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NATURAL EXAMPLES OF $\mathbf{\Pi}_5^0$-COMPLETE SETS IN ANALYSIS

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Abstract. The purpose of this paper is to show that given any non-negative real number α, the set of entire functions whose order is equal to α is $\mathbf{\Pi}_3^0$-complete, and the set of all sequences of entire functions whose orders converge to α is $\mathbf{\Pi}_5^0$-complete.

1. Introduction

On page 322 of [4], H. J. Rogers states that:

... the human mind seems limited in its ability to understand and visualize beyond four or five alternations of quantifier.

Moreover, on page 189 of [3], A. S. Kechris states the following:

In conclusion, we would like to mention that we do not know of any interesting "natural" examples of Borel sets in analysis or topology which are in one of the classes $\mathbf{\Sigma}_\xi^0$ or $\mathbf{\Pi}_\xi^0$ for $\xi \geq 5$, but not in a class with lower index.

The purpose of this paper is to give new natural examples of complex Borel sets originating from analysis that live in the third and the fifth level of the Borel hierarchy. In fact, we obtain the following result:

Theorem. *Given any non-negative real number α, the set of entire functions whose order is equal to α is $\mathbf{\Pi}_3^0$-complete, and the set of all sequences of entire functions whose orders converge to α is $\mathbf{\Pi}_5^0$-complete.*

Thus, we provide for the first time natural examples of complex Borel sets in analysis or topology that live in the fifth level of the Borel hierarchy, and require five alternations of quantifier for their definition.

2. Complex Borel sets associated with entire functions

Let $H(\mathbf{C})$ stand, as usual, for the Polish space of entire functions, equipped with the *topology of almost uniform convergence*, i.e., the topology of uniform convergence on compacts. If for any $r > 0$, we set $M(r; f) = \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})|$, then

$$\rho(f) = \limsup_{r \to \infty} \frac{\log \log M(r; f)}{\log r}$$

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is called the *order* of the entire function f (see, for example, page 182 of [2]), and if for any $n \in \mathbf{N}$, we set $c_n = \frac{f^{(n)}(0)}{n!}$, then $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for every $z \in \mathbf{C}$, and

$$\rho(f) = \limsup_{n\to\infty} \frac{n \cdot \log n}{\log \frac{1}{|c_n|}}$$

(see, for example, page 186 of [2]).

Theorem 2.1. *For any $\alpha \in [0,\infty)$, the set*

$$A_\alpha = \{f \in H(\mathbf{C}) : \rho(f) = \alpha\}$$

is $\mathbf{\Pi}^0_3$-complete, and the set

$$B_\alpha = \left\{(f_k)_{k\in\mathbf{N}} \in H(\mathbf{C})^{\mathbf{N}} : \lim_{k\to\infty} \rho(f_k) = \alpha\right\}$$

is $\mathbf{\Pi}^0_5$-complete.

Proof. If $f \in H(\mathbf{C})$ and $(f_k)_{k\in\mathbf{N}} \in H(\mathbf{C})^{\mathbf{N}}$, while $c_n = \frac{f^{(n)}(0)}{n!}$ and $c_{k,n} = \frac{f_k^{(n)}(0)}{n!}$, whenever k, n are in $\mathbf{N}$, then

$$\rho(f) = \alpha$$

$$\Updownarrow$$

$$\forall i \exists j \geq i \forall k \geq j \left(\frac{j \cdot \log j}{\log \frac{1}{|c_j|}} \geq \alpha - 2^{-i} \ \wedge \ \frac{k \cdot \log k}{\log \frac{1}{|c_k|}} \leq \alpha + 2^{-i} \right)$$

and

$$\lim_{k\to\infty} \rho(f_k) = \alpha$$

$$\Updownarrow$$

$$\forall i \exists j \forall k \geq j \forall l \exists m \geq l \forall n \geq m$$

