PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 130, Number 4, Pages 1065–1071
S 0002-9939(01)06315-8
Article electronically published on October 5, 2001

A NOTE ON IDEMPOTENTS IN FINITE AW*-FACTORS

GABRIEL NAGY

(Communicated by David R. Larson)

ABSTRACT. We prove that the value of the quasi-trace on an idempotent element in an AW*-factor of type II_1 is the same as the dimension of its left (or right) support.

It is a long-standing open problem (due to Kaplansky) to prove that *an AW*-factor of type II_1 is in fact a von Neumann algebra*. A remarkable answer, in the affirmative, was found by Haagerup ([Ha]), who proved that if an AW*-factor A is generated by an exact C*-algebra, then A is indeed a von Neumann algebra.

The main object, that was investigated in connection with Kaplansky's problem, is the *quasi-trace*, whose construction we briefly recall below.

One starts with an AW*-factor of type II_1, say A. Denote by $\mathcal{P}(A)$ the collection of projections in A, that is

$$\mathcal{P}(A) = \{p \in A \,:\, p = p^* = p^2\}.$$

A key fact is then the existence of a (unique) *dimension function* $D : \mathcal{P}(A) \to [0,1]$ with the following properties:

- $D(p) = D(q) \Longleftrightarrow p \sim q$;
- if $p \perp q$, then $D(p+q) = D(p) + D(q)$;
- $D(1) = 1$.

The symbol "$\sim$" denotes the Murray-von Neumann equivalence relation ($p \sim q \Leftrightarrow \exists\, x \in A$ with $p = x^*x$ and $q = xx^*$), while "$\perp$" denotes the orthogonality relation ($p \perp q \Leftrightarrow pq = 0$; this implies that $p + q$ is again a projection).

Once the dimension function is defined, it is extended to self-adjoint elements with finite spectrum. More explicitly, if $a \in A$ is self-adjoint with finite spectrum, then there are (real) numbers $\alpha_1, \dots, \alpha_n$ and pairwise orthogonal projections $p_1, \dots, p_n$, such that $a = \sum_{k=1}^n \alpha_k p_k$. We then define $d(a) = \sum_{k=1}^n \alpha_k D(p_k)$.

For an arbitrary self-adjoint element $a \in A$, one can approximate uniformly a with a sequence $(a_n)_{n\geq 1} \in \{a\}''$ of elements with finite spectrum. (Here $\{a\}''$ stands for the AW*-subalgebra generated by a and 1.) It turns out that the limit $q(a) = \lim_{n\to\infty} d(a_n)$ is independent of the particular choice of $(a_n)_{n\geq 1}$.

Finally, for an arbitrary element $x \in A$, one defines $Q(x) = q(\operatorname{Re} x) + iq(\operatorname{Im} x)$, where $\operatorname{Re} x = \frac{1}{2}(x + x^*)$ and $\operatorname{Im} x = \frac{1}{2i}(x - x^*)$.

Received by the editors October 4, 2000.
1991 *Mathematics Subject Classification.* Primary 46L10; Secondary 46L30.
Key words and phrases. AW*-algebra, quasi-trace, idempotent, projection, dimension function.
This work was partially supported by NSF grant DMS 9706858.

The map $Q : A \to \mathbb{C}$, defined this way, is the unique one with the properties:

(i) Q is *linear*, when restricted to *abelian* C*-subalgebras of A;
(ii) $Q(x^*x) = Q(xx^*) \geq 0$, for all $x \in A$;
(iii) $Q(x) = Q(\operatorname{Re} x) + iQ(\operatorname{Im} x)$, for all $x \in A$;
(iv) $Q(1) = 1$.

It is obvious that $Q\big|_{\mathcal{P}(A)} = D$. The map Q is called *the quasi-trace of* A.

It is well known that an AW*-factor of type II_1 is a von Neumann algebra if and only if its quasi-trace is *linear*. Haagerup's solution for Kaplansky's problem goes through the proof of the linearity of the quasi-trace.

