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STRONGLY MEAGER SETS OF REAL NUMBERS AND TREE FORCING NOTIONS

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ABSTRACT. We show that every strongly meager set has the l_0- and the m_0-property.

In [NW] it was proven that every strongly meager subset of 2^ω is a completely Ramsey null (CR_0) set. In this paper we show that an analogous result holds for Laver and Miller notions of forcing, i.e., every strongly meager set has both the l_0- and the m_0-property. Notice that the three classes of subsets of 2^ω mentioned above are not related as far as inclusion is concerned. This fact is due to J. Brendle (see [B]).

Our terminology is standard and can be found in [B] and [NW]. If T is a tree on $\omega^{<\omega\uparrow}$, the set of all increasing sequences of natural numbers, then the stem(T) of T is the unique $s \in T$ (if such s exists) with $\forall_{t\in T}\ s \subseteq t \vee t \subseteq s$ and $|\{n \in \omega : s \frown \langle n\rangle \in T\}| \geq 2$. Given $t \in T$, we define $\mathrm{succ}_T(t) = \{n \in \omega : t \frown \langle n\rangle \in T\}$ and we put $\mathrm{split}(T) = \{s \in T : |\mathrm{succ}_T(s)| \geq 2\}$. For $s \in T$, $\mathrm{Succ}_T(s) = \{t \in \omega^{<\omega\uparrow} : s \frown t \in \mathrm{split}(T) \wedge \forall_{t'\in \mathrm{split}(T)} s \subseteq t' \subseteq s \frown t \Rightarrow (t' = s \vee t' = s \frown t)\}$. Finally, $[T] = \{x \in \omega^{\omega\uparrow} : \forall_{n\in\omega} x\restriction n \in T\}$. We recall that a tree $T \subseteq \omega^{<\omega\uparrow}$ is said to be a <u>Laver tree</u> iff for every $s \in T$ with stem$(T) \subseteq s$, $\mathrm{succ}_T(s)$ is infinite. A tree $T \subseteq \omega^{<\omega\uparrow}$ is a <u>superperfect tree</u> iff for any given $s \in T$, there is $t \supseteq s$ such that $\mathrm{succ}_T(t)$ is infinite. Let $[\omega]^\omega$ be the set of all infinite subsets of ω. It is clear that an $x \in \omega^{\omega\uparrow}$ can be identified with an element of $[\omega]^\omega$ and vice versa. Thus, depending on the context, $\omega^{\omega\uparrow}$ is often conflated with $[\omega]^\omega$ or with the set $\{x : x \in 2^\omega \text{ and } \exists^\infty_n x(n) = 1\}$.

Throughout this paper we assume that the measure used on 2^ω is the product measure. By "+" we mean the usual modulo 2 coordinatewise addition in 2^ω and for sets $A, B \subseteq 2^\omega$ we define $A + B = \{a + b : a \in A, b \in B\}$.

Let us recall that $X \subseteq 2^\omega$ is <u>strongly meager</u> iff for every measure zero set $A \subseteq 2^\omega$, there is $t \in 2^\omega$, so that $(X + t) \cap A = \emptyset$.

Definition 1. We say that $X \subseteq [\omega]^\omega$ is an <u>l_0- set</u> iff for every Laver tree T, there exists a Laver tree $S \subseteq T$ such that $\{\mathrm{ran}(x) : x \in [S]\} \cap X = \emptyset$.

Definition 2. A set $X \subseteq [\omega]^\omega$ is called an <u>m_0- set</u> iff for every superperfect tree T, there is a superperfect tree $S \subseteq T$ such that $\{\mathrm{ran}(x) : x \in [S]\} \cap X = \emptyset$.

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Theorem 1. *Let $X \subseteq 2^\omega$ be a strongly meager set. Then X has the l_0-property.*

Proof. It suffices to show the following lemma. $\square$

Lemma 1. *For every Laver tree $T \subseteq \omega^{<\omega\uparrow}$, there exists a measure zero set $H \subseteq 2^\omega$ such that*

$$\forall_{t\in 2^\omega} \exists_{S\subseteq T} S \textit{ is a Laver tree } \& \operatorname{stem}(S) = \operatorname{stem}(T) \ \& \ \{\operatorname{ran}(x) : x \in [S]\} \subseteq H + t.$$

Proof. Before proving Lemma 1, we recall the following theorem due to Lorentz (see [L]).

Fact 1. *If $(G, +)$ is a finite group and if H is a nonempty subset of G, then there is a subset F of G such that*

1. *The cardinality of F is at most $|G| \cdot (\frac{1+\ln(|H|)}{|H|})$, and*
2. *$G = F + H$.*

To begin, we construct by induction a partition of ω into finite intervals as follows.

