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LINEAR SYSTEMS ON ABELIAN VARIETIES OF DIMENSION $2g + 1$

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Abstract. We show that polarisations of type $(1, ..., 1, 2g+2)$ on g-dimensional abelian varieties are *never* very ample, if $g \geq 3$. This disproves a conjecture of Debarre, Hulek and Spandaw. We also give a criterion for non-embeddings of abelian varieties into $2g + 1$-dimensional linear systems.

1. Introduction

Let L be an ample line bundle of type $\delta = (d_1, d_2, ..., d_g)$ on an abelian variety A of dimension g. Classical results of Lefschetz ($n \geq 3$) and Ohbuchi ($n = 2$) imply very ampleness of L^n, if $|L|$ has no fixed divisor when $n = 2$. Suppose L is an ample line bundle of type $(1, ..., 1, d)$ on A. When $g = 2$, Ramanan (see [4]) has shown that if $d \geq 5$ and the abelian surface does not contain elliptic curves, then L is very ample. When $g \geq 3$, Debarre, Hulek and Spandaw (see [3], Corollary 2.5, p. 201) have shown the following.

Theorem 1.1. *Let (A, L) be a generic polarized abelian variety of dimension g and type $(1, ..., 1, d)$. For $d > 2^g$, the line bundle L is very ample.*

They further conjecture that if $d \geq 2g + 2$, then the line bundle L is very ample (see [3], Conjecture 4, p. 184). In particular, when $g = 3$ and $d \geq 8$, their results (for $d \geq 9$) and conjecture (for $d = 8$) imply that L is very ample.

The results due to Barth ([1]) and Van de Ven ([5]) show

Theorem 1.2. *For $g \geq 3$, no abelian variety A_g can be embedded in $\mathbb{P}^d$, for $d \leq 2g$.*

In particular, it implies that line bundles of type $(1, ..., 1, d)$, $d \leq 2g + 1$, are never very ample.

We show

Theorem 1.3. *Suppose L is an ample line bundle of type $(1, ..., 1, d)$ on an abelian variety A, of dimension g. If $g \geq 3$ and $d \leq 2g + 2$, then L is never very ample.*

This disproves the conjecture of Debarre et. al when $d = 2g + 2$ and gives a different proof of Theorem 1.2, for morphisms into the complete linear system $|L|$. The proof of Theorem 1.3 also indicates the type of singularities of the image in $|L|$.

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Now any abelian variety A of dimension g can be embedded in a projective space of dimension $2g+1$.

Consider a morphism $A \longrightarrow |V|$, where $dim|V| = 2g+1$. Suppose the involution $i : A \longrightarrow A$, $a \mapsto -a$ lifts to an involution on the vector space V, hence on the linear system $|V|$. (Such a situation will arise, essentially, if A is embedded by a symmetric line bundle into its complete linear system, of dimension greater than $2g+1$. One may then project the abelian variety from a vertex which is invariant for the involution i to a projective space of dimension $2g+1$, and the involution i will then descend down to this projection.)

Then we show

Theorem 1.4. *Suppose there is a morphism* $A \xrightarrow{\phi} |V|$, *with* $dim|V| = 2g+1$ *and the involution* i *acting on the vector space* V. *If degree* $\phi(A) > 2^{2g}$ *and* $dimV_+ \neq dimV_-$, *then the morphism* ϕ *is never an embedding, for all* $g \geq 1$. *In fact,* ϕ *identifies some pairs* $\{a, -a\}$, *where* a *is not a* 2*-torsion element of* A. *Here* V_+ *and* V_- *denote the* ± 1*-eigenspaces of* V, *for the involution* i.

When $dimV_+ = dimV_-$, the morphism ϕ need not identify any pairs $\{a, -a\}$ in $|V|$ (see Remark 3.1 for counterexamples).

2. Proof of Theorem 1.3

Consider a pair (A, L), as in Theorem 1.3.