On the one hand, one can easily see that the linearity of the quasi-trace is equivalent to its scalar homogeneity (compare with (i) above):

$$\text{(H)} \qquad Q(\alpha x) = \alpha Q(x), \text{ for all } x \in A,\ \alpha \in \mathbb{C}.$$

Notice that (H) holds when either $\alpha \in \mathbb{R}$ or when x is *normal*. On the other hand, it is again easy to note that the linearity of the quasi-trace is equivalent to the similarity invariance property

$$\text{(S)} \qquad Q(sxs^{-1}) = Q(x), \text{ for all } x \in A,\ s \in GL(A).$$

(Here $GL(A)$ denotes the group of invertible elements in A.) Notice that (S) is true if s is *unitary*.

The purpose of this note is to prove that *both* (H) *and* (S) *hold if* $x \in A$ *is an idempotent* (i.e. $x^2 = x$).

1. *Notations.* If A is an AW*-algebra, for an element $x \in A$, we denote by $\mathbf{L}(x)$ (resp. $\mathbf{R}(x)$) the *left* (resp. *right*) *support* of x. Recall that both $\mathbf{L}(x)$ and $\mathbf{R}(x)$ are projections, and moreover we have

$$\mathbf{L}(x) \sim \mathbf{R}(x), \text{ for all } x \in A.$$

It is known (see [Ka]) that, for every $x \in A$, there exists a unique partial isometry v such that

(i) $x = v(x^*x)^{1/2}$;
(ii) $vv^* = \mathbf{L}(x)$;
(iii) $v^*v = \mathbf{R}(x)$.

(This property is referred to as the *Polar Decomposition.*)

2. *Remark.* If A is a finite AW*-algebra, then the group $GL(A)$, of invertible elements, is dense in A in the norm topology.

Indeed, on the one hand, since $\mathbf{L}(x) \sim \mathbf{R}(x)$, by the finiteness assumption we also have $1 - \mathbf{L}(x) \sim 1 - \mathbf{R}(x)$. In particular, there exists a partial isometry $w \in A$ such that $1 - \mathbf{L}(x) = ww^*$ and $1 - \mathbf{R}(x) = w^*w$. On the other hand, if we take $x = v(x^*x)^{1/2}$ to be the polar decomposition described above, then we obviously have $w^*v = 0$, so the element $u = v + w$ is *unitary*, and we still have $x = u(x^*x)^{1/2}$. Then, for every $\varepsilon > 0$ the positive element $(x^*x)^{1/2} + \varepsilon 1$ is invertible, and so is $u\{(x^*x)^{1/2} + \varepsilon 1\}$. The result then follows from the obvious equality $\|x - u\{(x^*x)^{1/2} + \varepsilon 1\}\| = \varepsilon$.

3. *Notation.* For a C*-algebra A and an integer $n \geq 2$, we denote by $M_n(A)$ the C*-algebra of $n \times n$ matrices with coefficients in A.

The key technical result in this paper is the following.

4. Lemma. *Suppose A is a unital C^*-algebra and $x \in GL(A)$. Then there exist a unitary element $U \in M_2(A)$ and elements $y, z \in A$, such that*

$$U^* \begin{bmatrix} 2 & x \\ 0 & 0 \end{bmatrix} U = \begin{bmatrix} 1 & y \\ z & 1 \end{bmatrix}. \tag{1}$$

Proof. Consider the function defined by

$$f(t) = \frac{(1-t)^2}{t}, \quad 0 < t < 1.$$

It is obvious that $f : (0,1) \to (0,\infty)$ is a homeomorphism. Since x is invertible, the spectrum of xx^* is contained in $(0,\infty)$, so by functional calculus there exists an invertible positive element $w \in A$, with $\|w\| < 1$, such that $f(w) = xx^*$, which means

$$xx^* = (1-w)^2 w^{-1}. \tag{2}$$

Define the elements

$$a = (1+w)^{-1/2} w^{1/2}, \qquad b = (1+w)^{-1/2},$$
$$c = x^{-1}(1+w)^{-1/2} w^{-1/2}(1-w), \qquad d = -x^{-1}(1+w)^{-1/2}(1-w).$$

