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REGULARITY OF VISCOSITY SOLUTIONS OF A DEGENERATE PARABOLIC EQUATION

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ABSTRACT. We study the Cauchy problem for the nonlinear degenerate parabolic equation of second order

$$\begin{cases} u_t = u\triangle u - \gamma|\nabla u|^2 & \text{in} \quad \Omega = R^N \times R^+, \\ u(x,0) = u_0(x) & \text{in} \quad R^N, \end{cases}$$

and present regularity results for the viscosity solutions.

1. INTRODUCTION

We study the Cauchy problem for the equation

$$u_t = u\triangle u - \gamma|\nabla u|^2 \quad \text{in} \quad \Omega = R^N \times R^+, \tag{1}$$

with the initial data

$$u(x,0) = u_0(x) \qquad \text{in} \quad R^N. \tag{2}$$

Here γ is a nonnegative constant and the initial function u_0 satisfies $u_0 \in C(R^N) \cap L^\infty(R^N)$ and $u_0 \geq 0$ in R^N.

Equation (1) arises in several applications in biology and physics [1, 2, 3]. It is degenerate parabolic at the points where u vanishes. Therefore the Cauchy problem (1), (2), in general, has no classical solutions. The weak solutions are defined as follows:

Definition 1. A function $u \in L^\infty(\Omega) \cap L^2_{Loc}([0,+\infty]; H^1_{Loc}(R^N))$ is called a weak solution of (1), (2) if $u \geq 0$ almost everywhere in Ω and

$$\int_{R^N} u_0\psi(0)dx + \int\int_\Omega (u\psi_t - u\nabla u \cdot \nabla\psi - (1+\gamma)|\nabla u|^2\psi)dxdt = 0 \tag{3}$$

for any $\psi \in C^{1,1}(\bar{\Omega})$ with compact support in $\bar{\Omega}$.

In [2] a weak solution is constructed by the well-known viscosity method: let $\omega_\epsilon(x,t)$ be the unique solution in $C^{2,1}(\Omega) \cap C(\overline{\Omega}) \cap L^\infty(\Omega)$ of the Cauchy problem with equation (1) replaced by

$$u_t = u\triangle u - \gamma|\nabla u|^2 + \epsilon\triangle u \tag{4}$$

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where $\epsilon > 0$ (see [4]). Then

$$u_\epsilon = \omega_\epsilon + \epsilon \tag{5}$$

is a classical solution of the Cauchy problem with u_0 replaced by $u_{0_\epsilon} = u_0 + \epsilon$. Since $u_\epsilon(x,t)$ is nonincreasing with respect to ϵ,

$$u(x,t) = \lim_{\epsilon \searrow 0} u_\epsilon(x,t) = \lim_{\epsilon \searrow 0} \omega_\epsilon(x,t) \tag{6}$$

is well definded for all $(x,t) \in \overline{\Omega}$.

It is proved in [2] that u is a weak solution of (1), (2).

Definition 2. The weak solution of (1), (2) constructed by the vanishing viscosity method is called the viscosity solution.

Let $u(x,t)$ be the viscosity solution of (1), (2). Then it is proved in [2] that

$$N = 1 \Rightarrow u \in C(\bar{\Omega})$$

and

$$\gamma > \frac{1}{2}N \Rightarrow u \in C(\bar{\Omega}) \cap C^{1,1}(\Omega).$$

So, it is natural to ask what the smoothness of u is if $N \geq 2$ and $0 \leq \gamma \leq \frac{N}{2}$. One result in [2] shows that if

$$N \geq 2, 0 \leq \gamma < 1,$$

then u is not necessarily continuous in $\bar{\Omega}$.

In this paper, by using the method developed in [5, 6, 7], we study the regularity of the viscosity solution of the Cauchy problem (1), (2) for $\gamma \geq \sqrt{2N} - 1$, and obtain certain conditions under which $u(x,t)$ is Lipschitz continuous. We also show that for $\gamma < \sqrt{2N} - 1$ this method will meet difficulty except for special cases.

The method is to apply the maximum principle to obtain uniform estimates after some suitable transforms to the original equations. A similar idea was also used by P.Z. Mkrtychyan to different problems. The readers may refer to the references [8], [9] for the details.

