PERIODS OF MIRRORS AND MULTIPLE ZETA VALUES

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Abstract. In a recent paper, A. Libgober showed that the multiplicative sequence \( \{Q_i(c_1, \ldots, c_i)\} \) of Chern classes corresponding to the power series \( Q(z) = \Gamma(1 + z)^{-1} \) appears in a relation between the Chern classes of certain Calabi-Yau manifolds and the periods of their mirrors. We show that the polynomials \( Q_i \) can be expressed in terms of multiple zeta values.

1. The Multiplicative Sequence

In [6], the Hirzebruch multiplicative sequence \( \{Q_i\} \) associated to the power series \( Q(z) = \Gamma(1 + z)^{-1} \) is considered in connection with mirror symmetry. If \( e_i \) denotes the \( i \)th elementary symmetric function in the variables \( t_1, t_2, \ldots \), then

\[
\sum_{i=0}^{\infty} Q_i(e_1, \ldots, e_i) = \prod_{i=1}^{\infty} \frac{1}{1 + t_i}.
\]

As shown in [6], the polynomials \( Q_i(c_1, \ldots, c_i) \) in the Chern classes of certain Calabi-Yau manifolds \( X \) are related to the coefficients of the generalized hypergeometric series expansion of the period (holomorphic at a maximum degeneracy point) of a mirror of \( X \). In particular, if \( X \) is a Calabi-Yau hypersurface of dimension 4 in a nonsingular toric Fano manifold, then

\[
\int_X Q_4(c_1, c_2, c_3, c_4) = \frac{1}{24} \sum_{ijkl} K_{ijkl} \frac{\partial^4 c(0, \ldots, 0)}{\partial \rho_i \partial \rho_j \partial \rho_k \partial \rho_l},
\]

where the \( c(\rho_1, \ldots, \rho_r) \) are coefficients in the expansion of the period and \( K_{ijkl} \) is the (suitably normalized) 4-point function corresponding to a mirror of \( X \). In [6] it is shown that the polynomials \( Q_i \) have the form

\[
Q_1(c_1) = \gamma c_1 \quad \text{and} \quad Q_i(c_1, \ldots, c_i) = \zeta(i) c_i + \cdots, i > 1.
\]

In this note we show that the polynomials \( Q_i \) have an explicit expression in terms of multiple zeta values (called multiple harmonic series in [3, 4]), which have previously appeared in connection with Kontsevich’s invariant in knot theory [8, 5], and in quantum field theory [1].

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2. SYMMETRIC AND QUASI-SYMMETRIC FUNCTIONS

For the convenience of the reader, we collect in this section the definitions about symmetric and quasi-symmetric functions we will need. Let \( t_1, t_2, \ldots \) be a countable sequence of indeterminates, each having degree 1, and let \( P \subset \mathbb{Q}[[t_1, t_2, \ldots]] \) be the set of formal power series in the \( t_i \) having bounded degree. Then \( P \) is a graded \( \mathbb{Q} \)-algebra. Let \( f \) be an element of \( P \). Then \( f \) is a symmetric function if

\[
\text{coefficient of } t_{n_1}^{i_1} t_{n_2}^{i_2} \cdots t_{n_k}^{i_k} \text{ in } f = \text{coefficient of } t_{m_1}^{i_1} t_{m_2}^{i_2} \cdots t_{m_k}^{i_k} \text{ in } f
\]

for any pair \((n_1, n_2, \ldots, n_k), (m_1, m_2, \ldots, m_k)\) of \( k \)-tuples of distinct positive integers. Following [2], we say \( f \) is a quasi-symmetric function if equation (2) holds whenever \( n_1 < n_2 < \cdots < n_k \) and \( m_1 < m_2 < \cdots < m_k \). The sets Sym and QSym of symmetric and quasi-symmetric functions respectively are both subalgebras of \( P \), with \( \text{Sym} \subset \text{QSym} \).

