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PERIODS OF MIRRORS AND MULTIPLE ZETA VALUES

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ABSTRACT. In a recent paper, A. Libgober showed that the multiplicative sequence $\{Q_i(c_1, \dots, c_i)\}$ of Chern classes corresponding to the power series $Q(z) = \Gamma(1+z)^{-1}$ appears in a relation between the Chern classes of certain Calabi-Yau manifolds and the periods of their mirrors. We show that the polynomials Q_i can be expressed in terms of multiple zeta values.

1. THE MULTIPLICATIVE SEQUENCE

In [6], the Hirzebruch multiplicative sequence $\{Q_i\}$ associated to the power series $Q(z) = \Gamma(1+z)^{-1}$ is considered in connection with mirror symmetry. If e_i denotes the ith elementary symmetric function in the variables $t_1, t_2, \dots$, then

$$\sum_{i=0}^{\infty} Q_i(e_1, \dots, e_i) = \prod_{i=1}^{\infty} \frac{1}{\Gamma(1+t_i)}. \tag{1}$$

As shown in [6], the polynomials $Q_i(c_1, \dots, c_i)$ in the Chern classes of certain Calabi-Yau manifolds X are related to the coefficients of the generalized hypergeometric series expansion of the period (holomorphic at a maximum degeneracy point) of a mirror of X. In particular, if X is a Calabi-Yau hypersurface of dimension 4 in a nonsingular toric Fano manifold, then

$$\int_X Q_4(c_1, c_2, c_3, c_4) = \frac{1}{24} \sum_{ijkl} K_{ijkl} \frac{\partial^4 c(0, \dots, 0)}{\partial \rho_i \partial \rho_j \partial \rho_k \partial \rho_l},$$

where the $c(\rho_1, \dots, \rho_r)$ are coefficients in the expansion of the period and K_{ijkl} is the (suitably normalized) 4-point function corresponding to a mirror of X. In [6] it is shown that the polynomials Q_i have the form

$$Q_1(c_1) = \gamma c_1 \quad \text{and} \quad Q_i(c_1, \dots, c_i) = \zeta(i) c_i + \cdots, i > 1.$$

In this note we show that the polynomials Q_i have an explicit expression in terms of multiple zeta values (called multiple harmonic series in [3, 4]), which have previously appeared in connection with Kontsevich's invariant in knot theory [8, 5], and in quantum field theory [1].

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2. Symmetric and quasi-symmetric functions

For the convenience of the reader, we collect in this section the definitions about symmetric and quasi-symmetric functions we will need. Let $t_1, t_2, \dots$ be a countable sequence of indeterminates, each having degree 1, and let $P \subset \mathbf{Q}[[t_1, t_2, \dots]]$ be the set of formal power series in the t_i having bounded degree. Then P is a graded $\mathbf{Q}$-algebra. Let f be an element of P. Then f is a symmetric function if

$$\text{coefficient of } t_{n_1}^{i_1} t_{n_2}^{i_2} \cdots t_{n_k}^{i_k} \text{ in } f = \text{coefficient of } t_{m_1}^{i_1} t_{m_2}^{i_2} \cdots t_{m_k}^{i_k} \text{ in } f \tag{2}$$

for any pair $(n_1, n_2, \dots, n_k)$, $(m_1, m_2, \dots, m_k)$ of k-tuples of distinct positive integers. Following [2], we say f is a quasi-symmetric function if equation (2) holds whenever $n_1 < n_2 < \cdots < n_k$ and $m_1 < m_2 < \cdots < m_k$. The sets Sym and QSym of symmetric and quasi-symmetric functions respectively are both subalgebras of P, with Sym $\subset$ QSym.

