

ACL HOMEOMORPHISMS AND LINEAR DILATATION

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ABSTRACT. We establish an integrability condition on the linear dilatation to guarantee ACL.

1. INTRODUCTION

Let D be a domain in \mathbf{R}^n , $n \geq 1$, and $f: D \rightarrow \mathbf{R}^n$ a homeomorphism. For $x \in D$ and $0 < r < d(x, \partial D)$ we set

$$L(x, f, r) = \sup\{|f(x) - f(y)| : y \in \partial B(x, r)\},$$

$$l(x, f, r) = \inf\{|f(x) - f(y)| : y \in \partial B(x, r)\}$$

where $B(x, r)$ stands for the open ball centered at x and radius r and $\partial B(x, r)$ for its boundary. The linear dilatation of f at x is defined as

$$H(x, f) = \limsup_{r \rightarrow 0} H(x, f, r)$$

where $H(x, f, r) = L(x, f, r)/l(x, f, r)$. At every point $x \in D$, $H(x, f) \in [1, \infty]$ and $H(x, f) = \|f'(x)\|/l(f'(x))$ provided that f is differentiable at x with $l(f'(x)) > 0$. Here the norm $\|f'(x)\|$ of the derivative $f'(x)$ of f at x is defined as

$$\|f'(x)\| = \sup_{|h|=1} |f'(x)h|$$

and the minimum norm $l(f'(x))$ as

$$l(f'(x)) = \inf_{|h|=1} |f'(x)h|.$$

A well-known result of Gehring [G1] says that if a homeomorphism f has its linear dilatation $H(x, f)$ uniformly bounded in D , then f is a quasiconformal mapping. In particular f is ACL in D , which means that f is absolutely continuous on almost every line segment parallel to the coordinate axis in D . In [T2] Tukia conjectured that the condition

$$(1.1) \quad m(\{x \in D : H(x, f) > t\}) < ct^{-\alpha}$$

for some $\alpha > 3$ is sufficient for the ACL property of a plane homeomorphism f . Indeed, this was proved in [FA] together with a space analogue. In this paper we show that $\alpha > 2$ in (1.1) implies the ACL property in the plane case with a

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corresponding improvement in space. Our proof is based on the Gehring method in [G1].

Theorem 1.2. *Suppose that a homeomorphism $f: D \rightarrow \mathbf{R}^n$, $D \subset \mathbf{R}^n$ a domain, a subset S of D and $s \in (1, \infty]$ satisfy the conditions:*

- (1.3) $s > n/(n - 1)$,
- (1.4) $H(x, f) < \infty$ for each $x \in D \setminus S$,
- (1.5) $H(x, f) \in L^s_{loc}(D)$,
- (1.6) S has σ -finite $(n - 1)$ -Hausdorff measure.

Then f is ACL in D .

Remarks 1.7. (a) The assumption (1.3) rules out the case $n = 1$, see Section 2.

(b) The assumption (1.6) means that the set S is of the form $S = \bigcup S_i$ where $H^{n-1}(S_i) < \infty$ and H^{n-1} refers to the $(n - 1)$ -dimensional Hausdorff measure. For the definition of the Hausdorff measure see e.g. [V, p. 99] or [G1].

(c) We do not know the optimal value for s in Theorem 1.2. An example shows that one cannot take $s < 1$. In this example the positive real axis, and lines parallel to it, are mapped to the curve $x \sin(1/x)$ and to its translations. This construction produces a non-ACL homeomorphism f of \mathbf{R}^2 which is the identity on the left half plane and satisfies $H(x, f) \in L^s_{loc}(\mathbf{R}^2)$ for any $s < 1$.

In Section 2 we consider some properties of homeomorphisms $f: D \rightarrow \mathbf{R}^n$ satisfying $H(x, f) < \infty$ a.e. in D . In Corollary 2.4 we show that $f' \in L^p_{loc}(D)$, $p = sn/(n - 1 + s)$, under the conditions of Theorem 1.2. In particular this implies that f is ACL^p. The section also contains some examples. Section 3 is devoted to the proof of Theorem 1.2.

