda\varepsilon + \frac{C\lambda\rho^2}{N} \right) + \varphi^2 |Du_{ee}|^2 \left( 2\lambda - \frac{8\Lambda}{\varepsilon} \right) \\
&\geq u_{ee}^2 |D\varphi|^2 \left( \frac{C\lambda\rho^2}{N} - 8\Lambda\varepsilon \right) + \varphi^2 |Du_{ee}|^2 \left( 2\lambda - \frac{8\Lambda}{\varepsilon} \right).
\end{aligned}$$

Now choose  $\varepsilon$  st  $\frac{C\lambda\rho^2}{N} - 8\Lambda\varepsilon = \frac{C\lambda\rho^2}{2N}$ . This gives

$$Lv \geq u_{ee}^2 |D\varphi|^2 \frac{C\lambda\rho^2}{2N} + \varphi^2 |Du_{ee}|^2 \left( 2\lambda - \frac{128\Lambda^2 N}{C\lambda\rho^2} \right).$$

Finally, choose  $C$  so large that  $2\lambda - \frac{128\Lambda^2 N}{C\lambda\rho^2} \geq 0$ . Observe that for

$$C = \max \left\{ \frac{2N}{\lambda\rho^2} (\Lambda d^{1/2} + 1), \frac{64\Lambda^2 N}{\lambda^2 \rho^2} \right\},$$

we have  $Lv \geq 0$  in  $Q_\rho(z_0) \cap \{u_{ee} \geq 0\}$ . But in  $Q_\rho(z_0) \cap \{u_{ee} \leq 0\}$ ,  $v = Cu_e^2$  and since  $Lu_e = 0$ , an easy calculation gives  $Lv = CL(u_e^2) \geq 2C\lambda|D(u_e)|^2 \geq 0$ . Thus  $Lv \geq 0$  in  $Q_\rho(z_0)$ . Hence by the maximum principle  $\sup_{Q_\rho(z_0)} v = \sup_{\partial' Q_\rho(z_0)} v$  and so

$(u_{ee}^+)^2(z_0) \leq C \sup_{Q_\rho(z_0)} u_e^2 = \frac{N_1(\lambda, \Lambda, d)}{\rho^2} \sup_{Q_\rho(z_0)} u_e^2 \leq \frac{N_1}{\rho^2} \sup_Q u_e^2$  and since  $z_0 \in Q^\rho$  was arbitrary, we have

$$(2.2) \quad \sup_{Q^\rho} u_{ee}^+ \leq \frac{N_2}{\rho} \sup_Q |u_e|.$$

Finally, to get an estimate for  $|u_e|$ , we apply Bernstein's technique to the function

$$w = \varphi^2 u_e^2 + C(M - u)^2$$

in  $Q_\rho(z_0)$ , where  $M := |u|_{0; Q_\rho(z_0)}$ . By Lemma 2.1,  $Lu_e = 0$ ,  $Lu \leq 0$  and hence

$$\begin{aligned}
Lw &= 2u_e^2 a^{ij} \varphi_{x_i} \varphi_{x_j} + 2\varphi u_e^2 (a^{ij} \varphi_{x_i x_j} - \varphi_t) + 8\varphi u_e a^{ij} \varphi_{x_i} u_{ex_j} + 2\varphi^2 a^{ij} u_{ex_j} u_{ex_i} \\
&\quad + 2\varphi^2 u_e [a^{ij} u_{ex_i x_j} - u_{et}] + 2Ca^{ij} u_{x_i} u_{x_j} - 2C(M - u) (a^{ij} u_{x_i x_j} - u_t) \\
&\geq 2u_e^2 a^{ij} \varphi_{x_i} \varphi_{x_j} + 2\varphi u_e^2 (a^{ij} \varphi_{x_i x_j} - \varphi_t) + 8\varphi u_e a^{ij} \varphi_{x_i} u_{ex_j} \\
&\quad + 2\varphi^2 a^{ij} u_{ex_j} u_{ex_i} + 2Ca^{ij} u_{x_i} u_{x_j} \\
&\geq 2u_e^2 \lambda |D\varphi|^2 + 2\varphi u_e^2 (a^{ij} \varphi_{x_i x_j} - \varphi_t) - 8\varphi \Lambda |D\varphi| \cdot |u_e| \cdot |D(u_e)| \\
&\quad + 2\varphi^2 \lambda |D(u_e)|^2 + 2C\lambda |Du|^2 \\
&\geq 2u_e^2 \lambda |D\varphi|^2 + 2\varphi u_e^2 (a^{ij} \varphi_{x_i x_j} - \varphi_t + C\lambda) - 8\varphi \Lambda |D\varphi| \cdot |u_e| \cdot |D(u_e)| \\
&\quad + 2\varphi^2 \lambda |D(u_e)|^2
\end{aligned}$$

