

OVERSAMPLING AND PRESERVATION OF TIGHTNESS IN AFFINE FRAMES

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ABSTRACT. The problem of how an oversampling of translations affects the bounds of an affine frame has been proposed by Chui and Shi. In particular, they proved that tightness is preserved if the oversampling factor is coprime with the dilation factor. In this paper we study, in the dyadic dilation case, oversampling of translation by factors which do not satisfy the above condition, and prove that tightness is preserved only in the case of affine frames generated by wavelets having frequency support with very particular properties.

1. INTRODUCTION

Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space. Let I be a countable set of indices. A sequence $\{x_i\}_{i \in I}$ in \mathbb{H} is a frame for \mathbb{H} if there exist constants $0 < A \leq B$ such that for every $y \in \mathbb{H}$ we have

$$A\|y\|_{\mathbb{H}}^2 \leq \sum_{i \in I} |\langle x_i, y \rangle|^2 \leq B\|y\|_{\mathbb{H}}^2.$$

If we can choose $A = B$, the frame is said to be *tight*, with constant A .

The main advantage of frames that are not Riesz bases is the combination of the stability of the system with the possibility of redundancy, which can be used effectively in many applications, such as removing uncorrelated noise.

For $\psi \in L^2(\mathbb{R})$, let

$$(1.1) \quad \Psi(a_0, b_0) = \left\{ \psi_{j,n} = a_0^{\frac{j}{2}} \psi(a_0^j \cdot - nb_0); j, n \in \mathbb{Z} \right\}.$$

In this paper we will mainly restrict our attention to the particular case $a_0 = 2$, $b_0 = 1$. The theorem in [1] implies that if $\Psi(2, b_0)$ is a tight frame, then the oversampled system $\Psi(2, \frac{b_0}{k})$ is also a tight frame when k is any odd integer. As observed there, this result is sharp, in the sense that if k is even, the oversampled system $\Psi(2, \frac{1}{k})$ is not necessarily a tight frame even when $\Psi(2, 1)$ is: the Haar wavelet provides an immediate counterexample.

Our aim is to study for which ψ it is true that if $\Psi(2, 1)$ is a tight frame, then the oversampled system $\Psi(2, \frac{1}{n2^r})$ is a tight frame. It turns out that conditions on

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the support of $\hat{\psi}$ are the key for solving this problem. We introduce the following definition:

Definition 1.1. A function $f \in L^2(\mathbb{R})$ is *band limited up to congruences with band width B* (we will say that f is *B -BLC*) if there exist bounded intervals $\{I_k, k \in \mathbb{Z}\}$ such that the translations $J_k = I_k - kB$ are disjoint, $J_k \subset [-B, B]$ and $\text{supp } \hat{f} \subset \bigcup_{k \in \mathbb{Z}} I_k$.

2. RESULTS

We will denote by \mathcal{F}_k the set of wavelets ψ such that the associated family $\Psi(2, \frac{1}{k})$ is a tight frame for $L^2(\mathbb{R})$ with constant k . We recall the following characterization of tight frame wavelets [11, 5, 7]:

Theorem 2.1. Let $\psi \in L^2(\mathbb{R})$. Then $\psi \in \mathcal{F}_k$ if and only if the following two conditions hold almost everywhere:

$$\begin{aligned} \text{i)} \quad & \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = k, \\ \text{ii)} \quad & t_q^k(\xi) := \sum_{j=0}^{\infty} \hat{\psi}(2^j \xi) \overline{\hat{\psi}(2^j(\xi - 2\pi kq))} = 0 \quad \forall q \in 2\mathbb{Z} + 1. \end{aligned}$$

These two equations will be the main tool in our investigation. They will allow us to prove the following results.

Theorem 2.2. For any $\psi \in \mathcal{F}_1$ the following three statements are equivalent:

- a) $\psi \in \bigcap_{n \in \mathbb{N}} \mathcal{F}_{2^n}$,
- b) $\psi \in \bigcap_{r \in \mathbb{N}} \mathcal{F}_{2^r}$,
- c) ψ is 2π -BLC.