We may assume, after suitable translation by an element of A, that L is a symmetric line bundle on A, i.e. there is an isomorphism $L \simeq i^*L$, for the involution $i : A \longrightarrow A$, $a \mapsto -a$. This induces an involution on the vector space $H^0(L)$, also denoted as i. Let $H^0(L)^+$ and $H^0(L)^-$ denote the $+1$ and -1-eigenspaces of $H^0(L)$, for the involution i and $h^0(L)^+$ and $h^0(L)^-$ denote their respective dimensions. Further, we assume that L is of characteristic 0. Then by [2], 4.6.6, $h^0(L)^{\pm} = \frac{h^0(L)}{2} \pm 2^{g-s-1}$, where s is the number of odd integers in the type δ of L. Choose a normalized isomorphism $\psi : L \simeq i^*L$, i.e. the fibre map $\psi(0) : L(0) \longrightarrow L(0)$ is $+1$.

Let A_2 denote the set of torsion 2 points of A. If $a \in A_2$, then $\psi(a) : L(a) \longrightarrow L(a)$ is either $+1$ or -1.

Let

$$A_2^+ = \{a \in A_2 : \psi(a) = +1\}$$

and

$$A_2^- = \{a \in A_2 : \psi(a) = -1\}$$

and $Card(A_2^+)$ and $Card(A_2^-)$ denote their respective cardinalities.

Consider the associated morphism $A \xrightarrow{\phi_L} \mathbb{P}H^0(L)$ and let

$$\mathbb{P}_+ = \mathbb{P}\{s = 0 : s \in H^0(L)^-\}$$

and

$$\mathbb{P}_- = \mathbb{P}\{s = 0 : s \in H^0(L)^+\}.$$

Then the involution i acts trivially on the subspaces $\mathbb{P}_+$ and $\mathbb{P}_-$ of $\mathbb{P}H^0(L)$. Moreover, $\phi_L(A_2^+) \subset \mathbb{P}_+$ and $\phi_L(A_2^-) \subset \mathbb{P}_-$.

Lemma 2.1. *If* $a \in A_2^+$, *then the intersection of the image* $\phi_L(A)$ *and* $\mathbb{P}_+$ *is transversal at the point* $\phi_L(a)$.

Proof. The action of the involution i at the tangent space, $T_{A,a}$, at a, is -1. If the intersection of $\phi_L(A)$ with $\mathbb{P}_+$ is not transversal at $\phi_L(a)$, then $\phi_{L*}(T_{A,a})$ intersects $\mathbb{P}_+$, giving a i-fixed non-trivial subspace of $T_{A,a}$, which is not true. (This argument was given by M. Gross.) □

Let $Z = \phi_L(A) \cap \mathbb{P}_+$ in $\mathbb{P}H^0(L)$. Then $\phi_L(A_2^+) \subset Z$. Suppose $dim Z > 0$. Since the involution i acts trivially on Z, the morphism ϕ_L restricts on $\phi_L^{-1}(Z) \longrightarrow Z$, as a morphism of degree at least 2, with its Galois group containing $\langle i \rangle$. If $dim Z = 0$, then by Lemma 2.1, the points of $\phi_L(A_2^+)$ have multiplicity 1 in Z. Let $r = deg Z - Card(A_2^+)$. Then there are $\frac{r}{2}$-points on $\phi_L(A)$ on which the involution i acts trivially, i.e. there are $\frac{r}{2}$-pairs $(a, -a)$, $a \in A - A_2$, which are identified transversally by ϕ_L. By $K(L)$-invariance of the image $\phi_L(A)$, there are more such pairs.

Remark 2.2. If $dim Z > 0$ or $r > 0$, then L is not very ample.

Case 1: $d = 2m$ and $m \leq g+1$.

By [2], 4.6.6, $h^0(L)^+ = m+1$ and $h^0(L)^- = m-1$.

Hence $dim \mathbb{P}_+ = m$ and $dim \mathbb{P}_- = m-2$.

a) If $m < g+1$, then $dim Z \geq g + m - 2m + 1 > 0$.

b) If $m = g+1$, by Riemann-Roch, $deg \phi_L(A) = (2g+2).g!$. If $dim Z = 0$, then since $\mathbb{P}_+$ and $\phi_L(A)$ have complementary dimensions in $\mathbb{P}H^0(L)$, $deg Z = (2g+2).g!$.