$$\left(\frac{m \cdot \log m}{\log \frac{1}{|c_{k,m}|}} \geq \alpha - 2^{-i} \ \wedge \ \frac{n \cdot \log n}{\log \frac{1}{|c_{k,n}|}} \leq \alpha + 2^{-i} + 2^{-l} \right)$$

which implies that the sets A_α and B_α are $\mathbf{\Pi}^0_3$ and $\mathbf{\Pi}^0_5$ respectively, since the mapping $H(\mathbf{C})^{\mathbf{N}} \ni (f_k)_{k\in\mathbf{N}} \mapsto f_\kappa \in H(\mathbf{C})$ is obviously continuous for every $\kappa \in \mathbf{N}$, and so is the mapping $H(\mathbf{C}) \ni f \mapsto f^{(\nu)}(0) \in \mathbf{C}$ for every $\nu \in \mathbf{N}$ (see, for example, page 192 of [1]).

To prove that A_α is $\mathbf{\Pi}^0_3$-hard, it is enough to show that a set which is known to be $\mathbf{\Pi}^0_3$-hard is Wadge reducible to A_α (see, for example, pages 156 and 169 of [3]), and as $P_3 = \left\{x \in 2^{\mathbf{N}\times\mathbf{N}} : \forall m \forall^\infty n (x(m,n) = 0)\right\}$ is $\mathbf{\Pi}^0_3$-complete (see, for example, page 179 of [3]), we will show that $A_\alpha \leq_W P_3$. So let

$$c_n^x = \begin{cases} \frac{1}{n^{n\cdot\phi_x(n)^{-1}}} & \text{if } n \in \mathbf{N}\setminus\{0\}\ , \\ 0 & \text{if } n = 0\ , \end{cases}$$

where

$$\phi_x([m,n]) = \begin{cases} \alpha + 2^{-m} & \text{if } x(m,n) = 1\ , \\ \alpha + 3^{-[m,n]} & \text{if } x(m,n) = 0 \end{cases}$$

whenever x is in $2^{\mathbf{N}\times\mathbf{N}}$ and m, n are in $\mathbf{N}$. We should mention that $[\cdot,\cdot]$ stands for the standard coding of the pairs of natural numbers by the natural numbers, and we denote by $(\cdot)_0$, $(\cdot)_1$ the associated decoding functions in the sense that $[(n)_0,(n)_1] = n$ for every $n \in \mathbf{N}$. Given any $x \in \mathbf{N}$, it is not difficult to prove that $|c_n^x|^{\frac{1}{n}} \to 0$ as $n \to \infty$. Therefore, by setting $f_x(z) = \sum_{n=0}^{\infty} c_n^x z^n$ for every $z \in \mathbf{C}$, we obtain an entire function (see, for example, page 118 of [1]) and what we want to show is that the mapping $2^{\mathbf{N}\times\mathbf{N}} \ni x \mapsto f_x \in H(\mathbf{C})$ is continuous. Indeed we need only remark that for any integer $N \geq 1$, if x, y are in $2^{\mathbf{N}\times\mathbf{N}}$ and $x((n)_0,(n)_1) = y((n)_0,(n)_1)$ for every $n \in \{0,...,N-1\}$, then $c_n^x = c_n^y$ for every $n \in \{0,...,N-1\}$, and since for any $u \in 2^{\mathbf{N}\times\mathbf{N}}$ and for any integer $n \geq N$, we have

$$|c_n^u| \leq \frac{1}{n^{\frac{n}{\alpha+1}}}$$

it follows that

$$\max_{|z|\leq R} |f_x(z) - f_y(z)| \leq 2 \cdot \sum_{n\geq N} \frac{R^n}{n^{\frac{n}{\alpha+1}}}$$

where

$$\sum_{n\geq N} \frac{R^n}{n^{\frac{n}{\alpha+1}}} \to 0$$

as $n \to \infty$ for every $R > 0$. What is left to show is that for any $x \in 2^{\mathbf{N}\times\mathbf{N}}$, we have $x \in P_3 \iff \rho(f_x) = \alpha$. If $x \notin P_3$, then there exists $m \in \mathbf{N}$ such that $\exists^\infty n(x(m,n) = 1)$, which implies that