Corollary 2.3. A wavelet $\psi \in \mathcal{F}_1$ is in \mathcal{F}_n for all $n \in \mathbb{N}$ if, and only if, it is 2π -BLC

For more general oversampling factors, we obtain the following

Theorem 2.4. Let $n_0, r_0 \in \mathbb{N}, \psi \in \mathcal{F}_1$. Then $\psi \in \bigcap_{r=r_0}^{\infty} \mathcal{F}_{2^r n_0}$ if, and only if, it is $2\pi 2^{r_0} n_0$ -BLC

Corollary 2.5. Let $n_0 \in \mathbb{N}, \psi \in \mathcal{F}_1$. If $\psi \in \mathcal{F}_m$ for all $m \geq n_0$, then ψ is $2\pi \bar{n}_0$ -BLC, where $\bar{n}_0 = n_0$ if n_0 is even and $\bar{n}_0 = n_0 + 1$ otherwise.

Our last result deals with oversampling of wavelets having compactly supported Fourier transform:

Theorem 2.6. Let ψ be a band limited wavelet and $a_0 > 1, b_0 > 0$ be such that $\Psi(a_0, b_0)$ is a frame with constants A, B . Then there exists $\lambda_0 > 0$ so that for any $\lambda \geq \lambda_0$ the family $\Psi(a_0, \frac{b_0}{\lambda})$ is also a frame, with admissible bounds $\lambda A, \lambda B$.

3. PROOFS

Proof of Theorem 2.2. Suppose $\Psi(2, 1)$ is a tight frame. We study first the conditions that guarantee that $\Psi(2, \frac{1}{2})$ is a tight frame. By Theorem 2.1 we need to impose, for any odd q ,

$$\begin{aligned} 0 = t_q^2(\xi) &= \sum_{j=0}^{\infty} \hat{\psi}(2^j \xi) \overline{\hat{\psi}(2^j(\xi - 2\pi 2q))} \\ &= -\hat{\psi}\left(\frac{\xi}{2}\right) + \sum_{j=0}^{\infty} \hat{\psi}\left(2^j \frac{\xi}{2}\right) \overline{\hat{\psi}\left(2^j\left(\frac{\xi}{2} - 2\pi q\right)\right)}. \end{aligned}$$

The last series sums to 0 a.e. By Theorem 2.1 and the assumption that $\Psi(2, 1)$ is a tight frame, so $\Psi(2, \frac{1}{2})$ is a tight frame if and only if

$$\hat{\psi}(\cdot) \overline{\hat{\psi}(\cdot - 2\pi q)} = 0$$

for every $q \in 2\mathbb{Z} + 1$. By induction one deduces the conditions for $\Psi(2, \frac{1}{2^r})$ to be a tight frame. For a fixed $h \in \mathbb{N}$, ψ is $2h\pi$ -BLc if and only if, for all $n \in \mathbb{Z}$, $\hat{\psi}(\cdot) \overline{\hat{\psi}(\cdot - 2\pi hn)} = 0$. The thesis and the various corollaries then follow by writing an arbitrary oversampling factor k in the form $2^r n$, for some $r \in \mathbb{N}$, and odd integer n . □

Proof of Theorem 2.6. Since the family $\Psi(a, b)$ is a frame for $L^2(\mathbb{R})$ with constants A, B , then ([3])

$$(3.1) \quad Ab \leq \sum_{j \in \mathbb{Z}} |\hat{\psi}(a^{-j}\xi)|^2 \leq Bb \quad \text{a.e.}$$

□

Let $f \in L^2(\mathbb{R})$, and take $\lambda \geq \lambda_0 > 0$ such that $\text{supp } \hat{\psi} \subset [-\frac{\pi\lambda_0}{b}, \frac{\pi\lambda_0}{b}]$. Then, by Plancherel's theorem

$$\begin{aligned} \langle f, \psi_{j,n}^\lambda \rangle &= \frac{1}{2\pi} \langle \hat{f}, \hat{\psi}_{j,n}^\lambda \rangle \\ &= \frac{1}{2\pi} a^{\frac{j}{2}} \sqrt{\frac{\lambda}{b}} \int_{-\frac{\pi\lambda}{b}}^{\frac{\pi\lambda}{b}} \hat{f}(a^j \xi) \overline{\hat{\psi}(\xi)} \sqrt{\frac{b}{\lambda}} e^{i\xi n \frac{b}{\lambda}} d\xi. \end{aligned}$$

Since $\{\sqrt{\frac{b}{\lambda}} e^{-in \frac{b}{\lambda}} \cdot ; n \in \mathbb{Z}\}$ is an orthonormal basis in $L^2([-\frac{\pi\lambda}{b}, \frac{\pi\lambda}{b}])$, the summation in n gives $\frac{1}{2\pi} \frac{\lambda}{b} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\hat{\psi}(a^{-j}\xi)|^2 d\xi$ and the sum in j then yields the condition of frame with the announced bounds.

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