For a composition (ordered sequence of positive integers) \( I = (i_1, i_2, \ldots, i_k) \), the corresponding monomial quasi-symmetric function \( M_I \in \text{QSym} \) is defined by

\[
M_I = \sum_{n_1 < n_2 < \cdots < n_k} t_{n_1}^{i_1} t_{n_2}^{i_2} \cdots t_{n_k}^{i_k}.
\]

Evidently \( \{M_I | I \text{ is a composition}\} \) is a basis for QSym as a vector space. For any composition \( I \), there is a partition \( \pi(I) \) given by forgetting the ordering. For any partition \( \lambda \), the monomial symmetric function \( m_\lambda \) is the sum of all the \( M_I \) over all distinct \( I \) such that \( \pi(I) = \lambda \), e.g., \( m_{(2,1)} = M_{(2,1)} + M_{(1,2)} \). The monomial symmetric functions are a vector space basis for Sym. The elementary symmetric function \( e_k \) is \( M_{I_k} = m_{\pi(I_k)} \), where \( I_k \) is the composition consisting of \( k \) 1’s, and the power-sum symmetric function \( p_k \) is \( M_{(k)} = m_k \). If for a partition \( \lambda = \pi(\lambda_1, \lambda_2, \ldots) \) we let \( e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots \) and \( p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \), then it is well known that \( \{e_\lambda | \lambda \text{ is a partition}\} \) and \( \{p_\lambda | \lambda \text{ is a partition}\} \) are vector space bases for Sym.

3. THE FORMULA FOR THE \( Q_i \)

In [3] it is shown (Theorem 5.1) that the homomorphism \( \zeta : \text{Sym} \rightarrow \mathbb{R} \) such that \( \zeta(p_1) = \gamma \) and \( \zeta(p_i) = \zeta(i) \) for \( i > 2 \) satisfies

\[
\zeta \left( \sum_{i \geq 0} e_i z^i \right) = \frac{1}{\Gamma(1 + z)}.
\]

Our main result expresses the polynomials \( Q_i \) in terms of \( \zeta \).

**Theorem.** For any partition \( \lambda \) of \( i \), the coefficient of \( e_\lambda \) in \( Q_i(e_1, \ldots, e_i) \) is \( \zeta(m_\lambda) \).

**Proof.** Using equations (1) and (3), we have

\[
\sum_{i \geq 0} Q_i(e_1, e_2, \ldots) = \prod_{i=1}^{\infty} \frac{1}{\Gamma(1 + t_i)} = \prod_{i=1}^{\infty} \sum_{j=0}^{\infty} \zeta(e_j) t_i^j = \sum_{\lambda} \zeta(e_\lambda) m_\lambda.
\]

Now the transition matrix \( M \) from the basis \( \{e_\lambda\} \) of Sym to the basis \( \{m_\lambda\} \), i.e.

\[
e_\lambda = \sum_\mu M_{\lambda \mu} m_\mu,
\]
is known to be symmetric (see Ch. I, §6 of [7]), so we have

$$\sum_{\lambda} \zeta(e_{\lambda}) m_{\lambda} = \sum_{\lambda} \sum_{\mu} M_{\lambda\mu} \zeta(m_{\mu}) m_{\lambda} = \sum_{\mu} \zeta(m_{\mu}) \sum_{\lambda} M_{\mu\lambda} m_{\lambda} = \sum_{\mu} \zeta(m_{\mu}) e_{\mu},$$

and the result follows.

4. Multiple zeta values

The homomorphism $\zeta : \text{Sym} \to \mathbb{R}$ above is the restriction to Sym of a homomorphism defined in [4] from QSym to $\mathbb{R}$. The definition in [4] is motivated by the multiple zeta values introduced in [3] and [8], i.e.

$$\zeta(i_1, i_2, \ldots, i_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} 1 / n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k},$$

where $i_1, i_2, \ldots, i_k$ are positive integers with $i_1 > 1$. To explain the connection, let $\mathfrak{S}^1$ be the rational vector space of polynomials in the noncommuting variables $z_1, z_2, \ldots$; then $\mathfrak{S}^1$ becomes isomorphic to QSym if it is given the (commutative) multiplication * defined by the inductive rule

$$z_i w_1 * z_j w_2 = z_i (w_1 * z_j w_2) + z_j (z_i w_1 * w_2) + z_{i+j} (w_1 * w_2)$$