First, we have

$$aa^* + bb^* = (1+w)^{-1/2}[w+1](1+w)^{-1/2} = 1, \tag{3}$$

and using (2) we also have

$$\begin{aligned} cc^* + dd^* &= x^{-1}(1+w)^{-1/2}(1-w)[w^{-1}+1](1-w)(1+w)^{-1/2}(x^*)^{-1} \\ &= x^{-1}(1-w)^2 w^{-1}(x^*)^{-1} = x^{-1}(xx^*)(x^*)^{-1} = 1. \end{aligned} \tag{4}$$

Secondly, since by taking inverses, (2) yields

$$(x^*)^{-1}x^{-1} = w(1-w)^{-2}, \tag{5}$$

so we also get

$$\begin{aligned} a^*a + c^*c &= w^{1/2}(1+w)^{-1/2}[1 + (1-w)(x^*)^{-1}x^{-1}(1-w)](1+w)^{-1/2}w^{1/2} \\ &= w^{1/2}(1+w)^{-1/2}[1+w^{-1}](1+w)^{-1/2}w^{1/2} = 1, \end{aligned} \tag{6}$$

$$\begin{aligned} b^*b + d^*d &= (1+w)^{-1/2}[1 + (1-w)(x^*)^{-1}x^{-1}(1-w)](1+w)^{-1/2} \\ &= (1+w)^{-1/2}[1+w](1+w)^{-1/2}1. \end{aligned} \tag{7}$$

Finally, we notice that

(8)
$$ac^* + bd^* = (1+w)^{-1/2}[w^{1/2}w^{-1/2} - 1](1-w)(1+w)^{-1/2}(x^*)^{-1} = 0,$$

and using (5) we also have

(9)
$$\begin{aligned} a^*b + c^*d &= (1+w)^{-1/2}[w^{1/2} - w^{-1/2}(1-w)(x^*)^{-1}x(1-w)](1+w)^{-1/2} \\ &= (1+w)^{-1/2}[w^{1/2} - w^{-1/2}w](1+w)^{-1/2} = 0. \end{aligned}$$

If we define the matrix $U = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, then (3), (4), and (8) give $UU^* = I$, while (6), (7), and (9) give $U^*U = I$, so U is indeed unitary. (Here I denotes the unit in $M_2(A)$.)

Let us now observe that

$$\begin{aligned}
2a + xc &= 2(1+w)^{-1/2}w^{1/2} + (1+w)^{1/2}(1-w)w^{-1/2} \\
&= (1+w)^{-1/2}(w^{1/2} + w^{-1/2}) = a + bw^{-1/2},
\end{aligned} \tag{10}$$

$$\begin{aligned}
2b + xd &= 2(1+w)^{-1/2} - (1+w)^{-1/2}(1-w) \\
&= (1+w)^{-1/2}(w+1) = aw^{1/2} + b,
\end{aligned} \tag{11}$$

$$c + dw^{-1/2} = x^{-1}(1+w)^{-1/2}[(1-w)w^{-1/2} - (1-w)w^{-1/2}] = 0, \tag{12}$$

$$cw^{1/2} + d = (c + dw^{-1/2})w^{1/2} = 0. \tag{13}$$

These equalities prove exactly that

$$\begin{bmatrix} 2 & x \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & w^{1/2} \\ w^{-1/2} & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

□

The next result is a particular case of the main result.

