

INTERPRETATION OF THE DEFORMATION SPACE
 OF A DETERMINANTAL BARLOW SURFACE
 VIA SMOOTHINGS

YONGNAM LEE

(Communicated by Michael Stillman)

ABSTRACT. In this present paper, we provide an interpretation of the deformation space of a determinantal Barlow surface via smoothings.

1. INTRODUCTION

Let (y_1, y_2, y_3, y_4) be a coordinate of \mathbb{P}^3 , and consider a $D_5 = \langle b, a \rangle$ action on \mathbb{P}^3 via

$$\begin{aligned} b &: (y_1, y_2, y_3, y_4) \rightarrow (\epsilon y_1, \epsilon^2 y_2, \epsilon^3 y_3, \epsilon^4 y_4), \\ a &: (y_1, y_2, y_3, y_4) \rightarrow (y_4, y_3, y_2, y_1) \end{aligned}$$

where ϵ is the primitive 5-th root of unity. In [C1], Catanese studied a four-dimensional family of surfaces that are double coverings of \mathbb{Z}_5 -quotients of the \mathbb{Z}_5 -invariant symmetric determinantal quintics. We will refer to these \mathbb{Z}_5 -quotients of the \mathbb{Z}_5 -invariant symmetric determinantal quintics as *determinantal Godeaux surfaces*. Inside of this four-dimensional family, there is a two-dimensional D_5 -invariant symmetric determinantal Godeaux surface. After providing a twisted \mathbb{Z}_2 -action on the double cover, we have a two-dimensional subfamily of the moduli space of Barlow's example [B]. We will refer to these surfaces as *determinantal Barlow surfaces*. Let Σ be a D_5 -invariant symmetric determinantal quintic surface in \mathbb{P}^3 and let $\sigma : Y \rightarrow \Sigma$ be a double cover. Consider a commutative diagram

$$(1.1) \quad \begin{array}{ccc} Z = Y/\langle b \rangle & \longrightarrow & B = Y/\langle b, a\sigma \rangle \\ \downarrow & & \\ X = Y/\langle b, \sigma \rangle = \Sigma/\langle b \rangle. & & \end{array}$$

In diagram (1.1), X is a D_5 -invariant determinantal Godeaux surface, and B is a determinantal Barlow surface. By the construction of a D_5 -invariant determinantal Godeaux surface, there is an extra involution. The fixed divisor of this involution is a -3 -curve D_X (\mathbb{P}^1 with $N_{\mathbb{P}^1|X} = \mathcal{O}_{\mathbb{P}^1}(-3)$) without passing through four nodes (the double cover $Z \rightarrow X$ is branched over these four nodes). So the preimage of D_X in Z is two disjoint -3 -curves, D_1, D_2 . Since a -action fixes each point on D_1

Received by the editors May 10, 2000 and, in revised form, October 17, 2000.

2000 *Mathematics Subject Classification*. Primary 14J10, 14J17.

This work was supported by grant 1999-2-102-002-3 from the Interdisciplinary Research Program of the KOSEF and by the Sogang University Research Grants in 2000.

and D_2 , \mathbb{Z}_2 -action of $Z \rightarrow B$ interchanges D_1 and D_2 . So these two -3 -curves go to a -3 -curve D_B in B without passing through four nodes (the double cover $Z \rightarrow B$ is branched over these four nodes) [B]. By parameterizing \mathbb{Z}_5 -invariant symmetric determinantal quintics with 20 nodes, we can easily see that the general element of those set do not have the preimage of D_X . This argument shows that the natural map of $H^1(T_X) \rightarrow H^1(N_{D_X|X})$ is surjective. Therefore we have two directions of the first order deformation space induced by the cohomology of $H^1(N_{D_X|X})$. In this paper, we will prove that the surjectivity of the map of $H^1(T_X) \rightarrow H^1(N_{D_X|X})$ produces the surjectivity of the map of $H^1(T_B) \rightarrow H^1(N_{D_B|B})$ by the lifting of the first order deformation space to the commutative diagram (1.1) and the vanishing of $H^2(T_Z) = 0$. This vanishing was proved by Catanese-Le Brun [CL] and by the author [L] independently. Then we provide an interpretation of the deformation space of a determinantal Barlow surface via smoothings :

Theorem. *Consider a determinantal Barlow surface. Then the extension of the two-dimensional family of the determinantal Barlow surface, in the recovering six directions, can be interpreted as follows:*

- (1) *one independent smoothing direction for each of the four nodes of B ,*
- (2) *two independent directions corresponding to the global smoothings induced by the surjectivity of the map $H^1(T_B) \rightarrow H^1(N_{D_B|B})$.*

Throughout we work over the complex number field \mathbb{C} . The notation here follows the standard textbook [H].

