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AN EXPRESSION OF SPECTRAL RADIUS VIA ALUTHGE TRANSFORMATION

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ABSTRACT. For an operator $T \in B(H)$, the Aluthge transformation of T is defined by $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. And also for a natural number n, the n-th Aluthge transformation of T is defined by $\widetilde{T_n} = (\widetilde{\widetilde{T_{n-1}}})$ and $\widetilde{T_1} = \widetilde{T}$. In this paper, we shall show

$$\lim_{n\to\infty} \|\widetilde{T_n}\| = r(T),$$

where $r(T)$ is the spectral radius.

1. Introduction

As a characterization of the spectral radius, it is well known that $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T)$. This result is very famous and quite useful. On the other hand, Aluthge [1] defined a transformation $\widetilde{T}$ of T by $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of T. $\widetilde{T}$ is called the *Aluthge transformation* of T. Many researchers have obtained their results by using Aluthge transformation, for example, [1], [2], [3], [4], [6], [7], [8]. It is easily obtained that $\|T\| \geq \|\widetilde{T}\| \geq r(\widetilde{T}) = r(T)$.

Recently [9], as a generalization of Aluthge transformation, for each natural number n, we defined a transformation $\widetilde{T_n}$ of T by

$$\widetilde{T_n} = (\widetilde{\widetilde{T_{n-1}}}) \quad \text{and} \quad \widetilde{T_1} = \widetilde{T}.$$

We call $\widetilde{T_n}$ the *n-th Aluthge transformation* of T.

In this paper, we shall show another characterization of the spectral radius by using n-th Aluthge transformation as follows:

Theorem 1. *Let $T \in B(H)$. Then $\lim_{n\to\infty} \|\widetilde{T_n}\| = r(T)$.*

2. Proof

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. To prove Theorem 1, we prepare the following results.

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Theorem A ([5]). *Let A and B be positive operators, and $X \in B(H)$. Then*

$$\|A^{\alpha} X B^{\alpha}\| \leq \|AXB\|^{\alpha} \|X\|^{1-\alpha}$$

holds for all $\alpha \in [0,1]$.

Lemma 2. *For a natural number n and $k = 0, 1, \cdots, n+1$, let*

$$ {}_nD_k = \frac{n!(n-2k+1)}{k!(n-k+1)!}. \tag{2.1}$$

Then the following assertions hold:

(i) ${}_nD_0 = 1$ *for all natural numbers n.*
(ii) ${}_nD_k + {}_nD_{k+1} = {}_{n+1}D_{k+1}$ *for all natural numbers n and $k = 0, 1, \cdots, n$.*
(iii) ${}_{2n+1}D_n = {}_{2n+2}D_{n+1}$ *for all natural numbers n.*
(iv) $\displaystyle\sum_{k=0}^{[\frac{n}{2}]} (n-2k+1)\, {}_nD_k = 2^n$,
where $[\frac{n}{2}]$ is the largest integer satisfying $[\frac{n}{2}] \leq \frac{n}{2}$.
(v) $\displaystyle\lim_{n\to\infty} \frac{(n-2k+1)\, {}_nD_k}{2^n} = 0$ *for all positive integers k.*

Proof. (i). By (2.1), we have

$${}_nD_0 = \frac{n!(n+1)}{0!(n+1)!} = 1.$$

(ii). By (2.1), we obtain

$$\begin{aligned} {}_nD_k + {}_nD_{k+1} &= \frac{n!(n-2k+1)}{k!(n-k+1)!} + \frac{n!(n-2k-1)}{(k+1)!(n-k)!} \\ &= \frac{n!\{(k+1)(n-2k+1) + (n-k+1)(n-2k-1)\}}{(k+1)!(n-k+1)!} \\ &= \frac{n!(n+1)(n-2k)}{(k+1)!(n-k+1)!} \\ &= \frac{(n+1)!(n-2k)}{(k+1)!(n-k+1)!} = {}_{n+1}D_{k+1}. \end{aligned}$$

