A NOTE ON IDEMPOTENTS IN FINITE AW*-FACTORS

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Abstract. We prove that the value of the quasi-trace on an idempotent element in an AW*-factor of type II
is the same as the dimension of its left (or right) support.

It is a long-standing open problem (due to Kaplansky) to prove that an AW*-factor of type II
is in fact a von Neumann algebra. A remarkable answer, in the affirmative, was found by Haagerup ([Ha]), who proved that if an AW*-factor $A$ is generated by an exact C*-algebra, then $A$ is indeed a von Neumann algebra.

The main object, that was investigated in connection with Kaplansky’s problem, is the quasi-trace, whose construction we briefly recall below.

One starts with an AW*-factor of type II, say $A$. Denote by $\mathcal{P}(A)$ the collection of projections in $A$, that is
$$\mathcal{P}(A) = \{ p \in A : p = p^* = p^2 \}.$$ A key fact is then the existence of a (unique) dimension function $D : \mathcal{P}(A) \to [0,1]$ with the following properties:

- $D(p) = D(q) \iff p \sim q$;
- if $p \perp q$, then $D(p + q) = D(p) + D(q)$;
- $D(1) = 1$.

The symbol “$\sim$” denotes the Murray-von Neumann equivalence relation ($p \sim q \iff \exists x \in A$ with $p = x^*x$ and $q = xx^*$), while “$\perp$” denotes the orthogonality relation ($p \perp q \iff pq = 0$; this implies that $p + q$ is again a projection).

Once the dimension function is defined, it is extended to self-adjoint elements with finite spectrum. More explicitly, if $a \in A$ is self-adjoint with finite spectrum, then there are (real) numbers $\alpha_1, \ldots, \alpha_n$ and pairwise orthogonal projections $p_1, \ldots, p_n$, such that $a = \sum_{k=1}^n \alpha_k p_k$. We then define $d(a) = \sum_{k=1}^n \alpha_k D(p_k)$.

For an arbitrary self-adjoint element $a \in A$, one can approximate uniformly $a$ with a sequence $(a_n)_{n \geq 1} \subseteq \{ a \}$ of elements with finite spectrum. (Here $\{ a \}$ stands for the AW*-subalgebra generated by $a$ and $1$.) It turns out that the limit $q(a) = \lim_{n \to \infty} d(a_n)$ is independent of the particular choice of $(a_n)_{n \geq 1}$.

Finally, for an arbitrary element $x \in A$, one defines $Q(x) = q(\text{Re} x) + iq(\text{Im} x)$, where $\text{Re} x = \frac{1}{2}(x + x^*)$ and $\text{Im} x = \frac{i}{2}(x - x^*)$.

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The map $Q : A \to \mathbb{C}$, defined this way, is the unique one with the properties:

(i) $Q$ is linear, when restricted to abelian C*-subalgebras of $A$;
(ii) $Q(x^*x) = Q(xx^*) \geq 0$, for all $x \in A$;
(iii) $Q(x) = Q(\Re x) + iQ(\Im x)$, for all $x \in A$;
(iv) $Q(1) = 1$.

It is obvious that $Q|_{P(A)} = D$. The map $Q$ is called the quasi-trace of $A$.

It is well known that an AW*-factor of type II$_1$ is a von Neumann algebra if and only if its quasi-trace is linear. Haagerup’s solution for Kaplansky’s problem goes through the proof of the linearity of the quasi-trace.

On the one hand, one can easily see that the linearity of the quasi-trace is equivalent to its scalar homogeneity (compare with (i) above):

$$(H) \quad Q(\alpha x) = \alpha Q(x), \text{ for all } x \in A, \alpha \in \mathbb{C}.\]

Notice that $(H)$ holds when either $\alpha \in \mathbb{R}$ or when $x$ is normal. On the other hand, it is again easy to note that the linearity of the quasi-trace is equivalent to the similarity invariance property

$$(S) \quad Q(sxs^{-1}) = Q(x), \text{ for all } x \in A, s \in GL(A).$$

(Here $GL(A)$ denotes the group of invertible elements in $A$.) Notice that $(S)$ is true if $s$ is unitary.

The purpose of this note is to prove that both $(H)$ and $(S)$ hold if $x \in A$ is an idempotent (i.e. $x^2 = x$).