Now by [2], Exercise 4.12 b)-Remark 4.7.7,

$$Card(A_2^+) \leq 2^{2g-(g-1)-1}(2^{g-1}+1)$$

$$= 2^g(2^{g-1}+1).$$

Since $g \geq 3$, $r \geq (2g+2).g! - 2^g(2^{g-1}+1) > 0$.

Hence by Remark 2.2, L is not very ample.

Case 2: $d = 2m-1$ and $m \leq g+1$.

Then $h^0(L)^+ = m$ and $h^0(L)^- = m-1$. Hence $dim \mathbb{P}_+ = m-1$ and $dim \mathbb{P}_- = m-2$.

a) If $m < g+1$, then $dim Z \geq g + m + 1 - 2m > 0$.

b) If $m = g+1$, as in **Case 1**, $deg \phi_L(A) = (2g+1)g!$, and $\mathbb{P}_+$ and $\phi_L(A)$ have complementary dimension in $\mathbb{P}H^0(L)$. Hence if $dim Z = 0$, then $deg Z = (2g+1)g!$. Also, in this case, $Card(A_2^+) \leq 2^{g-1}(2^g+1)$.

Since $g \geq 3$, $r \geq (2g+1)g! - 2^{g-1}(2^g+1) > 0$. Hence by Remark 2.2, L is not very ample. □

3. Morphisms into i-invariant linear systems

Proof of Theorem 1.4. Consider the morphism $A \xrightarrow{\phi} |V|$, with the involution i acting on the vector space V. Let

$$\mathbb{P}_+ = \mathbb{P}\{s = 0 : s \in V_-\}$$

and

$$\mathbb{P}_- = \mathbb{P}\{s = 0 : s \in V_+\},$$

where V_+ and V_- denote the $+1$ and -1-eigenspaces of the vector space V, for the involution i. Let $d = degree \phi(A)$.

Now $dim\mathbb{P}_+ > g$ or $dim\mathbb{P}_+ < g$ or $dim\mathbb{P}_+ = g$.

Case 1: $dim\mathbb{P}_+ > g$.

Consider the intersection $Z = \mathbb{P}_+ \cap \phi(A)$.

Then $dimZ \geq g + g + 1 - 2g - 1 \geq 0$.

As in the proof of Theorem 1.3, if $dimZ > 0$, then the restricted morphism $\phi^{-1}(Z) \longrightarrow Z$ is of degree at least 2, since i acts trivially on Z. Suppose $dimZ = 0$. Then the intersection of $\phi(A)$ and $\mathbb{P}_+$ is transversal at the image of torsion 2 points of A, by Lemma 2.1. Since $Card(A_2) = 2^{2g}$ and $degree(\phi(A)) > 2^{2g}$, there are pairs $\{a, -a\}$ on A which get identified transversally by the morphism ϕ.

Case 2: $dim\mathbb{P}_+ < g$.

In this situation, $dim\mathbb{P}_- > g$ and we can repeat the above argument.

Hence ϕ is never an embedding. □

Remark 3.1. When $dimV_+ = dimV_-$, the morphism ϕ need not identify any pair of points $\{a, -a\}$ in the linear system $|V|$. For example, consider a symmetric line bundle L, of type $(1, 1, 9)$, on a generic abelian threefold A. Then L is very ample and $dimH^0(L)_+ = 5$ and $dimH^0(L)_- = 4$. Hence $dim\mathbb{P}_+ = 4$ and $dim\mathbb{P}_- = 3$. Consider the scroll $S_A = \bigcup_{a \in A - A_2} l_{a,-a}$, where $l_{a,-a}$ is the line joining the points a and $-a$, in $|L|$. Then the line $l_{a,-a}$ is invariant for the involution i and has two fixed points, one of them, say $x \in \mathbb{P}_+$ and the other, $x' \in \mathbb{P}_-$. This defines a map $A - A_2 \longrightarrow \mathbb{P}_+$, $a \mapsto x$. Hence S_A intersects $\mathbb{P}_+$ in at most a $3-$dimensional subset. Now we can project from a point of $\mathbb{P}_+$, outside this subset, and the projection will have the fixed spaces of i to be equidimensional. Also, by the choice of the point of projection, there are no pairs $\{a, -a\}$ identified in the projection.

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