(iii). By (ii) and ${}_{2n+1}D_{n+1} = 0$, we have

$${}_{2n+2}D_{n+1} = {}_{2n+1}D_n + {}_{2n+1}D_{n+1} = {}_{2n+1}D_n.$$

(iv). We shall prove (iv) by induction on n.
(a) The case $n = 1$. By (2.1), we obtain

$$\sum_{k=0}^{[\frac{1}{2}]} (1-2k+1)\, {}_1D_k = 2\, {}_1D_0 = 2.$$

(b) Assume that

$$\sum_{k=0}^{[\frac{n-1}{2}]} (n-2k)\, {}_{n-1}D_k = 2^{n-1}. \tag{2.2}$$

(c-1) The case $n = 2m + 1$ for $m = 1, 2, \cdots$. Then $[\frac{n}{2}] = [\frac{n-1}{2}] = m$. Hence we obtain

$$\begin{aligned}
&\sum_{k=0}^{m}(n-2k+1)_nD_k \\
&= (n+1)_nD_0 + \sum_{k=1}^{m}(n-2k+1)_nD_k \\
&= (n+1)_{n-1}D_0 + \sum_{k=1}^{m}(n-2k+1)({}_{n-1}D_{k-1} + {}_{n-1}D_k) \quad \text{by (i) and (ii)} \\
&= (n+1)_{n-1}D_0 + \sum_{k=1}^{m}(n-2k+1)_{n-1}D_{k-1} + \sum_{k=1}^{m}(n-2k+1)_{n-1}D_k \\
&= \sum_{k=0}^{m-1}(n-2k-1)_{n-1}D_k + \sum_{k=0}^{m}(n-2k+1)_{n-1}D_k \\
&= 2\sum_{k=0}^{m-1}(n-2k)_{n-1}D_k + (n-2m+1)_{n-1}D_m \\
&= 2\sum_{k=0}^{m-1}(n-2k)_{n-1}D_k + 2\,{}_{n-1}D_m \quad \text{by } n = 2m+1 \\
&= 2\sum_{k=0}^{m}(n-2k)_{n-1}D_k = 2 \cdot 2^{n-1} = 2^n \quad \text{by (2.2).}
\end{aligned}$$

(c-2) The case $n = 2m+2$ for $m = 0, 1, 2, \cdots$. Then $[\frac{n}{2}] = m+1$ and $[\frac{n-1}{2}] = m$. Hence we obtain

$$\begin{aligned}
&\sum_{k=0}^{m+1}(n-2k+1)_nD_k \\
&= (n+1)_nD_0 + \sum_{k=1}^{m+1}(n-2k+1)_nD_k \\
&= (n+1)_{n-1}D_0 + \sum_{k=1}^{m+1}(n-2k+1)({}_{n-1}D_{k-1} + {}_{n-1}D_k) \quad \text{by (i) and (ii)} \\
&= (n+1)_{n-1}D_0 + \sum_{k=1}^{m+1}(n-2k+1)_{n-1}D_{k-1} + \sum_{k=1}^{m+1}(n-2k+1)_{n-1}D_k \\
&= \sum_{k=0}^{m}(n-2k-1)_{n-1}D_k + \sum_{k=0}^{m+1}(n-2k+1)_{n-1}D_k \\
&= 2\sum_{k=0}^{m}(n-2k)_{n-1}D_k + \{n-2(m+1)+1\}_{n-1}D_{m+1} \\
&= 2 \cdot 2^{n-1} = 2^n \quad \text{by (2.2) and } {}_{n-1}D_{m+1} = {}_{2m+1}D_{m+1} = 0.
\end{aligned}$$

(v). We remark that

$$\lim_{n\to\infty}\frac{n^\alpha}{2^n}=0 \quad \text{holds for fixed } \alpha\geq 0. \tag{2.3}$$

(a) The case $k=0$. We have

$$\lim_{n\to\infty}\frac{(n+1)_nD_0}{2^n}=2\lim_{n\to\infty}\frac{(n+1)}{2^{n+1}}=0 \quad \text{by (2.3).}$$

(b) The case $k=1$. We have

$$\lim_{n\to\infty}\frac{(n-1)_nD_1}{2^n}=\frac{1}{2}\lim_{n\to\infty}\frac{(n-1)^2}{2^{n-1}}=0 \quad \text{by (2.3).}$$