2. MAPPINGS WITH $H(x, f) < \infty$ A.E.

If a homeomorphism $f: D \rightarrow \mathbf{R}^n$ satisfies $H(x, f) < \infty$ for a.e. $x \in D$ or even $\text{ess sup}_{x \in D} H(x, f) < \infty$, then f need not be ACL. A well-known example is constructed from the Cantor staircase function $g: [0, 1] \rightarrow [0, 1]$, i.e. g is an increasing function with the property $g'(x) = 0$ for a.e. $x \in [0, 1]$. Now $f: (0, 1) \times (0, 1) \rightarrow (0, 2) \times (0, 1)$ defined as $f(x, y) = (g(x) + x, y)$ is a homeomorphism with $H(z, f) = 1$ a.e. but f is not ACL. Moreover, no boundedness condition, except $H(x, f) = 1$ for all x , in the case $n = 1$ implies absolute continuity. Indeed, an increasing homeomorphism $f: \mathbf{R} \rightarrow \mathbf{R}$ is called K -quasisymmetric if it satisfies

$$\frac{1}{K} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq K$$

for all $x \in \mathbf{R}$ and $t > 0$. If f is K -quasisymmetric, then $H(x, f) \leq K$ for all $x \in \mathbf{R}$. Beurling and Ahlfors [BA] constructed for each $K > 1$ a K -quasisymmetric mapping f which is not absolutely continuous. For more striking examples of such mappings see [T1]. Hence no integrability condition for $H(x, f)$ like (1.5) implies absolute continuity for $n = 1$.

However, homeomorphisms which satisfy $H(x, f) < \infty$ a.e. have some nice properties.

Theorem 2.1. *Suppose that a homeomorphism $f: D \rightarrow \mathbf{R}^n$ satisfies $H(x, f) < \infty$ a.e. in D . Then f is a.e. differentiable.*

Proof. Fix an open set $G \subset\subset D$ and let $\Phi(E) = |f(E)|$ for each Borel set $E \subset G$. Then Φ is a finite Borel measure on G and hence it has a finite derivative

$$\Phi'(x) = \lim_{r \rightarrow 0} \frac{\Phi(B(x, r))}{|B(x, r)|}$$

at a.e. $x \in G$. Here and in the following $|A|$ means the Lebesgue measure of a set $A \subset \mathbf{R}^n$.

Now at almost every point x of G , $\Phi'(x)$ exists and $H(x, f) < \infty$. Fix such a point x . Let $y \in G$ with $0 < |x - y| < d(x, \partial G)$. Now

$$\begin{aligned} \left(\frac{|f(x) - f(y)|}{|y - x|} \right)^n &\leq \left(\frac{L(x, f, |y - x|)}{l(x, f, |y - x|)} \right)^n \left(\frac{l(x, f, |y - x|)}{|y - x|} \right)^n \\ &\leq H(x, f, |y - x|)^n \frac{\Phi(B(x, |y - x|))}{|B(x, |y - x|)|} \end{aligned}$$

and letting $y \rightarrow x$ we see that

$$\limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|} \leq H(x, f) \Phi'(x)^{\frac{1}{n}} < \infty.$$

By the Rademacher-Stepanov theorem the mapping f is a.e. differentiable in G . The theorem follows.

Theorem 2.2. *Suppose that a homeomorphism $f: D \rightarrow \mathbf{R}^n$ satisfies $H(x, f) \in L^s_{loc}(D)$, $s \in [1, \infty]$. Then $f' \in L^p_{loc}(D)$ with $p = sn/(n - 1 + s)$ and $p = n$ if $s = \infty$.*

Proof. We may assume that f is sense-preserving. Since $H(x, f) < \infty$ a.e. in D , Theorem 2.1 implies that $f'(x)$ exists a.e. If f is differentiable at x and $H(x, f) < \infty$, then an elementary argument shows that

$$(2.3) \quad \|f'(x)\|^n \leq H(x, f)^{n-1} J(x, f)$$

where $J(x, f)$ is the Jacobian determinant of $f'(x)$.

Fix an open set $G \subset\subset D$. For $s < \infty$ (2.3) and the Hölder inequality imply

$$\begin{aligned} \int_G |f'(x)|^p dx &\leq \int_G H(x, f)^{\frac{p(n-1)}{n}} J(x, f)^{\frac{p}{n}} dx \\ &\leq \left[\int_G H(x, f)^{\frac{p(n-1)}{(n-p)}} dx \right]^{\frac{(n-p)}{n}} \left[\int_G J(x, f) dx \right]^{\frac{p}{n}} \\ &\leq \left[\int_G H(x, f)^s dx \right]^{\frac{(n-p)}{n}} |f(G)|^{\frac{p}{n}} < \infty \end{aligned}$$

as required. For $s = \infty$ the proof is similar. Note that the inequality

$$\int_G J(x, f) dx \leq |f(G)|$$

always holds for an a.e. differentiable homeomorphism; see [RR, p. 360].