2. PROOF OF THE THEOREM

Let V be a smooth projective surface and D a smooth curve in V . There is a natural map $n : T_V \rightarrow N_{D|V}$ induced by the composition of the maps $T_V \rightarrow T_V|_D \rightarrow N_{D|V}$. Define $T_{V,D}$ by the kernel of the map of n :

$$(2.1) \quad 0 \rightarrow T_{V,D} \rightarrow T_V \xrightarrow{n} N_{D|V} \rightarrow 0.$$

The short exact sequence (2.1) induces the long exact sequence

$$(2.2) \quad \begin{aligned} 0 \rightarrow H^0(T_{V,D}) \rightarrow H^0(T_V) \rightarrow H^0(N_{D|V}) \rightarrow H^1(T_{V,D}) \\ \rightarrow H^1(T_V) \rightarrow H^1(N_{D|V}) \rightarrow H^2(T_{V,D}) \rightarrow H^2(T_V). \end{aligned}$$

In the long exact sequence (2.2), each cohomology group relates the first order infinitesimal deformation or obstruction:

(2.2.1) $H^0(N_{D|V})$ classes the first order infinitesimal deformation of D in V and the obstruction lies in $H^1(N_{D|V})$,

(2.2.2) $H^1(T_{V,D})$ classes the first order infinitesimal deformation of the pair (V, D) and the obstruction lies in $H^2(T_{V,D})$,

(2.2.3) $H^1(T_V)$ classes the first order infinitesimal deformation of V and the obstruction lies in $H^2(T_V)$.

We also have the commutative diagram of the obstruction maps in (2.2):

$$(2.3) \quad \begin{array}{ccccccc} H^0(T_V) & \longrightarrow & H^0(N_{D|V}) & \longrightarrow & H^1(T_{V,D}) & \longrightarrow & H^1(T_V) \\ & & ob_1 \downarrow & & ob_2 \downarrow & & ob_3 \downarrow \\ & \xrightarrow{n} & H^1(N_{D|V}) & \longrightarrow & H^2(T_{V,D}) & \longrightarrow & H^2(T_V). \end{array}$$

The next lemma is easily obtained by the commutative diagram (2.3).

Lemma 1. (1) Assume that the natural map $n : H^1(T_V) \rightarrow H^1(N_{D|V})$ is surjective. Then $ob_3 = 0$ implies that $ob_2 = 0$.

(2) If $H^1(N_{D|V}) = 0$, then $ob_2 = 0$ if and only if $ob_3 = 0$.

The definition of $T_{V,D}$ induces a short exact sequence:

$$(2.4) \quad 0 \rightarrow T_V(I_D) \rightarrow T_{V,D} \xrightarrow{t} T_D \rightarrow 0.$$

It is interesting to find the cases when the natural maps $n : H^1(T_V) \rightarrow H^1(N_{D|V})$ or $t : H^1(T_{V,D}) \rightarrow H^1(T_D)$ are zero or surjective.

Definition 2. A simple normal crossing surface X (a projective surface with normal crossing, and each irreducible component is smooth) is smoothable if there is an analytic disc Δ in \mathbb{C} and a projective flat family of varieties $\pi : \mathcal{X} \rightarrow \Delta$ whose central fiber is X , and a general fiber $X_t = \pi^{-1}(t)$ is smooth for $t \neq 0$. Then $\pi : \mathcal{X} \rightarrow \Delta$ is called a smoothing of X .

Denote by T_X^i and \mathbb{T}_X^i the local and global deformation objects of Lichtenbaum-Schlessinger, respectively. In the case of reduced local complete intersection, we have

$$T_X^i = \mathcal{E}xt_X^i(\Omega_X, \mathcal{O}_X), \quad \mathbb{T}_X^i = \text{Ext}_X^i(\Omega_X, \mathcal{O}_X).$$

Since X is normal crossing, X is locally embedded in a smooth variety \mathcal{X} and we have

$$0 \rightarrow I_X/I_X^2 \rightarrow \Omega_{\mathcal{X}|X} \rightarrow \Omega_X \rightarrow 0$$

where I_X/I_X^2 is locally free \mathcal{O}_X -module. Therefore $T_X^2 = 0$.