1. Notations. If $A$ is an AW*-algebra, for an element $x \in A$, we denote by $L(x)$ (resp. $R(x)$) the left (resp. right) support of $x$. Recall that both $L(x)$ and $R(x)$ are projections, and moreover we have

$$L(x) \sim R(x), \text{ for all } x \in A.$$

It is known (see [Ka]) that, for every $x \in A$, there exists a unique partial isometry $v$ such that

(i) $x = v(x^*x)^{1/2}$;
(ii) $vv^* = L(x)$;
(iii) $v^*v = R(x)$.

(This property is referred to as the Polar Decomposition.)

2. Remark. If $A$ is a finite AW*-algebra, then the group $GL(A)$, of invertible elements, is dense in $A$ in the norm topology.

Indeed, on the one hand, since $L(x) \sim R(x)$, by the finiteness assumption we also have $1 - L(x) \sim 1 - R(x)$. In particular, there exists a partial isometry $w \in A$ such that $1 - L(x) = ww^*$ and $1 - R(x) = w^*w$. On the other hand, if we take $x = v(x^*x)^{1/2}$ to be the polar decomposition described above, then we obviously have $w^*v = 0$, so the element $u = v + w$ is unitary, and we still have $x = u(x^*x)^{1/2}$. Then, for every $\varepsilon > 0$ the positive element $(x^*x)^{1/2} + \varepsilon 1$ is invertible, and so is $u((x^*x)^{1/2} + \varepsilon 1)$. The result then follows from the obvious equality $\|x - u((x^*x)^{1/2} + \varepsilon 1)\| = \varepsilon$.

3. Notation. For a C*-algebra $A$ and an integer $n \geq 2$, we denote by $M_n(A)$ the C*-algebra of $n \times n$ matrices with coefficients in $A$. 
The key technical result in this paper is the following.

4. Lemma. Suppose $A$ is a unital C*-algebra and $x \in GL(A)$. Then there exist a unitary element $U \in M_2(A)$ and elements $y, z \in A$, such that

\[ U^* \begin{bmatrix} 2 & x \\ 0 & 0 \end{bmatrix} U = \begin{bmatrix} 1 & y \\ z & 1 \end{bmatrix}. \]

Proof. Consider the function defined by

\[ f(t) = \frac{(1-t)^2}{t}, \quad 0 < t < 1. \]

It is obvious that $f : (0, 1) \to (0, \infty)$ is a homeomorphism. Since $x$ is invertible, the spectrum of $xx^*$ is contained in $(0, \infty)$, so by functional calculus there exists an invertible positive element $w \in A$, with $\|w\| < 1$, such that $f(w) = xx^*$, which means

\[ xx^* = (1-w)^2 w^{-1}. \]

Define the elements

\[ a = (1+w)^{-1/2}w^{1/2}, \quad b = (1+w)^{-1/2}; \]
\[ c = x^{-1}(1+w)^{-1/2}w^{-1/2}(1-w), \quad d = -x^{-1}(1+w)^{-1/2}(1-w). \]

First, we have

\[ aa^* + bb^* = (1+w)^{-1/2}[w+1](1+w)^{-1/2} = 1, \]

and using (2) we also have

\[ cc^* + dd^* = x^{-1}(1+w)^{-1/2}(1-w)[w^{-1}+1](1-w)(1+w)^{-1/2}(x^*)^{-1}
\]
\[ = x^{-1}(1-w)^2 w^{-1}(x^*)^{-1} = x^{-1}(xx^*)(x^*)^{-1} = 1. \]

Secondly, since by taking inverses, (2) yields

\[ (x^*)^{-1}x^{-1} = w(1-w)^{-2}, \]

so we also get

\[ a^*a + c^*c = w^{1/2}(1+w)^{-1/2}[1 + (1-w)(x^*)^{-1}x^{-1}(1-w)](1+w)^{-1/2}w^{1/2}
\]
\[ = w^{1/2}(1+w)^{-1/2}[1 + w^{-1}](1+w)^{-1/2}w^{1/2} = 1, \]
\[ b^*b + d^*d = (1+w)^{-1/2}[1 + (1-w)(x^*)^{-1}x^{-1}x(1-w)](1+w)^{-1/2}
\]
\[ = (1+w)^{-1/2}[1 + w](1+w)^{-1/2}. \]