since  $0 \leq \varphi \leq 1$ ,  $|u_e| \leq |Du|$  and  $|a^{ij} \varphi_{x_i} u_{ex_j}| \leq \sqrt{a^{ij} \varphi_{x_i} \varphi_{x_j}} \sqrt{a^{ij} u_{ex_i} u_{ex_j}} \leq \Lambda |D\varphi| \cdot |D(u_e)|$ . First choose  $C$  so large that  $|a^{ij} \varphi_{x_i x_j} - \varphi_t| \leq \frac{C\lambda}{2}$ . By Young's



inequality  $\forall \varepsilon > 0$  we have

$$\begin{aligned} Lw &\geq 2u_e^2\lambda|D\varphi|^2 + \varphi u_e^2 C\lambda - 8\Lambda \left( \varepsilon u_e^2|D\varphi|^2 + \frac{\varphi^2|D(u_e)|^2}{\varepsilon} \right) + 2\varphi^2\lambda|D(u_e)|^2 \\ &\geq u_e^2|D\varphi|^2 \left( 2\lambda - 8\Lambda\varepsilon + \frac{C\lambda\rho^2}{N} \right) + \varphi^2|D(u_e)|^2 \left( 2\lambda - \frac{8\Lambda}{\varepsilon} \right) \\ &\geq u_e^2|D\varphi|^2 \left( \frac{C\lambda\rho^2}{N} - 8\Lambda\varepsilon \right) + \varphi^2|D(u_e)|^2 \left( 2\lambda - \frac{8\Lambda}{\varepsilon} \right). \end{aligned}$$

Now choose  $\varepsilon$  st  $\frac{C\lambda\rho^2}{N} - 8\Lambda\varepsilon = \frac{C\lambda\rho^2}{2N}$ . This gives

$$Lw \geq u_e^2|D\varphi|^2 \frac{C\lambda\rho^2}{2N} + \varphi^2|D(u_e)|^2 \left( 2\lambda - \frac{128\Lambda^2 N}{C\lambda\rho^2} \right).$$

Finally, choose  $C$  so large that  $2\lambda - \frac{128\Lambda^2 N}{C\lambda\rho^2} \geq 0$ . Observe that for

$$C = \max \left\{ \frac{2N}{\lambda\rho^2}(\Lambda d^{1/2} + 1), \frac{64\Lambda^2 N}{\lambda^2\rho^2} \right\},$$

we have  $Lw \geq 0$  in  $Q_\rho(z_0)$ . Hence by the maximum principle  $\sup_{Q_\rho(z_0)} w = \sup_{\partial'Q_\rho(z_0)} w$

and so  $u_e^2(z_0) \leq C \sup_{Q_\rho(z_0)} (M - u)^2 = \frac{N_1(\lambda, \Lambda, d)}{\rho^2} \sup_{Q_\rho(z_0)} (M - u)^2 \leq \frac{4N_1}{\rho^2} (|u|_{0;Q})^2$ . Since

$z_0 \in Q^\rho$  was arbitrary, we have

$$(2.3) \quad |u_e|_{0;Q^\rho} \leq \frac{N_2}{\rho} |u|_{0;Q}.$$

Note that by (2.3) with  $\frac{\rho}{2}$  in place of  $\rho$ , we have  $|u_e|_{0;Q^{\frac{\rho}{2}}} \leq \frac{N_2}{\frac{\rho}{2}} |u|_{0;Q}$ . Now by (2.2) using the domains  $Q^\rho \subset Q^{\frac{\rho}{2}} \subset Q$  we have

$$\sup_{Q^\rho} u_{ee}^+ \leq \frac{N_2}{\rho} |u_e|_{0;Q^{\frac{\rho}{2}}} \leq \frac{N_2}{\rho} \cdot \frac{N_2}{\frac{\rho}{2}} |u|_{0;Q} = \frac{2N_2^2}{\rho^2} |u|_{0;Q}.$$

□

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DEPARTMENT OF MATHEMATICAL SCIENCES, FLORIDA INSTITUTE OF TECHNOLOGY, MELBOURNE,  
FLORIDA 32901

*E-mail address:* `jkovats@zach.fit.edu`