There exists a spectral sequence $E_2^{p,q} = H^p(X, \mathcal{E}xt_X^q(\Omega_X, \mathcal{O}_X))$ and E_∞ goes to $\mathcal{E}xt_X^{p+q}(\Omega_X, \mathcal{O}_X)$. Also, there is a local-global exact sequence ([Go], §7.3)

$$\begin{aligned} 0 \rightarrow H^1(X, T_X^0) &\rightarrow \text{Ext}_X^1(\Omega_X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{E}xt_X^1(\Omega_X, \mathcal{O}_X)) \\ &\rightarrow H^2(X, T_X^0) \rightarrow \text{Ext}_X^2(\Omega_X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{E}xt_X^1(\Omega_X, \mathcal{O}_X)), \end{aligned}$$

so the following long exact sequence holds:

$$(2.5) \quad 0 \rightarrow H^1(T_X^0) \rightarrow \mathbb{T}_X^1 \rightarrow H^0(T_X^1) \rightarrow H^2(T_X^0) \rightarrow \mathbb{T}_X^2 \rightarrow H^1(T_X^1).$$

The space $H^1(T_X^0)$ classifies all “locally trivial” deformations of X , i.e. for which the singularities remain locally a product, and its obstruction lies in $H^2(T_X^0)$. So if there is a smoothing of X , then $\mathbb{T}_X^1 \rightarrow H^0(T_X^1)$ is not zero. Let $D = \text{Sing } X$. If there is a smoothing \mathcal{X} of X , then $T_X^1 \cong \mathcal{O}_D(E)$ for some effective divisor E . In addition, if \mathcal{X} is assumed to be smooth, then $T_X^1 \cong \mathcal{O}_D$. In general, T_X^1 is not \mathcal{O}_D [F, §2]. For example if $X = V \cup_D W$, then $T_X^1 \cong N_{D|V} \otimes N_{D|W}$.

Definition 3. Let X be a simple normal crossing variety. Assume $T_X^1 \cong \mathcal{O}_D$ for $D = \text{Sing } X$. Then X is called d -semistable.

We consider the simplest case: $X = V \cup_D W$ with $T_X^1 \cong \mathcal{O}_D$. Then there are relations between T_X^0 , and $T_{V,D}, T_{W,D}$ by definition of T_X^0 .

$$(2.6) \quad 0 \rightarrow T_X^0 \rightarrow T_{V,D} \oplus T_{W,D} \xrightarrow{\frac{1}{2}(t_1+t_2)^*} T_D \rightarrow 0,$$

$$(2.7) \quad 0 \rightarrow T_V(-D) \oplus T_W(-D) \rightarrow T_X^0 \rightarrow T_D \rightarrow 0.$$

Then the map $H^0(N_{D|V} \otimes N_{D|W}) \rightarrow H^2(T_V) \oplus H^2(T_W)$, that is a composition of the map $H^0(N_{D|V} \otimes N_{D|W}) \rightarrow H^2(T_X^0)$ and the maps induced by the exact sequences (2.6), (2.1), can be obtained via the following natural maps [DF].

Let ω_W be the obstruction class of the extension of the first-order neighborhood of D in W to the flat model, which is in $H^1(N_{D|W}^* \otimes T_D)$. The splitting of the exact sequence of $0 \rightarrow T_D \rightarrow T_W|_D \rightarrow N_{D|W} \rightarrow 0$ is the same as $\omega_W = 0$. Consider the natural map

$$\begin{aligned} H^0(N_{D|V} \otimes N_{D|W}) \otimes \omega_W &\rightarrow H^1(N_{D|V} \otimes T_D) \\ &\rightarrow H^1(N_{D|V} \otimes T_V|_D) \\ &\rightarrow H^2(T_V), \end{aligned}$$

induced by the exact sequence $0 \rightarrow T_V \rightarrow T_V(D) \rightarrow T_V(D)|_D \rightarrow 0$. The other direction is exactly the same as this. Then the next lemma is easily obtained by the observation.