For a composition (ordered sequence of positive integers) $I = (i_1, \dots, i_k)$, the corresponding monomial quasi-symmetric function $M_I \in$ QSym is defined by

$$M_I = \sum_{n_1 < n_2 < \cdots < n_k} t_{n_1}^{i_1} t_{n_2}^{i_2} \cdots t_{n_k}^{i_k}.$$

Evidently $\{M_I | I \text{ is a composition}\}$ is a basis for QSym as a vector space. For any composition I, there is a partition $\pi(I)$ given by forgetting the ordering. For any partition λ, the monomial symmetric function m_λ is the sum of all the M_I over all distinct I such that $\pi(I) = \lambda$, e.g., $m_{21} = M_{(2,1)} + M_{(1,2)}$. The monomial symmetric functions are a vector space basis for Sym. The elementary symmetric function e_k is $M_{I_k} = m_{\pi(I_k)}$, where I_k is the composition consisting of k 1's, and the power-sum symmetric function p_k is $M_{(k)} = m_k$. If for a partition $\lambda = \pi(\lambda_1, \lambda_2, \dots)$ we let $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$ and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$, then it is well known that $\{e_\lambda | \lambda \text{ is a partition}\}$ and $\{p_\lambda | \lambda \text{ is a partition}\}$ are vector space bases for Sym.

3. The formula for the Q_i

In [4] it is shown (Theorem 5.1) that the homomorphism $\zeta : \text{Sym} \to \mathbf{R}$ such that $\zeta(p_1) = \gamma$ and $\zeta(p_i) = \zeta(i)$ for $i > 2$ satisfies

$$\zeta\left(\sum_{i \ge 0} e_i z^i\right) = \frac{1}{\Gamma(1+z)}. \tag{3}$$

Our main result expresses the polynomials Q_i in terms of ζ.

Theorem. *For any partition λ of i, the coefficient of e_λ in $Q_i(e_1, \dots, e_i)$ is $\zeta(m_\lambda)$.*

Proof. Using equations (1) and (3), we have

$$\sum_{i \ge 0} Q_i(e_1, e_2, \dots) = \prod_{i=1}^{\infty} \frac{1}{\Gamma(1+t_i)} = \prod_{i=1}^{\infty} \sum_{j=0}^{\infty} \zeta(e_j) t_i^j = \sum_\lambda \zeta(e_\lambda) m_\lambda.$$

Now the transition matrix M from the basis $\{e_\lambda\}$ of Sym to the basis $\{m_\lambda\}$, i.e.

$$e_\lambda = \sum_\mu M_{\lambda\mu} m_\mu,$$

is known to be symmetric (see Ch. I, §6 of [7]), so we have

$$\sum_{\lambda} \zeta(e_\lambda) m_\lambda = \sum_{\lambda} \sum_{\mu} M_{\lambda\mu} \zeta(m_\mu) m_\lambda = \sum_{\mu} \zeta(m_\mu) \sum_{\lambda} M_{\mu\lambda} m_\lambda = \sum_{\mu} \zeta(m_\mu) e_\mu,$$

and the result follows.

4. Multiple zeta values

The homomorphism $\zeta : \mathrm{Sym} \to \mathbf{R}$ above is the restriction to Sym of a homomorphism defined in [4] from QSym to $\mathbf{R}$. The definition in [4] is motivated by the multiple zeta values introduced in [3] and [8], i.e.

$$\zeta(i_1, i_2, \dots, i_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}, \tag{4}$$

where $i_1, i_2, \dots, i_k$ are positive integers with $i_1 > 1$. To explain the connection, let $\mathfrak{H}^1$ be the rational vector space of polynomials in the noncommuting variables $z_1, z_2, \dots$; then $\mathfrak{H}^1$ becomes isomorphic to QSym if it is given the (commutative) multiplication * defined by the inductive rule

$$z_i w_1 * z_j w_2 = z_i(w_1 * z_j w_2) + z_j(z_i w_1 * w_2) + z_{i+j}(w_1 * w_2)$$

