

STRONGLY MEAGER SETS OF REAL NUMBERS AND TREE FORCING NOTIONS

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(Communicated by Carl G. Jockusch, Jr.)

ABSTRACT. We show that every strongly meager set has the l_0 - and the m_0 -property.

In [NW] it was proven that every strongly meager subset of 2^ω is a completely Ramsey null (CR_0) set. In this paper we show that an analogous result holds for Laver and Miller notions of forcing, i.e., every strongly meager set has both the l_0 - and the m_0 -property. Notice that the three classes of subsets of 2^ω mentioned above are not related as far as inclusion is concerned. This fact is due to J. Brendle (see [B]).

Our terminology is standard and can be found in [B] and [NW]. If T is a tree on $\omega^{<\omega^\uparrow}$, the set of all increasing sequences of natural numbers, then the $\text{stem}(T)$ of T is the unique $s \in T$ (if such s exists) with $\forall t \in T$ $s \subseteq t \vee t \subseteq s$ and $|\{n \in \omega : s \frown \langle n \rangle \in T\}| \geq 2$. Given $t \in T$, we define $\text{succ}_T(t) = \{n \in \omega : t \frown \langle n \rangle \in T\}$ and we put $\text{split}(T) = \{s \in T : |\text{succ}_T(s)| \geq 2\}$. For $s \in T$, $\text{Succ}_T(s) = \{t \in \omega^{<\omega^\uparrow} : s \frown t \in \text{split}(T) \wedge \forall t' \in \text{split}(T) s \subseteq t' \subseteq s \frown t \Rightarrow (t' = s \vee t' = s \frown t)\}$. Finally, $[T] = \{x \in \omega^{\omega^\uparrow} : \forall n \in \omega x \upharpoonright n \in T\}$. We recall that a tree $T \subseteq \omega^{<\omega^\uparrow}$ is said to be a Laver tree iff for every $s \in T$ with $\text{stem}(T) \subseteq s$, $\text{succ}_T(s)$ is infinite. A tree $T \subseteq \omega^{<\omega^\uparrow}$ is a superperfect tree iff for any given $s \in T$, there is $t \supseteq s$ such that $\text{succ}_T(t)$ is infinite. Let $[\omega]^\omega$ be the set of all infinite subsets of ω . It is clear that an $x \in \omega^{\omega^\uparrow}$ can be identified with an element of $[\omega]^\omega$ and vice versa. Thus, depending on the context, ω^{ω^\uparrow} is often conflated with $[\omega]^\omega$ or with the set $\{x : x \in 2^\omega \text{ and } \exists_n^\infty x(n) = 1\}$.

Throughout this paper we assume that the measure used on 2^ω is the product measure. By “+” we mean the usual modulo 2 coordinatewise addition in 2^ω and for sets $A, B \subseteq 2^\omega$ we define $A + B = \{a + b : a \in A, b \in B\}$.

Let us recall that $X \subseteq 2^\omega$ is strongly meager iff for every measure zero set $A \subseteq 2^\omega$, there is $t \in 2^\omega$, so that $(X + t) \cap A = \emptyset$.

Definition 1. We say that $X \subseteq [\omega]^\omega$ is an l_0 -set iff for every Laver tree T , there exists a Laver tree $S \subseteq T$ such that $\{\text{ran}(x) : x \in [S]\} \cap X = \emptyset$.

Definition 2. A set $X \subseteq [\omega]^\omega$ is called an m_0 -set iff for every superperfect tree T , there is a superperfect tree $S \subseteq T$ such that $\{\text{ran}(x) : x \in [S]\} \cap X = \emptyset$.

Received by the editors July 7, 2000 and, in revised form, October 2, 2000.
2000 *Mathematics Subject Classification.* Primary 03E15, 03E20, 28E15.
Key words and phrases. Strongly meager sets, Laver forcing, Miller forcing.
The first author was partially supported by KBN grant 2 P03A 047 09.

Theorem 1. *Let $X \subseteq 2^\omega$ be a strongly meager set. Then X has the l_0 -property.*

Proof. It suffices to show the following lemma. □

Lemma 1. *For every Laver tree $T \subseteq \omega^{<\omega^1}$, there exists a measure zero set $H \subseteq 2^\omega$ such that*

$$\forall t \in 2^\omega \exists S \subseteq T \text{ } S \text{ is a Laver tree \& } \text{stem}(S) = \text{stem}(T) \text{ \& } \{\text{ran}(x) : x \in [S]\} \subseteq H + t.$$

Proof. Before proving Lemma 1, we recall the following theorem due to Lorentz (see [L]).

Fact 1. *If $(G, +)$ is a finite group and if H is a nonempty subset of G , then there is a subset F of G such that*

1. *The cardinality of F is at most $|G| \cdot (\frac{1+\ln(|H|)}{|H|})$, and*
2. *$G = F + H$.*

To begin, we construct by induction a partition of ω into finite intervals as follows.