for any words $w_1, w_2$ in the $z_i$. (In fact the isomorphism is simply the map that sends $z_i, \ldots, z_{ik} \in \mathfrak{S}^1$ to $M_{(i_1, \ldots, i_k)} \in \text{QSym}$: see [4] for details.) The algebra Sym of symmetric functions can be identified with the subspace of $\mathfrak{S}^1$ generated by linear combinations of monomials invariant under permutation of subscripts, e.g., $z_2^2 = m_{22}$ and $z_1 z_2 + z_2 z_1 = m_{21}$; $z_i$ and $z_i^k$ correspond to $p_i$ and $e_i$ respectively. As an algebra $\mathfrak{S}^1$ is generated by Lyndon words in the $z_i$, i.e., monomials $w$ such that for any nontrivial decomposition $w = uv$ one has $v > w$, where the $z_i$ are ordered as $z_1 > z_2 > \cdots$ and this order is extended to monomials lexicographically. Then the only Lyndon word that starts with $z_1$ is $z_1$ itself, and we can define a homomorphism $\zeta : \mathfrak{S}^1 \to \mathbb{R}$ by giving its value on Lyndon words $w = z_{i_1} z_{i_2} \cdots z_{i_k}$:

$$\zeta(w) = \begin{cases} \gamma, & w = z_1, \\ \zeta(i_1, i_2, \ldots, i_k), & \text{otherwise}. \end{cases}$$

By the results of [4], $\zeta(z_{i_1} z_{i_2} \cdots z_{i_k})$ coincides with $\zeta(i_1, i_2, \ldots, i_k)$ as defined by equation (4) whenever $i_1 > 1$.

Since the power-sum symmetric functions $p_i$ generate the algebra Sym, we can compute $\zeta(m_{\lambda})$ by first expressing $m_{\lambda}$ in terms of power-sum functions (see [7], p. 109 for an explicit formula) and then applying the homomorphism $\zeta$. Hence the coefficient of each monomial $c_{\lambda}$ in $Q_i(c_1, \ldots, c_i)$ is a polynomial in the numbers $\gamma$ and $\zeta(i)$, $i \geq 2$. For example, since $m_3 = p_3$, $m_{21} = p_1 p_2 - p_3$ and $m_{111} = \frac{1}{6} (p_3^2 - 3p_1 p_2 + 2p_3)$, we have

$$Q_3(c_1, c_2, c_3) = \zeta(3)c_3 + (\gamma \zeta(2) - \zeta(3))c_1 c_2 + \frac{1}{6} (\gamma^3 - 3\gamma \zeta(2) + 2\zeta(3))c_1^3,$$

which corrects equation (1.5) of [4]. Similarly we can obtain equations (1.3), (1.4) (the coefficient of $c_1^2$ should be $\frac{1}{2}(\gamma^2 - \zeta(2))$), and (1.6) of [4].

If $m_3$ is a monomial symmetric function such that the partition $\lambda$ involves no 1’s, there is an alternative method of computing $\zeta(m_{\lambda})$: in this case $\zeta(m_{\lambda})$ is just a symmetric sum of ordinary multiple zeta values, e.g. $\zeta(m_{32}) = \zeta(2, 3) + \zeta(3, 2)$,
and such a symmetric sum can be written as a sum of products of the numbers $\zeta(i)$ by Theorem 2.2 of \cite{homan1}; one can use Euler’s formula for $\zeta(2n)$ to simplify further. Since $c_1 = 0$ on a Calabi-Yau manifold, only terms $\zeta(m_\lambda)c_\lambda$ with $\lambda$ having no 1’s appear in $Q_i$, and we have in this case

\[
Q_2(c_2) = \zeta(2)c_2, \\
Q_3(c_2, c_3) = \zeta(3)c_3, \\
Q_4(c_2, c_3, c_4) = \zeta(4)c_4 + \zeta(2, 2)c_2^2 = \zeta(4)c_4 + \frac{3}{4}\zeta(4)c_2^2, \\
Q_5(c_2, c_3, c_4, c_5) = \zeta(5)c_5 + (\zeta(3, 2) + \zeta(2, 3))c_2c_3 = \zeta(5)c_5 + (\zeta(2)\zeta(3) - \zeta(5))c_2c_3, \\
\text{and} \\
Q_6(c_2, c_3, c_4, c_5, c_6) = \zeta(6)c_6 + (\zeta(4, 2) + \zeta(2, 4))c_2c_4 + \zeta(2, 2, 2)c_2^2 + \zeta(3, 3)c_3^2 \\
= \zeta(6)c_6 + \frac{3}{4}\zeta(6)c_2c_4 + \frac{3}{16}\zeta(6)c_2^3 + \frac{1}{2}(\zeta(3)^2 - \zeta(6))c_3^2.
\]

References


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