$$\exists^\infty n \left(\frac{n \cdot \log n}{\log \frac{1}{|c_n^x|}} = \alpha + 2^{-m} \right)$$

and consequently $\rho(f_x) \geq \alpha + 2^{-m}$. If $x \in P_3$, then for any $m \in \mathbf{N}$, there exists $n_m \in \mathbf{N}$ such that for any integer $n > n_m$, we have $x(m,n) = 0$ and hence

$$\frac{[m,n] \cdot \log[m,n]}{\log \frac{1}{|c_{[m,n]}^x|}} = \alpha + 3^{-[m,n]} < \alpha + 2^{-n}.$$

Therefore, given $M \in \mathbf{N} \setminus \{0\}$, we have

$$\alpha < \frac{[m,n] \cdot \log[m,n]}{\log \frac{1}{|c_{[m,n]}^x|}} \leq \alpha + 2^{-M}$$

for every natural numbers m and n, apart from the values $0 \leq n \leq \max_{0\leq m<M} n_m$ and $0 \leq m < M$, which proves that $\rho(f_x) = \alpha$.

To prove that B_α is $\mathbf{\Pi}_5^0$-hard, as before, it is enough to show that a set which is known to be $\mathbf{\Pi}_5^0$-hard is Wadge reducible to B_α. Since

$$S_4 = \left\{ x \in 2^{\mathbf{N}\times\mathbf{N}} : \forall^\infty m \forall^\infty n(x(m,n) = 0) \right\}$$

is $\mathbf{\Sigma}_4^0$-complete (see, for example, page 181 of [3]), it is not difficult to prove that so is $S_4^* = \left\{ x \in 2^{\mathbf{N}} : \forall^\infty m \forall^\infty n(x([m,n]) = 0) \right\}$. Indeed we need only remark that the mapping $2^{\mathbf{N}} \ni x \mapsto (x([m,n]))_{(m,n)\in\mathbf{N}\times\mathbf{N}} \in 2^{\mathbf{N}\times\mathbf{N}}$ constitutes a homeomorphism whose inverse is $2^{\mathbf{N}\times\mathbf{N}} \ni x \mapsto (x((n)_0,(n)_1))_{n\in\mathbf{N}} \in 2^{\mathbf{N}}$. Therefore the set $P_5^* = \left\{ x \in 2^{\mathbf{N}\times\mathbf{N}} : \forall l (x_l \in S_4^*) \right\}$ is $\mathbf{\Pi}_5^0$-complete (see, for example, page 180 of [3]), and what we have to show is that $P_5^* = \left\{ x \in 2^{\mathbf{N}\times\mathbf{N}} : \forall l \forall^\infty m \forall^\infty n(x(l,[m,n]) = 0) \right\}$ is

Wadge reducible to B_α. So let

$$c^x_{[l,m],n} = \begin{cases} \frac{1}{n^{n\cdot\phi_x(l,[m,n])^{-1}}} & \text{if } n \in \mathbf{N}\setminus\{0\}\ , \\ 0 & \text{if } n = 0 \end{cases}$$

where

$$\phi_x(l,[m,n]) = \begin{cases} \alpha + 2^{-l} + n^{-1} & \text{if } x(l,[m,n]) = 1 \text{ and } n \in \mathbf{N}\setminus\{0\}\ , \\ \alpha + n^{-1} & \text{if } x(l,[m,n]) = 0 \text{ and } n \in \mathbf{N}\setminus\{0\} \end{cases}$$