Let I_0 be equal to an interval $[0, n)$, which satisfies

$$\max(I_0) > \max \operatorname{ran} (\operatorname{stem}(T)).$$

Having constructed $I_0, \dots, I_n$ we define I_{n+1}, so that $\min(I_{n+1}) = \max(I_n) + 1$ and for every $s \in T$ with the $\operatorname{stem}(T) \subseteq s$, $\max\operatorname{ran}(s) \le \max(I_n)$,

$$\left|\{k : \max\operatorname{ran}(s) < k \ \& \ s \frown \langle k\rangle \in T\} \cap I_{n+1}\right| \ge L_{n+1},$$

where L_{n+1} is any natural number satisfying

$$\frac{\ln(L_{n+1}) + 1}{L_{n+1}} \cdot 2^{\max I_n + 1} \le \frac{1}{2^{n+1}}.$$

Now, for every $s \in T$ with the $\operatorname{stem}(T) \subseteq s$, $\max\operatorname{ran}(s) \le \max(I_n)$, we apply Lorentz's theorem to get $\tilde{H}^s_{n+1} \subseteq 2^{I_{n+1}}$ such that:

1.
$$\tilde{H}^s_{n+1} + \{e_l : l \in \{k : \max\operatorname{ran}(s) < k \ \& \ s \frown \langle k\rangle \in T\} \cap I_{n+1}\} = 2^{I_{n+1}},$$
2.
$$|\tilde{H}^s_{n+1}| \le \frac{\ln(L_{n+1}) + 1}{L_{n+1}} \cdot 2^{|I_{n+1}|},$$

where e_l for $l \in I_{n+1}$ is an element of $2^{I_{n+1}}$ given by

$$e_l(j) = \begin{cases} 1 & \text{if } j = l, \\ 0 & \text{if } j \ne l. \end{cases}$$

We define

$$H_{n+1} = \bigcup_{s\in T, \operatorname{stem}(T)\subseteq s, \max\operatorname{ran}(s)\le\max(I_n)} \tilde{H}^s_{n+1}$$

and we have

$$|H_{n+1}| \le \frac{\ln(L_{n+1}) + 1}{L_{n+1}} \cdot 2^{|I_{n+1}|} \cdot 2^{\max I_n + 1} \le \frac{1}{2^{n+1}} \cdot 2^{|I_{n+1}|}.$$

Thus,

$$\sum_{n=0}^{\infty} \frac{|H_n|}{2^{|I_n|}} < \infty.$$

This implies that the set

$$H = \{x \in 2^\omega : \exists_n^\infty x \restriction I_n \in H_n\}$$

is of measure zero.

We verify the following property:

$$\forall_{t \in 2^\omega} \exists_{S \subseteq T} S \text{ is a Laver tree } \& \{\mathrm{ran}(x) : x \in [S]\} \subseteq H + t.$$

Fix $t \in 2^\omega$. We construct a system $\{s_r\}_{r \in \omega^{<\omega}}$ of elements from $\omega^{<\omega\uparrow}$ using induction on $|r|$. Suppose $s_\emptyset = \mathrm{stem}(T)$. Assume that we already have $\{s_r\}_{r \in \omega^{<n}}$. Let $r \in \omega^{n-1}$. We choose $p_r \in \omega$ with $\max \mathrm{ran}(s_r) < \min(I_{p_r})$. Then for every $p' \in \omega$, there exists $k \in I_{p_r+p'}$ satisfying:

1.
$$\exists_{h \in H_{p_r+p'}} e_k + h = t \restriction I_{p_r+p'},$$

2.
$$s_r \frown \langle k \rangle \in T.$$

Let $s_{r \frown \langle p' \rangle} = s_r \frown \langle k \rangle$, where k is as above. Thus, for every $r \in \omega^{n-1}$ and $p' \in \omega$, we have defined $s_{r \frown \langle p' \rangle}$. Finally, we put

$$S = \{s \in \omega^{<\omega\uparrow} : \exists_{r \in \omega^{<\omega}} s \subseteq s_r\}.$$

It is not hard to see that S is a Laver tree, $S \subseteq T$, and $\mathrm{stem}(S) = \mathrm{stem}(T)$. Also, if $x \in [S]$, then $\exists_n^\infty \big((t \restriction I_n) + \mathrm{ran}(x) \cap I_n\big) \in H_n$, so we have

$$\{\mathrm{ran}(x) : x \in [S]\} \subseteq H + t.$$

This completes our proof of Lemma 1 from which Theorem 1 easily follows. $\square$

Theorem 2. *Let $X \subseteq 2^\omega$ be a strongly meager set. Then X is an m_0-set.*

Proof. As before it is sufficient to show the following lemma.

Lemma 2. *For every superperfect tree $T \subseteq \omega^{<\omega\uparrow}$, there exists a measure zero set $H \subseteq 2^\omega$ such that*

$$\forall_{t \in 2^\omega} \exists_{S \subseteq T} S \text{ is a superperfect tree } \& \mathrm{stem}(S) = \mathrm{stem}(T) \wedge$$

$$\{\mathrm{ran}(x) : x \in [S]\} \subseteq H + t.$$

Proof. Suppose we are given a superperfect tree $T \subseteq \omega^{<\omega\uparrow}$. Using the standard pruning argument we can assume that $\forall_{t \in T} |\mathrm{succ}_T(t)| \in \{1, \omega\}$.