5. Lemma. *Let A be an AW^*-factor of type II_1, and let $e \in A$ be an idempotent with $D(\mathbf{L}(e)) = \frac{1}{2}$. Then, for any $\lambda \in \mathbb{C}$, we have*

$$Q(\lambda e) = \lambda/2. \tag{14}$$

Proof. Denote, for simplicity, the projection $\mathbf{L}(x)$ by p. The assumption is that $p \sim 1 - p$. Then we have a $*$-isomorphism $\Phi : A \to M_2(pAp)$, such that $\Phi(p) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. It is obvious that, since $e = pe$, there exists an element $x \in pAp$ such that $\Phi(e) = \left[\begin{smallmatrix} 1 & 0 \\ x & 0 \end{smallmatrix}\right]$. By Remark 2, we can find a sequence $(x_n)_{n\geq 1}$ of invertible elements in pAp (the unit in pAp is p) with $\lim_{n\to\infty} \|x_n - x\| = 0$. Define the sequence

$$E_n = \begin{bmatrix} 1 & x_n \\ 0 & 0 \end{bmatrix} \in M_2(pAp), \ \ n \geq 1.$$

By Lemma 4, one can find two sequences $(y_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ in pAp, and a sequence of unitaries $(U_n)_{n\geq 1} \subset M_2(pAp)$, such that

$$E_n = U_n \begin{bmatrix} \frac{1}{2} & y_n \\ z_n & \frac{1}{2} \end{bmatrix} U_n^*, \text{ for all } n \geq 1. \tag{15}$$

Define $e_n = \Phi^{-1}(E_n)$, $u_n = \Phi^{-1}(U_n)$, and $a_n = u_n^* e_n u_n - \frac{1}{2}1$, $n \geq 1$, so that we have

$$\Phi(a_n) = \begin{bmatrix} 0 & y_n \\ z_n & 0 \end{bmatrix}, \text{ for all } n \geq 1. \tag{16}$$

Now fix a complex number λ. On the one hand, we have

$$\lambda e_n = u_n(\tfrac{\lambda}{2} + a_n)u_n^* = \tfrac{\lambda}{2}1 + u_n a_n u_n^*, \text{ for all } n \geq 1.$$

This gives, for every $n \geq 1$, the equalities

$$\mathrm{Re}(\lambda e_n) = (\mathrm{Re}\tfrac{\lambda}{2})1 + \mathrm{Re}(\lambda u_n a_n u_n^*), \tag{17}$$

$$\mathrm{Im}(\lambda e_n) = (\mathrm{Im}\tfrac{\lambda}{2})1 + \mathrm{Im}(\lambda u_n a_n u_n^*). \tag{18}$$

Notice however that, using the *unitary invariance* (property (S) for s unitary), together with (17) and (18), gives, for every $n \geq 1$, the equalities

$$\begin{aligned} Q(\lambda e_n) &= Q\big((\mathrm{Re}\tfrac{\lambda}{2})1 + \mathrm{Re}(\lambda u_n a_n u_n^*)\big) + iQ\big((\mathrm{Im}\tfrac{\lambda}{2})1 + \mathrm{Im}(\lambda u_n a_n u_n^*)\big) \\ &= \mathrm{Re}\tfrac{\lambda}{2} + Q\big(\mathrm{Re}(\lambda u_n a_n u_n^*)\big) + i\mathrm{Im}\tfrac{\lambda}{2} + iQ\big(\mathrm{Im}(\lambda u_n a_n u_n^*)\big) \\ &= \tfrac{\lambda}{2} + Q(\lambda u_n a_n u_n^*) = \tfrac{\lambda}{2} + Q\big(u_n(\lambda a_n)u_n^*\big) = \tfrac{\lambda}{2} + Q\big(\lambda a_n\big). \end{aligned} \tag{19}$$

On the other hand, if we define $v = 1 - 2p$ (which is obviously a unitary in A), then for all $n \geq 1$ we have

$$\Phi\big(v(\lambda a_n)v^*\big) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & \lambda y_n \\ \lambda z_n & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -\lambda y_n \\ -\lambda z_n & 0 \end{bmatrix} = \Phi(-\lambda a_n).$$