Corollary 2.4. *Under the hypotheses of Theorem 1.2 f is a.e. differentiable and $f' \in L^p_{loc}(D)$, $p = sn/(n - 1 + s)$. In particular f is ACL^p in D .*

3. PROOF OF THEOREM 1.2

We prove Theorem 1.2 in the case $S = \emptyset$. By the theorem of Gross (see e.g. [V, p. 103]) the condition (1.1.2) implies that for a.e. line L parallel to some coordinate axis, $S \cap L$ is countable. For a continuous function a countable set E does not destroy absolute continuity if an estimate like (3.8) below holds for compact sets F in the complement of E . Thus the case $S \neq \emptyset$ does not lead to essential difficulties; see [G1].

Pick a closed cube $Q \subset\subset D$ whose sides are parallel to the coordinate axes and write $Q' = \frac{1}{2}Q$ for the cube with the same center as Q and side length half of that of Q . In order to show that f is ACL it suffices to show that f is absolutely continuous on almost every line segment of Q' parallel to the coordinate axes. Renormalizing we may assume that $Q = [-2, 2]^n$ and by symmetry it is sufficient to consider segments parallel to the x_n -axis. Let $P: \mathbf{R}^n \rightarrow \mathbf{R}^{n-1}$ denote the projection $P(x) = x - x \cdot e_n e_n$ and for $y \in P(Q) \subset \mathbf{R}^{n-1}$ write $I = I(y) = Q' \cap P^{-1}(y)$ for the line segment parallel to the x_n -axis in Q' .

Next for a Borel set $E \subset P(Q)$ write

$$\Phi(E) = |f(Q \cap P^{-1}(E))| \leq |f(Q)| < \infty.$$

Then Φ is a finite Borel measure on $P(Q)$ and hence it has a finite derivative $\Phi'(y)$ for almost all $y \in P(Q')$. We choose $y \in P(Q')$ such that (i) $\Phi'(y)$ exists and (ii) $H(x, f) \in L^s(I(y))$. The last assertion follows from the Fubini theorem. It suffices to show that f is absolutely continuous on $I(y)$.

To this end let $F \subset I(y)$ be a compact set. For each $k = 0, 1, 2, \dots$ write

$$F_k = \{x \in F : 2^k \leq H(x, f) < 2^{k+1}\}.$$

Then F_k is a Borel set and $F = \bigcup F_k$ because of (1.4) and our assumption $S \neq \emptyset$. Note also that $H(x, f) \geq 1$ for every x . We first derive the estimate

$$(3.1) \quad H^1(fF_k) \leq C 2^k H^1(F_k)^{\frac{n-1}{n}}$$

where $C = (2^{2n+1}\Phi'(y))^{1/n}$.

For (3.1) fix k and for each $j = 1, 2, \dots$ consider the set

$$F_{k,j} = \{x \in F_k : L(x, f, r)^n \leq 2^{n(k+1)}|fB(x, r)|/\Omega_n \text{ for } 0 < r < 1/j\}$$

where $\Omega_n = |B(0, 1)|$. The sets $F_{k,j}$ are Borel sets and $F_{k,j} \subset F_{k,j+1}$ with

$$(3.2) \quad F_k = \bigcup_{j=1}^{\infty} F_{k,j}.$$

To see (3.2) let $x \in F_k$. Then $H(x, f) < 2^{k+1}$ and hence there is j such that

$$L(x, f, r)/l(x, f, r) < 2^{k+1}$$

for all $0 < r < 1/j$ and we obtain

$$L(x, f, r)^n < 2^{n(k+1)}l(x, f, r)^n \leq 2^{n(k+1)}|fB(x, r)|/\Omega_n.$$

This shows that $x \in F_{k,j}$ and (3.2) follows.