2. Theorem and proof

The main result is the following.

Theorem. *If $\gamma \geq \sqrt{2N} - 1$, $|\nabla(u_0^{1+\frac{\alpha}{2}})| \leq M$, where M is a positive constant, $\alpha^2 + (\gamma+1)\alpha + \frac{N}{2} \leq 0$, then the viscosity solution $u(x,t)$ of equation* (1), (2) *satisfies $|\nabla(u^{1+\frac{\alpha}{2}})| \leq M$ in $\overline{\Omega}$.*

Proof. We may add a small positive perturbation ϵ to the initial data (2) and the "viscosity " term $\epsilon \triangle u$ to the right-hand side of equation (1), and then resolve the Cauchy problem for this new strictly parabolic equation with positive initial data and obtain the approximated solutions u_ϵ. If u_ϵ satisfy some regular estimates, then there exists a subsequence u_{ϵ_l} converging uniformly as $\epsilon_l \to 0^+$, on any bounded region, to the solution u of the Cauchy problem (1), (2). This technique is standard. For simplicity, we omit the details and only give the uniform estimate.

Let

$$w = \frac{1}{2}\sum_{i=1}^{N} u_{x_i}^2. \tag{7}$$

From (7) we obtain that

$$
\begin{aligned}
w_t &= \sum_{i=1}^{N} u_{x_i}[u_{x_i}\triangle u + u(\sum_{j=1}^{N} u_{x_ix_jx_j}) - \gamma(2w)_{x_i}] \\
&= 2w\Delta u + u\sum_{i,j=1}^{N}[(u_{x_i}u_{x_ix_j})_{x_j} - u^2_{x_ix_j}] - 2\gamma\sum_{i=1}^{N} u_{x_i}w_{x_i} \\
&= 2w\Delta u + u\Delta w - u\sum_{i,j=1}^{N} u^2_{x_ix_j} - 2\gamma\sum_{i=1}^{N} u_{x_i}w_{x_i}. \qquad (8)
\end{aligned}
$$

Set

$$z = f(u)w. \qquad (9)$$

Then

$$w_{x_i} = (f^{-1})_{x_i}z + f^{-1}z_{x_i} \qquad (10)$$

and

$$w_{x_ix_i} = (f^{-1})_{x_ix_i}z + 2(f^{-1})_{x_i}z_{x_i} + f^{-1}z_{x_ix_i}. \qquad (11)$$

These imply that

$$\Delta w = f^{-1}\Delta z - 2f^{-2}f'\sum_{i=1}^{N} u_{x_i}z_{x_i} + 2(\frac{2f'^2 - ff''}{f^4})z^2 - \frac{f'}{f^2}z\Delta u. \qquad (12)$$

So from $(8), (9)$ and (12) we get

$$
\begin{aligned}
z_t &= 2f(u)w\Delta u - uf(u)\sum_{i,j=1}^{N} u^2_{x_ix_j} + f'(u)(u\Delta u - 2\gamma w)w \\
&+ f(u)\Delta w - 2\gamma f(u)\sum_{i=1}^{N} u_{x_i}w_{x_i} \\
&= u\Delta z - (2f^{-1}uf' + 2\gamma)\sum_{i=1}^{N} u_{x_i}z_{x_i} \\
&+ (\frac{4uf'^2}{f^3} - \frac{2uf''}{f^2} + \frac{2\gamma f'}{f^2})z^2 \\
&+ 2z\Delta u - uf(u)\sum_{i,j=1}^{N} u^2_{x_ix_j}. \qquad (13)
\end{aligned}
$$

We choose in (13)

$$f(u) = u^{\alpha}. \qquad (14)$$

Since

$$\sum_{i,j=1}^{N} u^2_{x_ix_j} \geq \frac{1}{N}(\Delta u)^2, \qquad (15)$$

then from (13), (14) and (15)

$$z_t \le u\Delta z - 2(\alpha+\gamma)\sum_{i=1}^{N} u_{x_i} z_{x_i}$$
$$+ 2\alpha(\alpha+\gamma+1)u^{-\alpha-1}z^2 + 2z\Delta u - \frac{u^{\alpha+1}}{N}(\Delta u)^2. \tag{16}$$