(c) The case $k\geq 2$. For sufficiently large n,

$$\begin{aligned}0\leq\frac{(n-2k+1)_nD_k}{2^n}&=\frac{n!(n-2k+1)^2}{2^nk!(n-k+1)!}\\&=\frac{\overbrace{n(n-1)\cdots(n-k+2)}^{k-1}(n-2k+1)^2}{2^nk!}\\&=\frac{n^{k+1}1\cdot(1-\frac{1}{n})\cdots(1-\frac{k-2}{n})(1-\frac{2k-1}{n})^2}{2^nk!}\leq\frac{n^{k+1}}{2^n}.\end{aligned}$$

Hence we obtain (v) by (2.3). □

Lemma 3. *Let $T\in B(H)$. Then*

$$\|\widetilde{T}^n\|\leq\|T^{n+1}\|^{\frac{1}{2}}\|T^{n-1}\|^{\frac{1}{2}}$$

holds for all natural numbers n.

Proof. Let $T=U|T|$ be the polar decomposition of T. Then we have

$$\begin{aligned}\|\widetilde{T}^n\|&=\|(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})^n\|=\||T|^{\frac{1}{2}}(U|T|)^{n-1}U|T|^{\frac{1}{2}}\|\\&\leq\||T|(U|T|)^{n-1}U|T|\|^{\frac{1}{2}}\|(U|T|)^{n-1}U\|^{\frac{1}{2}} \quad \text{by Theorem A}\\&=\|T^{n+1}\|^{\frac{1}{2}}\|T^{n-1}\|^{\frac{1}{2}}.\end{aligned}$$

□

Lemma 4. *Let $T\in B(H)$ and $m=[\frac{n}{2}]$. Then*

$$\|\widetilde{T_n}\|\leq\|T^{n+1}\|^{\frac{{}_nD_0}{2^n}}\|T^{n-1}\|^{\frac{{}_nD_1}{2^n}}\cdots\|T^{n-2k+1}\|^{\frac{{}_nD_k}{2^n}}\cdots\|T^{n-2m+1}\|^{\frac{{}_nD_m}{2^n}}.$$

Proof. We shall prove Lemma 4 by induction on n.

(a) $\|\widetilde{T}\|\leq\|T^2\|^{\frac{1}{2}}$ holds by Lemma 3.

(b) Assume that

$$\begin{aligned}\|\widetilde{T_{n-1}}\|\leq&\|T^n\|^{\frac{{}_{n-1}D_0}{2^{n-1}}}\|T^{n-2}\|^{\frac{{}_{n-1}D_1}{2^{n-1}}}\\&\times\cdots\times\|T^{n-2k}\|^{\frac{{}_{n-1}D_k}{2^{n-1}}}\cdots\|T^{n-2m}\|^{\frac{{}_{n-1}D_m}{2^{n-1}}},\end{aligned} \tag{2.4}$$

where $m=[\frac{n-1}{2}]$.

(c-1) The case $n = 2m + 1$ for $m = 1, 2, \cdots$. Then $[\frac{n}{2}] = [\frac{n-1}{2}] = m$. Hence by (2.4), we have