Let I_0 be equal to an interval $[0, n)$, which satisfies

$$\max(I_0) > \max \text{ran}(\text{stem}(T)).$$

Having constructed I_0, \dots, I_n we define I_{n+1} , so that $\min(I_{n+1}) = \max(I_n) + 1$ and for every $s \in T$ with the $\text{stem}(T) \subseteq s$, $\max \text{ran}(s) \leq \max(I_n)$,

$$|\{k : \max \text{ran}(s) < k \text{ \& } s \frown \langle k \rangle \in T\} \cap I_{n+1}| \geq L_{n+1},$$

where L_{n+1} is any natural number satisfying

$$\frac{\ln(L_{n+1}) + 1}{L_{n+1}} \cdot 2^{\max I_{n+1}} \leq \frac{1}{2^{n+1}}.$$

Now, for every $s \in T$ with the $\text{stem}(T) \subseteq s$, $\max \text{ran}(s) \leq \max(I_n)$, we apply Lorentz's theorem to get $\tilde{H}_{n+1}^s \subseteq 2^{I_{n+1}}$ such that:

1. $\tilde{H}_{n+1}^s + \{e_l : l \in \{k : \max \text{ran}(s) < k \text{ \& } s \frown \langle k \rangle \in T\} \cap I_{n+1}\} = 2^{I_{n+1}},$
- 2.

$$|\tilde{H}_{n+1}^s| \leq \frac{\ln(L_{n+1}) + 1}{L_{n+1}} \cdot 2^{|I_{n+1}|},$$

where e_l for $l \in I_{n+1}$ is an element of $2^{I_{n+1}}$ given by

$$e_l(j) = \begin{cases} 1 & \text{if } j = l, \\ 0 & \text{if } j \neq l. \end{cases}$$

We define

$$H_{n+1} = \bigcup_{s \in T, \text{stem}(T) \subseteq s, \max \text{ran}(s) \leq \max(I_n)} \tilde{H}_{n+1}^s$$

and we have

$$|H_{n+1}| \leq \frac{\ln(L_{n+1}) + 1}{L_{n+1}} \cdot 2^{|I_{n+1}|} \cdot 2^{\max I_{n+1}} \leq \frac{1}{2^{n+1}} \cdot 2^{|I_{n+1}|}.$$

Thus,

$$\sum_{n=0}^{\infty} \frac{|H_n|}{2^{|I_n|}} < \infty.$$

This implies that the set

$$H = \{x \in 2^\omega : \exists_n^\infty x \upharpoonright I_n \in H_n\}$$

is of measure zero.

We verify the following property:

$$\forall_{t \in 2^\omega} \exists_{S \subseteq T} S \text{ is a Laver tree } \& \{ \text{ran}(x) : x \in [S] \} \subseteq H + t.$$

Fix $t \in 2^\omega$. We construct a system $\{s_r\}_{r \in \omega^{<\omega}}$ of elements from $\omega^{<\omega^\uparrow}$ using induction on $|r|$. Suppose $s_\emptyset = \text{stem}(T)$. Assume that we already have $\{s_r\}_{r \in \omega^{<n}}$. Let $r \in \omega^{n-1}$. We choose $p_r \in \omega$ with $\max \text{ran}(s_r) < \min(I_{p_r})$. Then for every $p' \in \omega$, there exists $k \in I_{p_r+p'}$ satisfying:

1.

$$\exists_{h \in H_{p_r+p'}} e_k + h = t \upharpoonright I_{p_r+p'},$$

2.

$$s_r \frown \langle k \rangle \in T.$$

Let $s_{r \frown \langle p' \rangle} = s_r \frown \langle k \rangle$, where k is as above. Thus, for every $r \in \omega^{n-1}$ and $p' \in \omega$, we have defined $s_{r \frown \langle p' \rangle}$. Finally, we put

$$S = \{s \in \omega^{<\omega^\uparrow} : \exists_{r \in \omega^{<\omega}} s \subseteq s_r\}.$$

It is not hard to see that S is a Laver tree, $S \subseteq T$, and $\text{stem}(S) = \text{stem}(T)$. Also, if $x \in [S]$, then $\exists_n^\infty ((t \upharpoonright I_n) + \text{ran}(x) \cap I_n) \in H_n$, so we have

$$\{ \text{ran}(x) : x \in [S] \} \subseteq H + t.$$

This completes our proof of Lemma 1 from which Theorem 1 easily follows. □

Theorem 2. *Let $X \subseteq 2^\omega$ be a strongly meager set. Then X is an m_0 -set.*

Proof. As before it is sufficient to show the following lemma.

Lemma 2. *For every superperfect tree $T \subseteq \omega^{<\omega^\uparrow}$, there exists a measure zero set $H \subseteq 2^\omega$ such that*

$$\forall_{t \in 2^\omega} \exists_{S \subseteq T} S \text{ is a superperfect tree } \& \text{stem}(S) = \text{stem}(T) \wedge$$

$$\{ \text{ran}(x) : x \in [S] \} \subseteq H + t.$$

Proof. Suppose we are given a superperfect tree $T \subseteq \omega^{<\omega^\uparrow}$. Using the standard pruning argument we can assume that $\forall_{t \in T} |\text{succ}_T(t)| \in \{1, \omega\}$.