This gives

$$v(\lambda a_n)v^* = -\lambda a_n, \text{ for all } n \geq 1. \tag{20}$$

Using the unitary invariance, combined with *real scalar homogeneity* (property (H) with $\alpha \in \mathbb{R}$), equality (20) gives

$$Q(\lambda a_n) = Q(v(\lambda a_n)v^*) = Q(-\lambda a_n) = -Q(\lambda a_n),$$

which forces

$$Q(\lambda a_n) = 0, \text{ for all } n \geq 1. \tag{21}$$

Combining (21) with (19) gives

$$Q(\lambda e_n) = \lambda/2, \text{ for all } n \geq 1. \tag{22}$$

It is obvious that, by construction, we have $\lim_{n\to\infty} \|E_n - \Phi(e)\| = 0$, which means that $\lim_{n\to\infty} \|e_n - e\| = 0$. Using the norm continuity of the quasi-trace (see [BH]), combined with (22), gives the desired result. $\square$

We are now ready to prove the main result.

6. Theorem. *Let A be an AW*-factor of type II_1, and let $e \in A$ be an idempotent element. Then, for any $\lambda \in \mathbb{C}$, one has the equality*

$$Q(\lambda e) = \lambda D\big(\mathbf{L}(e)\big). \tag{23}$$

Proof. The proof will be carried out in two steps.

PARTICULAR CASE: *Assume $D\big(\mathbf{L}(e)\big) \leq \frac{1}{2}$.*

Denote, for simplicity $\mathbf{L}(e)$ by p, and $\mathbf{R}(e)$ by q. By the *Paralellogram Law* (see [Ka]) we have

$$p \vee q - p \sim q - p \wedge q.$$

Since $p \sim q$, we get

$$D(p \vee q - p) = D(q - p \wedge q) = D(q) - D(p \wedge q) \leq D(q) = D(p).$$

Using the intermediate value property for D, we can find a projection $r \leq 1 - p \vee q$ such that $D(r) + D(p \vee q - p) = D(p)$. Put $q_0 = r + p \vee q - p$. We have $p \perp q_0$ and $p + q_0 \geq p \vee q$. Let us work in the AW*-algebra $A_0 = (p + q_0)A(p + q_0)$. Obviously A_0 is again an AW*-factor of type II_1, so it carries its quasi-trace Q_0. By the uniqueness of the quasi-trace, it is obvious that

$$Q_0(x) = \frac{Q(x)}{D(p + q_0)}, \text{ for all } x \in A_0. \tag{24}$$

Notice that $e \in (p \vee q)A(p \vee q)$, so in particular e belongs to A_0. In A_0, we have $\mathbf{L}(e) = p$, and $p \sim q_0 = 1_{A_0} - p$, which means that $D_0(p) = \frac{1}{2}$. (Here D_0 denotes the dimension function in A_0). By Lemma 5, we get

$$Q_0(\lambda e) = \lambda/2,$$

which combined with (24) yields

$$\frac{\lambda}{2} = \frac{Q(\lambda e)}{D(p+q_0)} = \frac{Q(\lambda e)}{2D(p)},$$

which obviously proves (23).

General Case. One knows (see [Be]) that $M_2(A)$ is also an AW*-factor of type II_1. Moreover, if we denote by $Q^{(2)}$ the quasi-trace of $M_2(A)$, we have (as above)

$$Q(x) = 2Q^{(2)}\Big(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\Big), \text{ for all } x \in A. \tag{25}$$

Define the idempotent $E = \left[\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right] \in M_2(A)$, and the projection $P = \left[\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right] \in M_2(A)$. Using (25) we have $Q(e) = 2Q^{(2)}(E)$. If we denote by $D^{(2)}$ the dimension function of $M_2(A)$, then by (25) we also have $D(p) = 2D^{(2)}(P)$, which gives $D^{(2)}(P) \leq \frac{1}{2}$. Using the obvious equality $\mathbf{L}(E) = P$, by the particular case above applied to $M_2(A)$, we get