$$\begin{aligned}
\|\widetilde{T_n}\| &= \|\widetilde{(\widetilde{T})_{n-1}}\| \\
&\leq \|\widetilde{T}^n\|^{\frac{{}_{n-1}D_0}{2^{n-1}}} \|\widetilde{T}^{n-2}\|^{\frac{{}_{n-1}D_1}{2^{n-1}}} \\
&\quad \times \cdots \times \|\widetilde{T}^{n-2k+2}\|^{\frac{{}_{n-1}D_{k-1}}{2^{n-1}}} \|\widetilde{T}^{n-2k}\|^{\frac{{}_{n-1}D_k}{2^{n-1}}} \cdots \|\widetilde{T}^3\|^{\frac{{}_{n-1}D_{m-1}}{2^{n-1}}} \|\widetilde{T}\|^{\frac{{}_{n-1}D_m}{2^{n-1}}} \\
&\leq \left(\|T^{n+1}\|^{\frac{1}{2}} \|T^{n-1}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_0}{2^{n-1}}} \left(\|T^{n-1}\|^{\frac{1}{2}} \|T^{n-3}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_1}{2^{n-1}}} \\
&\quad \times \cdots \times \left(\|T^{n-2k+3}\|^{\frac{1}{2}} \|T^{n-2k+1}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_{k-1}}{2^{n-1}}} \\
&\quad \times \left(\|T^{n-2k+1}\|^{\frac{1}{2}} \|T^{n-2k-1}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_k}{2^{n-1}}} \\
&\quad \times \cdots \times \left(\|T^4\|^{\frac{1}{2}} \|T^2\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_{m-1}}{2^{n-1}}} \|T^2\|^{\frac{{}_{n-1}D_m}{2^n}} \\
&= \|T^{n+1}\|^{\frac{{}_{n-1}D_0}{2^n}} \|T^{n-1}\|^{\frac{{}_{n-1}D_0 + {}_{n-1}D_1}{2^n}} \\
&\quad \times \cdots \times \|T^{n-2k+1}\|^{\frac{{}_{n-1}D_{k-1} + {}_{n-1}D_k}{2^n}} \cdots \|T^2\|^{\frac{{}_{n-1}D_{m-1} + {}_{n-1}D_m}{2^n}} \\
&= \|T^{n+1}\|^{\frac{{}_nD_0}{2^n}} \|T^{n-1}\|^{\frac{{}_nD_1}{2^n}} \cdots \|T^{n-2k+1}\|^{\frac{{}_nD_k}{2^n}} \cdots \|T^2\|^{\frac{{}_nD_m}{2^n}},
\end{aligned}$$

by (i) and (ii) of Lemma 2, and the last inequality holds by Lemma 3.

(c-2) The case $n = 2m + 2$ for $m = 0, 1, 2, \cdots$. Then $[\frac{n}{2}] = m + 1$ and $[\frac{n-1}{2}] = m$. Hence by (2.4), we have

$$\begin{aligned}
\|\widetilde{T_n}\| &= \|\widetilde{(\widetilde{T})_{n-1}}\| \\
&\leq \|\widetilde{T}^n\|^{\frac{{}_{n-1}D_0}{2^{n-1}}} \|\widetilde{T}^{n-2}\|^{\frac{{}_{n-1}D_1}{2^{n-1}}} \\
&\quad \times \cdots \times \|\widetilde{T}^{n-2k+2}\|^{\frac{{}_{n-1}D_{k-1}}{2^{n-1}}} \|\widetilde{T}^{n-2k}\|^{\frac{{}_{n-1}D_k}{2^{n-1}}} \cdots \|\widetilde{T}^4\|^{\frac{{}_{n-1}D_{m-1}}{2^{n-1}}} \|\widetilde{T}^2\|^{\frac{{}_{n-1}D_m}{2^{n-1}}} \\
&\leq \left(\|T^{n+1}\|^{\frac{1}{2}} \|T^{n-1}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_0}{2^{n-1}}} \left(\|T^{n-1}\|^{\frac{1}{2}} \|T^{n-3}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_1}{2^{n-1}}} \\
&\quad \times \cdots \times \left(\|T^{n-2k+3}\|^{\frac{1}{2}} \|T^{n-2k+1}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_{k-1}}{2^{n-1}}} \\
&\quad \times \left(\|T^{n-2k+1}\|^{\frac{1}{2}} \|T^{n-2k-1}\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_k}{2^{n-1}}} \\
&\quad \times \cdots \times \left(\|T^5\|^{\frac{1}{2}} \|T^3\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_{m-1}}{2^{n-1}}} \left(\|T^3\|^{\frac{1}{2}} \|T\|^{\frac{1}{2}}\right)^{\frac{{}_{n-1}D_m}{2^{n-1}}} \\
&= \|T^{n+1}\|^{\frac{{}_{n-1}D_0}{2^n}} \|T^{n-1}\|^{\frac{{}_{n-1}D_0 + {}_{n-1}D_1}{2^n}} \\
&\quad \times \cdots \times \|T^{n-2k+1}\|^{\frac{{}_{n-1}D_{k-1} + {}_{n-1}D_k}{2^n}} \cdots \|T^3\|^{\frac{{}_{n-1}D_{m-1} + {}_{n-1}D_m}{2^n}} \|T\|^{\frac{{}_{n-1}D_m}{2^n}} \\
&= \|T^{n+1}\|^{\frac{{}_nD_0}{2^n}} \|T^{n-1}\|^{\frac{{}_nD_1}{2^n}} \cdots \|T^{n-2k+1}\|^{\frac{{}_nD_k}{2^n}} \cdots \|T^3\|^{\frac{{}_nD_m}{2^n}} \|T\|^{\frac{{}_nD_{m+1}}{2^n}},
\end{aligned}$$