Lemma 4. *Assume that the exact sequence $0 \rightarrow T_D \rightarrow T_W|_D \rightarrow N_{D|W} \rightarrow 0$ splits and that $H^2(T_W)$ vanishes. Then the map $H^0(T_X^1) \rightarrow H^2(T_V) \oplus H^2(T_W)$ is zero. In particular, if W is a ruled surface and D is a section in W , then the map $H^0(T_X^1) \rightarrow H^2(T_V) \oplus H^2(T_W)$ is zero.*

The next example in [PP] shows that the vanishing of the above map is not enough for the obstruction map to be zero even for a d -semistable case.

Example 5. Let V be a smooth projective surface and D a smooth curve in V . Choose an element $\eta \in H^1(N_{D|V})$. Since $H^1(N_{D|V}) = \text{Ext}^1(\mathcal{O}_D, N_{D|V})$, η corresponds to a vector bundle \mathcal{E} of rank two over D by the extension

$$0 \rightarrow N_{D|V} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_D \rightarrow 0.$$

Let $W = \mathbb{P}_D(\mathcal{E})$ and let $X = V \cup_D W$. Then X satisfies d -semistability by its construction. In [PP, §2], Persson and Pinkham prove that the image of the map $H^0(T_X^1) \rightarrow H^2(T_{V,D})$, induced by the composition of the maps $H^0(T_X^1) \rightarrow H^2(T_X^0) \rightarrow H^2(T_{V,D}) + H^2(T_{W,D}) \rightarrow H^2(T_{V,D})$ where the last map is the projection, is same as the image of η in $H^2(T_{V,D})$ induced by the exact sequence (2.1). Therefore if η is not in the image of the map $n : H^1(T_V) \rightarrow H^1(N_{D|V})$, then there is no smoothing of X .

Lemma 6. *In Example 5, if we assume that the map $H^1(T_W) \rightarrow H^1(N_{D|W})$ is surjective, then the obstruction map $H^0(T_X^1) \rightarrow H^2(T_X^0)$ is zero if and only if η is the image of n for $\eta \in H^1(N_{D|V})$.*

Proof. Since W is a ruled surface, we have the vanishing $H^2(T_W) = 0$. Then the assumption implies that $H^2(T_{W,D}) = 0$ by the exact sequence (2.1). Since W is a ruled surface, the map $H^1(T_W) \rightarrow H^1(T_D)$, induced by the splitting of the exact sequence (2.1), is surjective. This splitting also applies to the surjectivity of the map $t : H^1(T_{W,D}) \rightarrow H^1(T_D)$ induced by the exact sequence (2.4). Therefore $H^2(T_X^0) = H^2(T_{V,D})$ by the exact sequence (2.6). \square

Lemma 7. *In Example 5, if we assume that $H^2(T_V) = 0$, then X has a smoothing if and only if η is the image of n .*

Proof. If X has a smoothing, then the map $H^0(T_X^1) \rightarrow H^2(T_X^0)$ is zero. So η is in the image of n by the argument in Example 5. Assume that η is in the image of n . So there is an element θ in $H^1(T_V)$ such that $n(\theta) = \eta$. Since $H^2(T_V) = 0$, we have the θ -direction deformation of V i.e. there is a flat family $\pi : \mathcal{V} \rightarrow \Delta$ with $\pi^{-1}(0) = V$ and θ corresponds to the extension $0 \rightarrow T_V \rightarrow T_{\mathcal{V}|_V} \rightarrow \mathcal{O}_V \rightarrow 0$. Blow up D in \mathcal{V} ; then we have a smoothing of X . \square

Example 8. Let V_1, W_1 be smooth projective surfaces. Assume that $\sigma : V_1 \rightarrow W_1$ is a double covering branched over a finite disjoint union of irreducible smooth curves. Assume that there is a smooth curve D in W_1 outside of branch loci. Write $\sigma^{-1}(D) = D_1 + D_2, D \cong D_i$ for $i = 1, 2$. Then the analytic neighborhood of D in W_1 is isomorphic to that of D_1 in V_1 , in particular $N_{D|V_1} \cong N_{D|W_1}$. For convenience we also write $D_1 = D$.