$$Q(\lambda e) = 2Q^{(2)}(\lambda E) = 2\lambda D^{(2)}(P) = \lambda D(p).$$

□

7. Corollary. *Let A be an AW*-factor of type II_1, and let $e \in A$ be an idempotent. Then, for every $s \in GL(A)$, we have*

$$Q(ses^{-1}) = Q(e). \tag{26}$$

Proof. Put $p = \mathbf{L}(e)$. If we define $t = 1 - e(1-p)$, then t is invertible, and $e = tpt^{-1}$. This computation shows that it is enough to prove (26) in the case when $e = e^*$. Let $q = \mathbf{L}(ses^{-1})$. Arguing as above, there exists some $x \in GL(A)$ such that $ses^{-1} = xqx^{-1}$. So if we put $y = x^{-1}s$, we have $yey^{-1} = q$, with both e and q self-adjoint idempotents. Since this obviously forces $e \sim q$, we get $D(e) = D(q) = Q(ses^{-1})$. □

8. Corollary. *Let A be an AW*-factor of type II_1 and let $e_1, e_2 \in A$ be idempotents, such that $e_1e_2 = e_2e_1 = 0$. (This implies that e_1+e_2 is again an idempotent.) Then*

$$Q(e_1 + e_2) = Q(e_1) + Q(e_2).$$

Proof. Let $p = \mathbf{L}(e_1 + e_2)$. As in the proof of the preceding corollary, there exists an element $s \in GL(A)$, such that $p = s(e_1 + e_2)s^{-1}$. Put $f_k = se_ks^{-1}$, $k = 1, 2$, so that $f_1 + f_2 = p$. But now we also have $pf_k = f_kp = f_k$, $k = 1, 2$, which means in particular that $f_1, f_2 \in pAp$. So, if we work in the AW*-factor (again of type II_1) $A_0 = pAp$, we will have $f_1 + f_2 = 1_{A_0}$. On the one hand, using the notations from the proof of Theorem 6, we have

$$Q_0(f_1) + Q_0(f_2) = 1.$$

On the other hand, we have

$$Q_0(f_k) = \frac{Q(f_k)}{D(p)}, \ \ k = 1, 2,$$

so we get

$$D(p) = Q(f_1) + Q(f_2).$$

Finally, using the preceding corollary, we get

$$Q(e_1 + e_2) = D(p) = Q(f_1) + Q(f_2) = Q(se_1s^{-1}) + Q(se_2s^{-1}) = Q(e_1) + Q(e_2).$$

□

9. Comment. A. Theorem 6 can be analyzed from a different point of view, as follows. In principle, one can extend the dimension function D to the collection of all idempotents by $\tilde{D}(e) = D\big(\mathbf{L}(e)\big)$. One can easily prove that this extended dimension function will have the same properties as the usual dimension function (when Murray-von Neumann equivalence is extended to idempotents). The point of Theorem 6 is then the fact that $\tilde{D} = Q$.

B. It is interesting to note that the linearity of the quasi-trace is equivalent to the following condition:

($*$) *For any family of idempotents $e_1, \dots, e_n \in A$, such that $e_j e_k = 0$ for all $j, k \in \{1, \dots, n\}$ with $j \neq k$, and any family of numbers $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, one has:*

$$Q(\alpha_1 e_1 + \cdots + \alpha_n e_n) = \alpha_1 Q(e_1) + \cdots + \alpha_n Q(e_n).$$

This will be discussed in a future paper.

References

[Be] S. Berberian, *Baer $*$-rings*, Springer, 1972 MR **46:**7294

[BH] B. Blackadar, D. Handelman, *Dimension functions and traces on C*-algebras*, J. Funct. Anal. 45 (1982), no. 3, 297–340. MR **83g:**46050

[Ha] U. Haagerup, *Quasi-traces on exact C*-algebras are traces*, Preprint, 1991

[Ka] I. Kaplansky, *Rings of operators*, Benjamin, 1968 MR **39:**6092

Department of Mathematics, Kansas State University, Manhattan, Kansas 66506

E-mail address: nagy@math.ksu.edu