Finally, we notice that

\[ ac^* + bd^* = (1+w)^{-1/2}[w^{1/2}w^{-1/2} - 1](1-w)(1+w)^{-1/2}(x^*)^{-1} = 0, \]

and using (5) we also have

\[ a^*b + c^*d = (1+w)^{-1/2}[w^{1/2} - w^{-1/2}(1-w)(x^*)^{-1}x(1-w)](1+w)^{-1/2}
\]
\[ = (1+w)^{-1/2}[w^{1/2} - w^{-1/2}w](1+w)^{-1/2} = 0. \]

If we define the matrix $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then (3), (4), and (5) give $UU^* = I$, while (6), (7), and (8) give $U^*U = I$, so $U$ is indeed unitary. (Here $I$ denotes the unit in $M_2(A)$.)
Let us now observe that
\[
2a + xc = 2(1 + w)^{-1/2}w^{1/2} + (1 + w)^{1/2}(1 - w)w^{-1/2} \\
= (1 + w)^{-1/2}(w^{1/2} + w^{-1/2}) = a + bw^{-1/2},
\]
\[(11)\]
\[
2b + xd = 2(1 + w)^{-1/2} - (1 + w)^{-1/2}(1 - w) \\
= (1 + w)^{-1/2}(w + 1) = aw^{1/2} + b,
\]
\[(12)\]
\[
c + dw^{-1/2} = x^{-1}(1 + w)^{-1/2}[(1 - w)w^{-1/2} - (1 - w)w^{-1/2}] = 0,
\]
\[(13)\]
\[
cw^{1/2} + d = (c + dw^{-1/2})w^{1/2} = 0.
\]

These equalities prove exactly that
\[
\begin{bmatrix} 2 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & w^{1/2} \\ w^{-1/2} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.
\]

\[
\square
\]

The next result is a particular case of the main result.

5. Lemma. Let \( A \) be an AW*-factor of type II\(_1\), and let \( e \in A \) be an idempotent with \( D(L(e)) = \frac{1}{2} \). Then, for any \( \lambda \in \mathbb{C} \), we have
\[
Q(\lambda e) = \lambda/2.
\]

Proof. Denote, for simplicity, the projection \( L(x) \) by \( p \). The assumption is that \( p \sim 1 - p \). Then we have a *-isomorphism \( \Phi : A \to M_2(pAp) \), such that \( \Phi(p) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). It is obvious that, since \( e = pe \), there exists an element \( x \in pAp \) such that \( \Phi(e) = \begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix} \). By Remark 2, we can find a sequence \( (x_n)_{n \geq 1} \) of invertible elements in \( pAp \) (the unit in \( pAp \) is \( p \)) with \( \lim_{n \to \infty} \|x_n - x\| = 0 \). Define the sequence
\[
E_n = \begin{bmatrix} 1 & x_n \\ 0 & 0 \end{bmatrix} \in M_2(pAp), \quad n \geq 1.
\]

By Lemma 4, one can find two sequences \( (y_n)_{n \geq 1} \) and \( (z_n)_{n \geq 1} \) in \( pAp \), and a sequence of unitaries \( (U_n)_{n \geq 1} \subset M_2(pAp) \), such that
\[
E_n = U_n \begin{bmatrix} \frac{1}{2} & y_n \\ z_n & \frac{1}{2} \end{bmatrix} U_n^*, \quad \text{for all } n \geq 1.
\]

Define \( e_n = \Phi^{-1}(E_n) \), \( u_n = \Phi^{-1}(U_n) \), and \( a_n = u_n^*e_nu_n - \frac{1}{2} \), \( n \geq 1 \), so that we have
\[
\Phi(a_n) = \begin{bmatrix} 0 & y_n \\ z_n & 0 \end{bmatrix}, \quad \text{for all } n \geq 1.
\]

Now fix a complex number \( \lambda \). On the one hand, we have
\[
\lambda e_n = u_n \left( \frac{1}{2} + a_n \right) u_n^* = \frac{1}{2} + u_n a_n u_n^*, \quad \text{for all } n \geq 1.
\]