whenever x is in $2^{\mathbf{N}\times\mathbf{N}}$ and l, m are in $\mathbf{N}$. Given any natural numbers l and m, it is not difficult to prove that for any $x \in 2^{\mathbf{N}\times\mathbf{N}}$, we have $|c^x_{[l,m],n}|^{\frac{1}{n}} \to 0$ as $n \to \infty$. Therefore, by setting $f^x_{[l,m]}(z) = \sum\limits_{n=0}^{\infty} c^x_{[l,m],n} z^n$ for every $z \in \mathbf{C}$, we obtain an entire function (see, for example, page 118 of [1]), and what we want to show is that the mapping $2^{\mathbf{N}\times\mathbf{N}} \ni x \mapsto f^x_{[l,m]} \in H(\mathbf{C})$ is continuous. Indeed we need only remark that for any integer $N \geq 2^l$, if x, y are in $2^{\mathbf{N}\times\mathbf{N}}$ and $x(l,[m,n]) = y(l,[m,n])$ for every $n \in \{0,...,N-1\}$, then since for any $u \in 2^{\mathbf{N}\times\mathbf{N}}$ and for any integer $n \geq N$, we have

$$|c^u_{[l,m],n}| \leq \frac{1}{n^{\frac{n}{\alpha+2^{-l}+1}}}$$

it follows that

$$\max_{|z|\leq R} |f^x_{[l,m]}(z) - f^y_{[l,m]}(z)| \leq 2 \cdot \sum_{n\geq N} \frac{R^n}{n^{\frac{n}{\alpha+2^{-l}+1}}}$$

where

$$\sum_{n\geq N} \frac{R^n}{n^{\frac{n}{\alpha+2^{-l}+1}}} \to 0$$

as $N \to \infty$ for every $R > 0$. Therefore the definition of the product topology is easily seen to imply that the mapping $2^{\mathbf{N}\times\mathbf{N}} \ni x \mapsto (f^x_k)_{k\in\mathbf{N}} \in H(\mathbf{C})^{\mathbf{N}}$ is continuous, and what is left to show is that for any $x \in 2^{\mathbf{N}\times\mathbf{N}}$, we have $x \in P^*_5 \iff (f^x_k)_{k\in\mathbf{N}} \in B_\alpha$; in other words, we have $x \in P^*_5 \iff \lim\limits_{k\to\infty} \rho(f^x_k) = \alpha$. If $x \notin P^*_5$, then there exist $l \in \mathbf{N}$ and natural numbers $m_0 < m_1 < ...$ such that $\exists^\infty n(x(l,[m_i,n]) = 1)$ for every $i \in \mathbf{N}$, which implies that

$$\exists^\infty n \left(\frac{n\cdot\log n}{\log\frac{1}{|c^x_{[l,m_i],n}|}} = \alpha + 2^{-l} + n^{-1} \right)$$

for every $i \in \mathbf{N}$, and consequently

$$\rho(f^x_{[l,m_i]}) = \limsup_{n\to\infty} \frac{n\cdot\log n}{\log\frac{1}{|c^x_{[l,m_i],n}|}} \geq \alpha + 2^{-l}$$

for every $i \in \mathbf{N}$, which implies in its turn that the sequence $(\rho(f^x_k))_{k\in\mathbf{N}}$ does not converge to α. So let $x \in P^*_5$ and let $l \in \mathbf{N}$. Then there exists $m_l \in \mathbf{N}$ such that $\forall m \geq m_l \forall^\infty n(x(l,[m,n]) = 0)$, which implies that

$$\forall m \geq m_l \forall^\infty n \left(\frac{n\cdot\log n}{\log\frac{1}{|c^x_{[l,m],n}|}} = \alpha + n^{-1} \right)$$

and consequently for any integer $m \geq m_l$, we have

$$\rho(f^x_{[l,m]}) = \limsup_{n\to\infty} \frac{n \cdot \log n}{\log \frac{1}{|c^x_{[l,m],n}|}} = \lim_{n\to\infty} (\alpha + n^{-1}) = \alpha$$

while if $0 \leq m < m_l$ and $n \in \mathbf{N} \setminus \{0\}$, then

$$\frac{n \cdot \log n}{\log \frac{1}{|c^x_{[l,m],n}|}} = \begin{cases} \alpha + 2^{-l} + n^{-1} & \text{if } x(l,[m,n]) = 1, \\ \alpha + n^{-1} & \text{if } x(l,[m,n]) = 0 \end{cases}$$

which is easily seen to imply that

$$\alpha \leq \rho(f^x_{[l,m]}) = \limsup_{n\to\infty} \frac{n \cdot \log n}{\log \frac{1}{|c^x_{[l,m],n}|}} \leq \alpha + 2^{-l}.$$