by (i), (ii) and (iii) of Lemma 2, and the last inequality holds by Lemma 3. □

Lemma 5. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence satisfying* $\lim_{n\to\infty} a_n = a$*, and for each natural number* n*, let* $\{\alpha_{n,k}\}_{k=1}^{n}$ *be a positive sequence satisfying*

$$\alpha_{n,1} + \cdots + \alpha_{n,k} + \cdots + \alpha_{n,n} = 1 \quad \text{for all natural numbers } n \tag{2.5}$$

and $\lim_{n\to\infty} \alpha_{n,k} = 0$ *for fixed* $k = 1, 2, \cdots$. *Then*

$$\lim_{n\to\infty} (\alpha_{n,1}a_1 + \cdots + \alpha_{n,k}a_k + \cdots + \alpha_{n,n}a_n) = a.$$

Proof. For any $\varepsilon > 0$, there exists $k > 0$ such that $|a_n - a| < \varepsilon$ and $\alpha_{n,1}|a_1 - a| + \cdots + \alpha_{n,k}|a_k - a| < \varepsilon$ for all natural numbers $n > k$ by the assumptions $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} \alpha_{n,k} = 0$. Then we have

$$\begin{aligned}
&|(\alpha_{n,1}a_1 + \cdots + \alpha_{n,k}a_k + \alpha_{n,k+1}a_{k+1} + \cdots \alpha_{n,n}a_n) - a| \\
&= |\alpha_{n,1}(a_1 - a) + \cdots + \alpha_{n,k}(a_k - a) \\
&\qquad + \alpha_{n,k+1}(a_{k+1} - a) + \cdots + \alpha_{n,n}(a_n - a)| \quad \text{by (2.5)} \\
&\leq \alpha_{n,1}|a_1 - a| + \cdots + \alpha_{n,k}|a_k - a| \\
&\qquad + \alpha_{n,k+1}|a_{k+1} - a| + \cdots + \alpha_{n,n}|a_n - a| \\
&< \alpha_{n,1}|a_1 - a| + \cdots + \alpha_{n,k}|a_k - a| \\
&\qquad + \{1 - (\alpha_{n,1} + \cdots + \alpha_{n,k})\}\varepsilon \quad \text{by (2.5)} \\
&< 2\varepsilon.
\end{aligned}$$

□

Proof of Theorem 1. Let $m = [\frac{n}{2}]$. Then by Lemma 4, (iv) of Lemma 2 and Arithmetic mean-Geometric mean inequality, we have

$$\begin{aligned}
r(T) = r(\widetilde{T_n}) \leq \|\widetilde{T_n}\| &\leq \|T^{n+1}\|^{\frac{{}_nD_0}{2^n}} \|T^{n-1}\|^{\frac{{}_nD_1}{2^n}} \\
&\qquad \cdots \|T^{n-2k+1}\|^{\frac{{}_nD_k}{2^n}} \cdots \|T^{n-2m+1}\|^{\frac{{}_nD_m}{2^n}} \\
&\leq \frac{(n+1)_nD_0}{2^n}\|T^{n+1}\|^{\frac{1}{n+1}} + \frac{(n-1)_nD_1}{2^n}\|T^{n-1}\|^{\frac{1}{n-1}} \\
&\qquad + \cdots + \frac{(n-2k+1)_nD_k}{2^n}\|T^{n-2k+1}\|^{\frac{1}{n-2k+1}} \\
&\qquad\qquad + \cdots + \frac{(n-2m+1)_nD_m}{2^n}\|T^{n-2m+1}\|^{\frac{1}{n-2m+1}} \\
&\longrightarrow r(T) \quad \text{as } n \to \infty
\end{aligned}$$

by $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T)$, (iv) and (v) of Lemma 2 and Lemma 5. □

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