Now we proceed as in the proof of Lemma 1 to get a sequence $\{I_m\}_{m \in \omega}$ of finite disjoint intervals in ω such that:

1.

$$\min I_{m+1} = \max I_m + 1,$$

2.

$$\forall_{s \in \text{split}(T), \max \text{ran}(s) \leq \max I_m} |\{t : t \in \text{Succ}_T(s), \text{ran}(t) \subseteq I_{m+1}\}| \geq L_{m+1},$$

where $L_{m+1} \in \omega$ fulfills the following condition:

$$\frac{\ln(L_{m+1}) + 1}{L_{m+1}} \cdot 2^{\max I_{m+1}} \leq \frac{1}{2^{m+1}}.$$

Let us denote

$$H_{m+1}^s = \{t : t \in \text{Succ}_T(s), \text{ran}(t) \subseteq I_{m+1}\}.$$

For every $s \in \text{split}(T)$ with $\max \text{ran}(s) \leq \max(I_m)$, we apply Lorentz’s theorem to get $\tilde{H}_{m+1}^s \subseteq 2^{I_{m+1}}$ such that

$$H_{m+1}^s + \{t : t \in \tilde{H}_{m+1}^s\} = 2^{I_{m+1}}.$$

We define

$$H_{m+1} = \bigcup_{\max \text{ran}(s) \leq \max(I_m)} \tilde{H}_{m+1}^s.$$

This completes the recursive construction of the sequence $\{H_m\}_{m \in \omega}$ and we argue as before to get a measure zero set H .

Let $t \in 2^\omega$. We construct, by induction on $|r|$ a sequence $\{s_r\}_{r \in \omega^{<\omega \uparrow}}$, so that $s_r \in \text{split}(T)$. So assume that we already have $\{s_r\}_{r \in \omega^{m-1}}$. First choose $p_r \in \omega$ such that $\max(s_r) < \min(I_{p_r})$. Then for every $p' \in \omega$, there is \tilde{t} , with $\text{ran}(\tilde{t}) \subseteq I_{p_r+p'}$, which satisfies

1.

$$\exists h \in H_{p_r+p'} \tilde{t} + h = t \upharpoonright I_{p_r+p'},$$

2.

$$s_r \frown \tilde{t} \in T.$$

We put $s_{r \frown p'} = s_r \frown \tilde{t}$, where \tilde{t} is as above and we set

$$S = \{s \in \omega^{<\omega \uparrow} : \exists r \in \omega^{<\omega \uparrow} s \subseteq s_r\}.$$

Let $x \in [S]$. Clearly, $\exists_n^\infty ((t \upharpoonright I_n) + \text{ran}(x) \cap I_n) \in H_n$. □

Remark. We shall say that an $X \subseteq 2^\omega$ is a **very meager** (VFC) set iff for every $H \subseteq 2^\omega$ of measure zero there are sequences $\{X_n\}_{n \in \omega}, \{t_n\}_{n \in \omega}$ such that $X \subseteq \bigcup_{n \in \omega} X_n$ and $t_n \in X_n + H$, for $n \in \omega$. It is easy to modify the proof of Theorem 4 from [NW] to show that sets with the latter property are CR_0 . Also one can check that Theorem 1 and Theorem 2 of this paper remain true with a strongly meager set replaced by a VFC set. To see this, let $\{A_n\}_{n \in \omega}$ be a partition of ω into disjoint infinite sets and assume that $\{t_n\}_{n \in \omega}$ is a sequence of reals. We construct a system $\{s_r\}_{r \in \omega^{<\omega}}$ of elements from $\omega^{<\omega \uparrow}$ by induction on $|r|$ using the same method as in the proof of Lemma 1 with condition 1 replaced by the following requirement. If $\{s_r\}_{r \in \omega^{<n}}$ are already constructed and $n \in A_m$ for some $m \in \omega$, then we choose $p_r \in \omega$ satisfying $\max \text{ran}(s_r) < \min(I_{p_r})$ and for every $p' \in \omega$ we pick $k \in I_{p_r+p'}$, so that:

1.

$$\exists h \in H_{p_r+p'} e_k + h = t_m \upharpoonright I_{p_r+p'},$$

2.

$$s_r \frown \langle k \rangle \in T.$$

Clearly, we will have that $\{\text{ran}(x) : x \in [S]\} \subseteq H + t_m$, for every $m \in \omega$, where $S = \{s \in \omega^{<\omega \uparrow} : \exists r \in \omega^{<\omega} s \subseteq s_r\}$. Thus if we assume that $X \subseteq \bigcup_{m \in \omega} X_m$ and $X_m \cap (H + t_m) = \emptyset$, for every $m \in \omega$, then $\{\text{ran}(x) : x \in [S]\} \cap X = \emptyset$.

ADDED AFTER POSTING

It was noticed by M. Kysiak that Theorem 2 holds for a wider class of sets, namely perfectly meager subsets of 2^ω . Let us notice also that the definition of VFC sets is due to M. Kysiak.

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