By the monotonicity and (3.2) it suffices to prove (3.1) for $F_{k,j}$ instead of F_k . Fix j and let F' be an arbitrary compact subset of $F_{k,j}$. Let $\epsilon > 0$ and $t > 0$. The continuity of the mapping $(x, r) \mapsto L(x, f, r)$ gives δ , $0 < \delta < 1/j$, such that $L(x, f, r) < t/2$ for $0 < r < \delta$ and for all $x \in F'$. By a well-known covering lemma for sets on a real line (see [G1, Lemma 1, p. 6]), for each sufficiently small

$r > 0, 0 < r < \delta$, there exists a covering of F' by a finite number of open balls $B_i = B(x_i, r), i = 1, \dots, l$, where (a) $x_i \in F', i = 1, \dots, l$, (b) each point of \mathbf{R}^n lies in at most two B_i and (c) $lr \leq H^1(F') + \epsilon$. Note that the normalizing condition gives

$$(3.3) \quad B_i \subset Q \cap P^{-1}(B)$$

where $B = B^{n-1}(y, r)$.

The union of the sets $f(B_i)$ covers $f(F')$ and

$$\text{dia}(fB_i) \leq 2L(x_i, f, r) < t.$$

Hence

$$H_t^1(fF') \leq \sum_{i=1}^l \text{dia}(fB_i)$$

where

$$H_t^1(A) = \inf\left\{ \sum \text{dia}(A_i) : \bigcup A_i \supset A, \text{dia}(A_i) < t \right\}$$

and the Hölder inequality together with the definition of $F_{k,j}$ yields

$$(3.4) \quad \begin{aligned} H_t^1(fF')^n &\leq \left(\sum_{i=1}^l \text{dia}(fB_i) \right)^n \leq l^{n-1} \sum_{i=1}^l \text{dia}(fB_i)^n \\ &\leq l^{n-1} 2^n \sum_{i=1}^l L(x_i, f, r)^n \leq \frac{l^{n-1} 2^n 2^{n(k+1)}}{\Omega_n} \sum_{i=1}^l |fB_i|. \end{aligned}$$

Since f is a homeomorphism, we obtain from (b) and (3.3) that

$$\sum_{i=1}^l |fB_i| \leq 2 \left| \bigcup_{i=1}^l fB_i \right| \leq 2\Phi(B)$$

and thus (3.4) and (c) yield

$$\begin{aligned} H_t^1(fF')^n &\leq 2^{n(k+2)+1} (H^1(F') + \epsilon)^{n-1} \Phi(B) / H^{n-1}(B) \\ &\leq 2^{n(k+2)+1} (H^1(F_{k,j}) + \epsilon)^{n-1} \Phi(B) / H^{n-1}(B). \end{aligned}$$

Since $H_t^1(fF') \rightarrow H^1(fF')$ as $t \rightarrow 0$, letting first $r \rightarrow 0$, then $\epsilon \rightarrow 0$, and finally $t \rightarrow 0$ we obtain

$$(3.5) \quad H^1(fF')^n \leq 2^{n(k+2)+1} H^1(F_{k,j})^{n-1} \Phi'(y).$$

Now F' is an arbitrary compact subset of $F_{k,j}$ and hence (3.5) holds for $F_{k,j}$ on the left-hand side of (3.5). This leads to the estimate (3.1).

Since $fF = \bigcup fF_k$, (3.1) implies

$$(3.6) \quad H^1(fF) \leq \sum H^1(fF_k) \leq C \sum 2^k H^1(F_k)^{\frac{n-1}{n}}.$$

The sets $F_k, k = 1, 2, \dots$, are disjoint and hence the integral estimate

$$(3.7) \quad \sum_{k=0}^{\infty} 2^{ks} H^1(F_k) \leq \int_F H(x, f)^s dx_n$$

is elementary. From (3.6), (3.7) and from the Hölder inequality we obtain

$$\begin{aligned}
 H^1(fF) &\leq C_1 \left(\sum_{k=0}^{\infty} 2^{ks} H^1(F_k) \right)^{\frac{n-1}{n}} \left(\sum_{k=0}^{\infty} 2^{k(n-s(n-1))} \right)^{\frac{1}{n}} \\
 (3.8) \qquad &\leq C_2 \left(\int_F H(x, f)^s dx_n \right)^{\frac{n-1}{n}}
 \end{aligned}$$

where C_2 depends only on n , s and $\Phi'(y)$. Note that the series

$$\sum_{k=0}^{\infty} 2^{k(n-s(n-1))}$$

converges because $s > n/(n-1)$ and hence $n - s(n-1) < 0$. Inequality (3.8) shows that f is absolutely continuous on $I(y)$ as required.

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