Choose a nonzero element η in $H^1(N_{D|V_1}) = H^1(N_{D|W_1})$. Construct $W_2 = \mathbb{P}_D(\mathcal{E})$ where \mathcal{E} corresponds to the extension sheaf of η (Example 5). Let $X = V_1 \cup_D W_2$ and $Y = W_1 \cup_D W_2$. Then one may guess the following statement: X has a smoothing if and only if Y has a smoothing. But this is not true in general by the next example.

Example 9. Choose a smooth curve C with $g(C) \geq 2$ and a finite morphism $f : C \rightarrow \mathbb{P}^1$. This induces a finite morphism \hat{f} and a commutative diagram

$$\begin{CD} C \times D @>\hat{f}>> \mathbb{P}^1 \times D \\ @VVV @VVV \\ C @>f>> \mathbb{P}^1 \end{CD}$$

where D is a smooth curve with $g(D) \geq 2$. Then \hat{f} is branched over finite fibers D_1, \dots, D_n . Choose a fiber D_0 such that D_0 is outside of $\{D_1, \dots, D_n\}$. Let $V_1 = C \times D, W_1 = \mathbb{P}^1 \times D$. Since we choose D_0 outside of the branch loci,

$$N_{D_0|V_1} \cong N_{D_0|W_1} \cong \mathcal{O}_{D_0}.$$

Choose a nonzero element $\eta \in H^1(\mathcal{O}_{D_0})$. Construct $W_2 = \mathbb{P}_{D_0}(\mathcal{E})$. Then $X = V_1 \cup_{D_0} W_2$ has no smoothing by Lemma 10 and Example 5, but $Y = W_1 \cup_{D_0} W_2$ has a smoothing by Lemma 11.

Lemma 10. *Let C, D be smooth curves of genus $g \geq 2$, and let $X = C \times D$. Then the map $n : H^1(T_X) \rightarrow H^1(N_{C|X})$ is zero.*

Proof. It holds that $H^1(T_X) = H^1(T_C) \oplus H^1(T_D)$. The map n restricted on $H^1(T_C)$ to $H^1(N_{C|X})$ is derived from the splitting of the long exact sequence of $0 \rightarrow T_C \rightarrow T_X|_C \rightarrow N_{C|X} \rightarrow 0$. Also, the map n restricted on $H^1(T_D)$ to $H^1(N_{C|X})$ is zero since D lies on the fiber of $X \rightarrow C$. \square

Lemma 11. *In Example 9, $Y = W_1 \cup_{D_0} W_2$ has a smoothing. And the natural map $n : H^1(T_{W_1}) \rightarrow H^1(N_{D_0|W_1})$ is surjective.*

Proof. Choose any $\eta \in H^1(N_{D_0|W_1}) = H^1(\mathcal{O}_{D_0})$. Consider a trivial family \mathcal{W} of $W_2 = \mathbb{P}_{D_0}(\mathcal{E})$ where \mathcal{E} corresponds to an extension sheaf of η . Blow up the central fiber; then we have $Y = W_1 \cup_{D_0} W_2$ in the central fiber. So Y has a smoothing. By Lemma 7, η is in the image of n . \square

According to the above example, the same analytic neighborhood of the double curve is not enough to determine whether a smoothing exists or not. Let us go back to the question in Example 8. Let $\sigma : V_1 \rightarrow W_1$ be a double covering branched over C where C consists of a finite disjoint union of smooth curves. There is a relation between the cohomology of the tangent sheaf of V_1 and the cohomology of the tangent sheaf of W_1 [C2]:

$$(2.8) \quad 0 \rightarrow \sigma_*(\sigma^*\Omega_{W_1}(K_{V_1})) \rightarrow \sigma_*(\Omega_{V_1}(K_{V_1})) \rightarrow \mathcal{O}_C(K_{W_1}) \rightarrow 0$$

and it holds the long exact sequence by Serre duality

$$0 \rightarrow H^0(T_{V_1}) \rightarrow H^0(T_{W_1}) \oplus H^0(T_{W_1}(-L)) \rightarrow H^0(\mathcal{O}_C(C)) \rightarrow H^1(T_{V_1}) \rightarrow \dots$$

where $2L \sim C$. In particular, $H^2(T_{V_1}) = 0$ implies that $H^2(T_{W_1}) = 0$.