For $\gamma \ge \sqrt{2N} - 1$, if α satisfies

$$\alpha^2 + (\gamma+1)\alpha + \frac{N}{2} \le 0, \tag{17}$$

then

$$2\alpha(\alpha+\gamma+1)u^{-\alpha-1}z^2 + 2z\Delta u - \frac{u^{\alpha+1}}{N}(\Delta u)^2 \le 0. \tag{18}$$

Therefore from (16) and (18) we have

$$z_t \le u\Delta z - 2(\alpha+\gamma)\sum_{i=1}^{N} u_{x_i} z_{x_i}. \tag{19}$$

By applying the maximum principle to (19), we have $|z|_\infty \le |z_0|_\infty$. From (7), (9) and (14) we get $|\nabla(u^{1+\frac{\alpha}{2}})| \le M$ provided $|\nabla(u_0^{1+\frac{\alpha}{2}})| \le M$ where α satisfies (17) and M is a positive constant. □

By the Theorem, we can easily draw some conclusions about the smoothness of u if $\gamma \ge \sqrt{2N} - 1$.

Corollary. *Under the same conditions as in the Theorem, the viscosity solution $u(x,t)$ of equations (1), (2) is Lipschitz continuous with respect to x and Hölder continuous with exponent $\frac{1}{2}$ with respect to t in $\overline{\Omega}$.*

Proof. From the construction of the viscosity solution $u(x,t)$, we know that it can be approximated from above by the classical solution u_ϵ. So from the Theorem,

$$|u_{x_i}| \le M u^{-\frac{\alpha}{2}} < M_1, \tag{20}$$

where $M_1 > 0$ is a constant.

The Hölder continuity in t follows directly from a result by Gilding [10]. □

Remark 1. Since $\sqrt{2N} - 1 \le \frac{N}{2}$, the result improves the continuity result for $\gamma \ge \frac{N}{2}$ in [2].

Remark 2. The smoothness of u if $0 \le \gamma < \sqrt{2N} - 1$ remains an open problem. By the above approach it can only be solved for very special cases.

From (13) we find that if $f(u)$ satisfies

$$u^2 f f'' - \gamma u f f' - 2u^2 f'^2 \ge \frac{N}{2} f^2, \tag{21}$$

then

$$\left(\frac{4uf'^2}{f^3} - \frac{2uf''}{f^2} + \frac{2\gamma f'}{f^2}\right)z^2 + 2z\Delta u - uf(u)\sum_{i,j=1}^{N} u_{x_i x_j}^2 \ge 0. \tag{22}$$

Thus we have from (13) and (22) that

$$z_t \leq u\Delta z - (2f^{-1}uf' + 2\gamma)\sum_{i=1}^{N} u_{x_i} z_{x_i}. \tag{23}$$

Applying the maximum principle to (23), we have $|z|_\infty \leq |z_0|_\infty$. Therefore we may have

$$f(u)\sum_{i=1}^{N} u_{x_i}^2 \leq M \tag{24}$$

when

$$f(u_0)\sum_{i=1}^{N} u_{0x_i}^2 \leq M. \tag{25}$$

If we can find some other $f(u)$ satisfying (21), we may hope to get some new estimates. However, it is not successful. In the special case of (21) when it turns to be an equation, we may set

$$F = \frac{d\ln f(u)}{du}. \tag{26}$$

Then

$$u^2F' = u^2F^2 + \gamma uF + \frac{N}{2}. \tag{27}$$

Set again

$$h(u) = uF(u). \tag{28}$$

It turns out to be

$$\frac{dh}{(h + \frac{\gamma+1}{2})^2 + \frac{2N-(\gamma+1)^2}{4}} = \frac{du}{u}. \tag{29}$$

If $\gamma < \sqrt{2N} - 1$, the solutions of (21) are given by

$$f(u) = u^{-\frac{\gamma+1}{2}}|cos(\alpha \ln u + c)|^{-1}, \tag{30}$$

where

$$\alpha^2 = \frac{2N - (\gamma+1)^2}{4}.$$

But (25) can only be satisfied for very special u_0.

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