This gives, for every \( n \geq 1 \), the equalities
\[
\begin{align*}
\text{Re}(\lambda e_n) &= (\text{Re}\Phi)1 + \text{Re}(\lambda u_n a_n u_n^*), \\
\text{Im}(\lambda e_n) &= (\text{Im}\Phi)1 + \text{Im}(\lambda u_n a_n u_n^*).
\end{align*}
\]
Notice however that, using the **unitary invariance** (property (S) for $s$ unitary), together with (17) and (18), gives, for every $n \geq 1$, the equalities
\[
Q(\lambda e_n) = Q((\operatorname{Re} \frac{1}{n} + \operatorname{Re}(\lambda u_n a_n u_n^*)) + iQ((\operatorname{Im} \frac{1}{n} + \operatorname{Im}(\lambda u_n a_n u_n^*))
\]
\[
= \lambda^2 + Q(\lambda u_n a_n u_n^*) + iQ(\operatorname{Im}(\lambda u_n a_n u_n^*))
\]
\[
= \frac{\lambda^2}{2} + Q(u_n(\lambda a_n) u_n^*) = \frac{\lambda^2}{2} + Q(\lambda a_n).
\]

Using the unitary invariance, combined with **real scalar homogeneity** (property (H) with $\alpha \in \mathbb{R}$), equality (20) gives
\[
Q(\lambda a_n) = Q(v(\lambda a_n) v^*) = Q(-\lambda a_n) = -Q(\lambda a_n),
\]
which forces
\[
Q(\lambda a_n) = 0,
\]
for all $n \geq 1$.

Combining (21) with (19) gives
\[
Q(\lambda e_n) = \lambda/2,
\]
for all $n \geq 1$.

It is obvious that, by construction, we have $\lim_{n \to \infty} \|E_n - \Phi(e)\| = 0$, which means that $\lim_{n \to \infty} \|e_n - e\| = 0$. Using the norm continuity of the quasi-trace (see [BH]), combined with (22), gives the desired result.

We are now ready to prove the main result.

**6. Theorem.** Let $A$ be an AW*-factor of type II$_1$, and let $e \in A$ be an idempotent element. Then, for any $\lambda \in \mathbb{C}$, one has the equality
\[
Q(\lambda e) = \lambda D(L(e)).
\]

**Proof.** The proof will be carried out in two steps.

**PARTICULAR CASE:** Assume $D(L(e)) \leq \frac{1}{r}$.

Denote, for simplicity $L(e)$ by $p$, and $R(e)$ by $q$. By the **Parallelogram Law** (see [Ku]) we have
\[
p \vee q - p \sim q - p \wedge q.
\]

Since $p \sim q$, we get
\[
D(p \vee q - p) = D(q - p \wedge q) = D(q) - D(p \wedge q) \leq D(q) = D(p).
\]

Using the intermediate value property for $D$, we can find a projection $r \leq 1 - p \vee q$ such that $D(r) + D(p \vee q - p) = D(p)$. Put $q_0 = r + p \vee q - p$. We have $p \perp q_0$ and $p + q_0 \geq p \vee q$. Let us work in the AW*-algebra $A_0 = (p + q_0) A (p + q_0)$. Obviously $A_0$ is again an AW*-factor of type II$_1$, so it carries its quasi-trace $Q_0$.

By the uniqueness of the quasi-trace, it is obvious that
\[
Q_0(x) = \frac{Q(x)}{D(p + q_0)},
\]
for all $x \in A_0$. 
Notice that \( e \in (p \vee q)A(p \vee q) \), so in particular \( e \) belongs to \( A_0 \). In \( A_0 \), we have 
\[ L(e) = p, \text{ and } p \sim q_0 = 1_{A_0} - p, \] 
which means that \( D_0(p) = \frac{1}{2} \). (Here \( D_0 \) denotes the dimension function in \( A_0 \)). By Lemma 5, we get 
\[ Q_0(\lambda e) = \lambda/2, \] 
which combined with (24) yields 
\[ \frac{\lambda}{2} = \frac{Q(\lambda e)}{D(p + q_0)} = \frac{Q(\lambda e)}{2D(p)}, \] 
which obviously proves (23).

**General Case.** One knows (see [12]) that \( M_2(A) \) is also an AW*-factor of type II\(_1\). Moreover, if we denote by \( Q^{(2)} \) the quasi-trace of \( M_2(A) \), we have (as above) 
\[ Q(x) = 2Q^{(2)}(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}), \] 
for all \( x \in A \).