Therefore, given $N \in \mathbf{N}$, we have $\alpha \leq \rho(f^x_{[l,m]}) \leq \alpha + 2^{-N}$ for every value of the natural numbers l and m, apart from the values $0 \leq m < m_l$ and $0 \leq l < N$, which proves that $\lim\limits_{k\to\infty} \rho(f^x_k) = \alpha$. □

Corollary 2.2. *The order of an entire function is a Baire class two function which is not Baire class one.*

Proof. Since for any particular $\alpha \in [0,\infty)$, the set $\rho^{-1}[\{\alpha\}] = \{f \in H(\mathbf{C}) : \rho(f) = \alpha\}$ is $\mathbf{\Pi}^0_3$-complete and therefore not $\mathbf{\Pi}^0_2$, the function ρ is not Baire class one, while since if $f \in H(\mathbf{C})$ and $c_n = \frac{f^{(n)}(0)}{n!}$ for every $n \in \mathbf{N}$, then

$$\rho(f) = \lim_{n\to\infty} \sup_{m\geq n} \frac{m \cdot \log m}{\log \frac{1}{|c_m|}}$$

in order to prove that the function ρ is Baire class two, it is enough to prove that for any $n \in \mathbf{N}$, the function

$$s_n : H(\mathbf{C}) \ni f \mapsto \sup_{m\geq n} \frac{m \cdot \log m}{\log \frac{m!}{|f^{(m)}(0)|}} \in [0,\infty]$$

is Baire class one. But if $U \subseteq [0,\infty]$ is non-empty open and $\infty \notin U$, then given $f \in H(\mathbf{C})$, we have

$$s_n(f) \in U$$
$$\Updownarrow$$
$$\exists r,s \in \mathbf{Q} \left([r-s, r+s] \subseteq U \ \wedge \ \forall m \geq n \right.$$
$$\left. \left(\frac{m \cdot \log m}{\log \frac{m!}{|f^{(m)}(0)|}} \leq r+s \right) \ \wedge \ \exists m \geq n \left(\frac{m \cdot \log m}{\log \frac{m!}{|f^{(m)}(0)|}} \geq r-s \right) \right)$$

and consequently the set $\{f \in H(\mathbf{C}) : s_n(f) \in U\}$ is $\mathbf{\Sigma}^0_2$. If $\infty \in U$, then $U = V \cup (\alpha,\infty]$, where $V \subseteq [0,\alpha)$ is non-empty open and $0 < \alpha < \infty$; hence

$$s_n(f) \in U \iff \left(s_n(f) \in V \ \vee \ \exists m \geq n \left(\frac{m \cdot \log m}{\log \frac{m!}{|f^{(m)}(0)|}} > \alpha \right) \right)$$

which implies that the set $\{f \in H(\mathbf{C}) : s_n(f) \in U\}$ is $\mathbf{\Sigma}^0_2$ and consequently the function s_n is Baire class one. □

It is not difficult to prove that the set $A_\infty = \{f \in H(\mathbf{C}) : \rho(f) = \infty\}$ is $\mathbf{\Pi}^0_2$-complete, and a straightforward computation shows that the set $B_\infty = \left\{(f_k)_{k\in\mathbf{N}} \in H(\mathbf{C})^{\mathbf{N}} : \lim_{k\to\infty} \rho(f_k) = \infty\right\}$ is $\mathbf{\Pi}^0_4$.

Open Problem. Is the set B_∞ $\mathbf{\Pi}^0_4$-complete?

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