Now we proceed as in the proof of Lemma 1 to get a sequence $\{I_m\}_{m \in \omega}$ of finite disjoint intervals in ω such that:

1.
$$\min I_{m+1} = \max I_m + 1,$$

2.
$$\forall_{s \in \mathrm{split}(T), \max \mathrm{ran}(s) \leq \max I_m} |\{t : t \in \mathrm{Succ}_T(s), \mathrm{ran}(t) \subseteq I_{m+1}\}| \geq L_{m+1},$$

where $L_{m+1} \in \omega$ fulfills the following condition:

$$\frac{\ln(L_{m+1}) + 1}{L_{m+1}} \cdot 2^{\max I_m + 1} \leq \frac{1}{2^{m+1}}.$$

Let us denote

$$H^s_{m+1} = \{t : t \in \mathrm{Succ}_T(s), \mathrm{ran}(t) \subseteq I_{m+1}\}.$$

For every $s \in \mathrm{split}(T)$ with $\max\mathrm{ran}(s) \leq \max(I_m)$, we apply Lorentz's theorem to get $\tilde{H}^s_{m+1} \subseteq 2^{I_{m+1}}$ such that

$$H^s_{m+1} + \{t : t \in \tilde{H}^s_{m+1}\} = 2^{I_{m+1}}.$$

We define

$$H_{m+1} = \bigcup_{\max\mathrm{ran}(s) \leq \max(I_m)} \tilde{H}^s_{m+1}.$$

This completes the recursive construction of the sequence $\{H_m\}_{m\in\omega}$ and we argue as before to get a measure zero set H.

Let $t \in 2^\omega$. We construct, by induction on $|r|$ a sequence $\{s_r\}_{r\in\omega^{<\omega\uparrow}}$, so that $s_r \in \mathrm{split}(T)$. So assume that we already have $\{s_r\}_{r\in\omega^{m-1}}$. First choose $p_r \in \omega$ such that $\max(s_r) < \min(I_{p_r})$. Then for every $p' \in \omega$, there is $\tilde{t}$, with $\mathrm{ran}(\tilde{t}) \subseteq I_{p_r+p'}$, which satisfies

1.
$$\exists_{h\in H_{p_r+p'}} \tilde{t} + h = t \restriction I_{p_r+p'},$$

2.
$$s_r \frown \tilde{t} \in T.$$

We put $s_{r\frown p'} = s_r \frown \tilde{t}$, where $\tilde{t}$ is as above and we set

$$S = \{s \in \omega^{<\omega\uparrow} : \exists_{r\in\omega^{<\omega\uparrow}} s \subseteq s_r\}.$$

Let $x \in [S]$. Clearly, $\exists^\infty_n \big((t\restriction I_n) + \mathrm{ran}(x) \cap I_n\big) \in H_n$. □

Remark. We shall say that an $X \subseteq 2^\omega$ is a very meager (VFC) set iff for every $H \subseteq 2^\omega$ of measure zero there are sequences $\{X_n\}_{n\in\omega}$, $\{t_n\}_{n\in\omega}$ such that $X \subseteq \bigcup_{n\in\omega} X_n$ and $t_n \in X_n + H$, for $n \in \omega$. It is easy to modify the proof of Theorem 4 from [NW] to show that sets with the latter property are CR_0. Also one can check that Theorem 1 and Theorem 2 of this paper remain true with a strongly meager set replaced by a VFC set. To see this, let $\{A_n\}_{n\in\omega}$ be a partition of ω into disjoint infinite sets and assume that $\{t_n\}_{n\in\omega}$ is a sequence of reals. We construct a system $\{s_r\}_{r\in\omega^{<\omega}}$ of elements from $\omega^{<\omega\uparrow}$ by induction on $|r|$ using the same method as in the proof of Lemma 1 with condition 1 replaced by the following requirement. If $\{s_r\}_{r\in\omega^{<n}}$ are already constructed and $n \in A_m$ for some $m \in \omega$, then we choose $p_r \in \omega$ satisfying $\max\mathrm{ran}(s_r) < \min(I_{p_r})$ and for every $p' \in \omega$ we pick $k \in I_{p_r+p'}$, so that:

1.
$$\exists_{h\in H_{p_r+p'}} e_k + h = t_m \restriction I_{p_r+p'},$$

2.
$$s_r \frown \langle k\rangle \in T.$$

Clearly, we will have that $\{\mathrm{ran}(x) : x \in [S]\} \subseteq H + t_m$, for every $m \in \omega$, where $S = \{s \in \omega^{<\omega\uparrow} : \exists_{r\in\omega^{<\omega}} s \subseteq s_r\}$. Thus if we assume that $X \subseteq \bigcup_{m\in\omega} X_m$ and $X_m \cap (H + t_m) = \emptyset$, for every $m \in \omega$, then $\{\mathrm{ran}(x) : x \in [S]\} \cap X = \emptyset$.

Added after posting

It was noticed by M. Kysiak that Theorem 2 holds for a wider class of sets, namely perfectly meager subsets of 2^ω. Let us notice also that the definition of VFC sets is due to M. Kysiak.

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