Define the idempotent \( E = [0 0] \in M_2(A) \), and the projection \( P = [0 0] \in M_2(A) \). Using (25) we have \( Q(e) = 2Q^{(2)}(E) \). If we denote by \( D^{(2)} \) the dimension function of \( M_2(A) \), then by (25) we also have \( D(p) = 2D^{(2)}(P) \), which gives \( D^{(2)}(P) \leq \frac{1}{2} \).
Using the obvious equality \( L(E) = P \), by the particular case above applied to \( M_2(A) \), we get 
\[ Q(\lambda e) = 2Q^{(2)}(\lambda E) = 2\lambda D^{(2)}(P) = \lambda D(p). \]

\( \square \)

**7. Corollary.** Let \( A \) be an AW*-factor of type II\(_1\), and let \( e \in A \) be an idempotent. Then, for every \( s \in GL(A) \), we have 
\[ Q(ses^{-1}) = Q(e). \]

**Proof.** Put \( p = L(e) \). If we define \( t = 1 - e(1 - p) \), then \( t \) is invertible, and \( e = tpt^{-1} \). This computation shows that it is enough to prove (26) in the case when \( e = e^* \). Let \( q = L(ses^{-1}) \). Arguing as above, there exists some \( x \in GL(A) \) such that 
\[ ses^{-1} = xqf^{-1}. \] 
So if we put \( y = x^{-1}s \), we have \( ye^{-1} = q \), with both \( e \) and \( q \) self-adjoint idempotents. Since this obviously forces \( e \sim q \), we get 
\( D(e) = D(q) = Q(ses^{-1}) \).

\( \square \)

**8. Corollary.** Let \( A \) be an AW*-factor of type II\(_1\) and let \( e_1, e_2 \in A \) be idempotents, such that \( e_1e_2 = e_2e_1 = 0 \). (This implies that \( e_1 + e_2 \) is again an idempotent.) Then 
\[ Q(e_1 + e_2) = Q(e_1) + Q(e_2). \]

**Proof.** Let \( p = L(e_1 + e_2) \). As in the proof of the preceding corollary, there exists an element \( s \in GL(A) \), such that \( p = s(e_1 + e_2)s^{-1} \). Put \( f_k = ses^{-1}, k = 1, 2 \), so that \( f_1 + f_2 = p \). But now we also have \( pf_k = f_kp = f_k, k = 1, 2 \), which means in particular that \( f_1, f_2 \in pAp \). So, if we work in the AW*-factor (again of type II\(_1\)) \( A_0 = pAp \), we will have \( f_1 + f_2 = 1_{A_0} \). On the one hand, using the notations from the proof of Theorem 6, we have 
\[ Q_0(f_1) + Q_0(f_2) = 1. \]
On the other hand, we have 
\[ Q_0(f_k) = \frac{Q(f_k)}{D(p)}, \] 
\( k = 1, 2, \)
so we get

\[ D(p) = Q(f_1) + Q(f_2). \]

Finally, using the preceding corollary, we get

\[ Q(e_1 + e_2) = D(p) = Q(f_1) + Q(f_2) = Q(se_1 s^{-1}) + Q(se_2 s^{-1}) = Q(e_1) + Q(e_2). \]

9. Comment. A. Theorem 6 can be analyzed from a different point of view, as follows. In principle, one can extend the dimension function \( D \) to the collection of all idempotents by \( \tilde{D}(e) = D(L(e)) \). One can easily prove that this extended dimension function will have the same properties as the usual dimension function (when Murray-von Neumann equivalence is extended to idempotents). The point of Theorem 6 is then the fact that \( \tilde{D} = Q \).

B. It is interesting to note that the linearity of the quasi-trace is equivalent to the following condition:

\[ (\ast) \text{ For any family of idempotents } e_1, \ldots, e_n \in A, \text{ such that } e_j e_k = 0 \text{ for all } j, k \in \{1, \ldots, n\} \text{ with } j \neq k, \text{ and any family of numbers } \alpha_1, \ldots, \alpha_n \in \mathbb{R}, \text{ one has:} \]

\[ Q(\alpha_1 e_1 + \cdots + \alpha_n e_n) = \alpha_1 Q(e_1) + \cdots + \alpha_n Q(e_n). \]

This will be discussed in a future paper.

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