Proposition 12. *Consider Example 8 under the assumption $H^2(T_{V_1}) = 0$. Let C be a branched divisor. Assume that $D = \mathbb{P}^1$ or L itself is effective for $2L \sim C$. Then we have the following :*

- (1) *If X has a smoothing, then Y has a smoothing.*
- (2) *If Y has a smoothing and the preimage of η in $H^1(T_{W_1})$ goes to zero in $H^1(N_{C|W_1})$, then X has a smoothing.*

Proof. Since we have the vanishing $H^2(T_{V_1}) = H^2(T_{W_1}) = 0$, the smoothability of X and Y is determined by the image of the following natural maps by Lemma 7:

$$n_1 : H^1(T_{V_1}) \rightarrow H^1(N_{D|V_1}) \quad n_2 : H^1(T_{W_1}) \rightarrow H^1(N_{D|W_1}).$$

Since $D = \mathbb{P}^1$ or L is effective and since D is outside of the branched divisor C , we have $\sigma_*\sigma^*N_{D|W_1}(-L) = N_{D|W_1}$.

From the commutative diagram (in the diagram, the vertical map is induced by the dual sequence of the exact sequence (2.8))

$$(2.9) \quad \begin{array}{ccc} \sigma_*T_{V_1} & \longrightarrow & \sigma_*N_{D|V_1} \\ \downarrow & & \parallel \\ \sigma_*\sigma^*T_{W_1}(-L) & \longrightarrow & \sigma_*\sigma^*N_{D|W_1}(-L) = N_{D|W_1} \\ \downarrow & & \\ N_{C|W_1} & & \end{array}$$

the following commutative diagram in the long exact sequence of cohomology holds:

$$(2.10) \quad \begin{array}{ccc} H^1(T_{V_1}) & \longrightarrow & H^1(N_{D|V_1}) \\ \downarrow & & \parallel \\ H^1(T_{W_1}) \oplus H^1(T_{W_1}(-L)) & \longrightarrow & H^1(N_{D|W_1}) \\ \downarrow & & \\ H^1(N_{C|W_1}) & & \end{array}$$

By the splitting of the eigenspaces of \mathbb{Z}_2 -action, the maps of $H^1(T_{W_1}(-L)) \rightarrow H^1(N_{D|W_1})$ and of $H^1(T_{W_1}(-L)) \rightarrow H^1(N_{C|W_1})$ are zero. Then the proof is obtained by the commutative diagram (2.10) and Lemma 7. □

By Proposition 12, we finish the proof of the Theorem.

REFERENCES

- [B] R. Barlow, *A simply connected surface of general type with $p_g = 0$* , Invent. Math. **79** (1985), 293–301. MR **87a**:14033
- [C1] F. Catanese, *Babbage’s conjecture, contact of surfaces, symmetric determinantal varieties and applications*, Invent. Math. **63** (1981), 433–465. MR **83c**:14026
- [C2] F. Catanese, *On the moduli spaces of surfaces of general type*, J. Diff. Geom. **19** (1984), 483–515. MR **86h**:14031
- [CL] F. Catanese and C. Le Brun, *On the scalar curvature of Einstein manifolds*, Math. Res. Lett. **4**, No. 6 (1997), 843–854. MR **98k**:53057
- [DF] S. Donaldson and R. Friedman, *Connected sums of self-dual manifolds and deformations of singular spaces*, Nonlinearity **2** (1989), 197–239. MR **90e**:32027
- [F] R. Friedman, *Global smoothings of varieties with normal crossings*, Ann. Math. **118** (1983), 75–114. MR **85g**:32029
- [Go] R. Godement, *Topologie Algébrique et Théorie des Faisceau*, Hermann, Paris, 1958. MR **49**:9831
- [H] R. Hartshorne, *Algebraic Geometry, Graduate Texts in Math.*, vol. 52, Springer-Verlag, 1977. MR **57**:3116
- [L] Y. Lee, *Bicanonical pencil of a determinantal Barlow surface*, Trans. Amer. Math. Soc. **353** (2001), 893–905. MR **2001f**:14075
- [PP] U. Persson and H. Pinkham, *Some examples of nonsmoothable varieties with normal crossings*, Duke Math. J. **50**, No. 2 (1983), 477–486. MR **84k**:14010

DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SINSU-DONG, MAPO-GU, SEOUL 121-742, KOREA

E-mail address: ynlee@ccs.sogang.ac.kr