for any words w_1, w_2 in the z_i. (In fact the isomorphism is simply the map that sends $z_{i_1} \cdots z_{i_k} \in \mathfrak{H}^1$ to $M_{(i_k, \dots, i_1)} \in \mathrm{QSym}$: see [4] for details.) The algebra Sym of symmetric functions can be identified with the subspace of $\mathfrak{H}^1$ generated by linear combinations of monomials invariant under permutation of subscripts, e.g., $z_2^2 = m_{22}$ and $z_1 z_2 + z_2 z_1 = m_{21}$; z_i and z_1^i correspond to p_i and e_i respectively. As an algebra $\mathfrak{H}^1$ is generated by Lyndon words in the z_i, i.e., monomials w such that for any nontrivial decomposition $w = uv$ one has $v > w$, where the z_i are ordered as $z_1 > z_2 > \cdots$ and this order is extended to monomials lexicographically. Then the only Lyndon word that starts with z_1 is z_1 itself, and we can define a homomorphism $\zeta : \mathfrak{H}^1 \to \mathbf{R}$ by giving its value on Lyndon words $w = z_{i_1} z_{i_2} \cdots z_{i_k}$:

$$\zeta(w) = \begin{cases} \gamma, & w = z_1, \\ \zeta(i_1, i_2, \dots, i_k), & \text{otherwise.} \end{cases}$$

By the results of [4], $\zeta(z_{i_1} z_{i_2} \cdots z_{i_k})$ coincides with $\zeta(i_1, i_2, \dots, i_k)$ as defined by equation (4) whenever $i_1 > 1$.

Since the power-sum symmetric functions p_i generate the algebra Sym, we can compute $\zeta(m_\lambda)$ by first expressing m_λ in terms of power-sum functions (see [7], p. 109 for an explicit formula) and then applying the homomorphism ζ. Hence the coefficient of each monomial c_λ in $Q_i(c_1, \dots, c_i)$ is a polynomial in the numbers γ and $\zeta(i)$, $i \geq 2$. For example, since $m_3 = p_3$, $m_{21} = p_1 p_2 - p_3$ and $m_{111} = \frac{1}{6}(p_1^3 - 3p_1 p_2 + 2p_3)$, we have

$$Q_3(c_1, c_2, c_3) = \zeta(3) c_3 + (\gamma\zeta(2) - \zeta(3)) c_1 c_2 + \frac{1}{6}(\gamma^3 - 3\gamma\zeta(2) + 2\zeta(3)) c_1^3,$$

which corrects equation (1.5) of [6]. Similarly we can obtain equations (1.3), (1.4) (the coefficient of c_1^2 should be $\frac{1}{2}(\gamma^2 - \zeta(2))$), and (1.6) of [6].

If m_λ is a monomial symmetric function such that the partition λ involves no 1's, there is an alternative method of computing $\zeta(m_\lambda)$: in this case $\zeta(m_\lambda)$ is just a symmetric sum of ordinary multiple zeta values, e.g. $\zeta(m_{32}) = \zeta(2,3) + \zeta(3,2)$, and such a symmetric sum can be written as a sum of products of the numbers $\zeta(i)$ by Theorem 2.2 of [3]; one can use Euler's formula for $\zeta(2n)$ to simplify further. Since $c_1 = 0$ on a Calabi-Yau manifold, only terms $\zeta(m_\lambda)c_\lambda$ with λ having no 1's appear in Q_i, and we have in this case

$$Q_2(c_2) = \zeta(2)c_2,$$
$$Q_3(c_2, c_3) = \zeta(3)c_3,$$
$$Q_4(c_2, c_3, c_4) = \zeta(4)c_4 + \zeta(2,2)c_2^2 = \zeta(4)c_4 + \frac{3}{4}\zeta(4)c_2^2,$$
$$Q_5(c_2, c_3, c_4, c_5) = \zeta(5)c_5 + (\zeta(3,2) + \zeta(2,3))c_2c_3 = \zeta(5)c_5 + (\zeta(2)\zeta(3) - \zeta(5))c_2c_3,$$

and

$$\begin{aligned} Q_6(c_2, c_3, c_4, c_5, c_6) &= \zeta(6)c_6 + (\zeta(4,2) + \zeta(2,4))c_2c_4 + \zeta(2,2,2)c_2^3 + \zeta(3,3)c_3^2 \\ &= \zeta(6)c_6 + \frac{3}{4}\zeta(6)c_2c_4 + \frac{3}{16}\zeta(6)c_2^3 + \frac{1}{2}(\zeta(3)^2 - \zeta(6))c_3^2. \end{aligned}$$

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