A NOTE ON IDEMPOTENTS IN FINITE AW*-FACTORS

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Abstract. We prove that the value of the quasi-trace on an idempotent element in an AW*-factor of type II$_1$ is the same as the dimension of its left (or right) support.

It is a long-standing open problem (due to Kaplansky) to prove that an AW*-factor of type II$_1$ is in fact a von Neumann algebra. A remarkable answer, in the affirmative, was found by Haagerup ([Ha]), who proved that if an AW*-factor $A$ is generated by an exact C*-algebra, then $A$ is indeed a von Neumann algebra.

The main object, that was investigated in connection with Kaplansky’s problem, is the quasi-trace, whose construction we briefly recall below.

One starts with an AW*-factor of type II$_1$, say $A$. Denote by $\mathcal{P}(A)$ the collection of projections in $A$, that is 

$$\mathcal{P}(A) = \{ p \in A : p = p^* = p^2 \}.$$

A key fact is then the existence of a (unique) dimension function $D : \mathcal{P}(A) \to [0, 1]$ with the following properties:

- $D(p) = D(q) \iff p \sim q$;
- if $p \perp q$, then $D(p + q) = D(p) + D(q)$;
- $D(1) = 1$.

The symbol “$\sim$” denotes the Murray-von Neumann equivalence relation ($p \sim q \iff \exists x \in A$ with $p = x^*x$ and $q = xx^*$), while “$\perp$” denotes the orthogonality relation ($p \perp q \iff pq = 0$; this implies that $p + q$ is again a projection).

Once the dimension function is defined, it is extended to self-adjoint elements with finite spectrum. More explicitly, if $a \in A$ is self-adjoint with finite spectrum, then there are (real) numbers $\alpha_1, \ldots, \alpha_n$ and pairwise orthogonal projections $p_1, \ldots, p_n$, such that $a = \sum_{k=1}^n \alpha_k p_k$. We then define $d(a) = \sum_{k=1}^n \alpha_k D(p_k)$.

For an arbitrary self-adjoint element $a \in A$, one can approximate uniformly $a$ with a sequence $(a_n)_{n \geq 1} \subseteq \{ a \}^d$ of elements with finite spectrum. (Here $\{ a \}^d$ stands for the AW*-subalgebra generated by $a$ and 1.) It turns out that the limit $q(a) = \lim_{n \to \infty} d(a_n)$ is independent of the particular choice of $(a_n)_{n \geq 1}$.

Finally, for an arbitrary element $x \in A$, one defines $Q(x) = q(Re x) + iq(Im x)$, where $Re x = \frac{1}{2} (x + x^*)$ and $Im x = \frac{1}{2i} (x - x^*)$. 

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The map $Q : A \rightarrow \mathbb{C}$, defined this way, is the unique one with the properties:

(i) $Q$ is linear, when restricted to abelian C*-subalgebras of $A$;
(ii) $Q(x^*x) = Q(xx^*) \geq 0$, for all $x \in A$;
(iii) $Q(x) = Q(\text{Re} x) + iQ(\text{Im} x)$, for all $x \in A$;
(iv) $Q(1) = 1$.

It is obvious that $Q|_{P(A)} = D$. The map $Q$ is called the quasi-trace of $A$.

It is well known that an AW*-factor of type $\text{II}_1$ is a von Neumann algebra if and only if its quasi-trace is linear. Haagerup’s solution for Kaplansky’s problem goes through the proof of the linearity of the quasi-trace.

On the one hand, one can easily see that the linearity of the quasi-trace is equivalent to its scalar homogeneity (compare with (i) above):

(H) \[ Q(\alpha x) = \alpha Q(x), \text{for all } x \in A, \ \alpha \in \mathbb{C}. \]

Notice that (H) holds when either $\alpha \in \mathbb{R}$ or when $x$ is normal. On the other hand, it is again easy to note that the linearity of the quasi-trace is equivalent to the similarity invariance property

(S) \[ Q(sxs^{-1}) = Q(x), \text{for all } x \in A, \ s \in GL(A). \]

(Here $GL(A)$ denotes the group of invertible elements in $A$.) Notice that (S) is true if $s$ is unitary.

The purpose of this note is to prove that both (H) and (S) hold if $x \in A$ is an idempotent (i.e. $x^2 = x$).

1. Notations. If $A$ is an AW*-algebra, for an element $x \in A$, we denote by $L(x)$ (resp. $R(x)$) the left (resp. right) support of $x$. Recall that both $L(x)$ and $R(x)$ are projections, and moreover we have

$L(x) \sim R(x)$, for all $x \in A$.

It is known (see [Ka]) that, for every $x \in A$, there exists a unique partial isometry $v$ such that

(i) \[ x = v(x^*x)^{1/2}; \]
(ii) \[ vv^* = L(x); \]
(iii) \[ v^*v = R(x). \]

(This property is referred to as the Polar Decomposition.)

2. Remark. If $A$ is a finite AW*-algebra, then the group $GL(A)$, of invertible elements, is dense in $A$ in the norm topology.

Indeed, on the one hand, since $L(x) \sim R(x)$, by the finiteness assumption we also have $1 - L(x) \sim 1 - R(x)$. In particular, there exists a partial isometry $w \in A$ such that $1 - L(x) = w w^*$ and $1 - R(x) = w^* w$. On the other hand, if we take $x = v(x^*x)^{1/2}$ to be the polar decomposition described above, then we obviously have $w^* v = 0$, so the element $u = v + w$ is unitary, and we still have $x = u(x^*x)^{1/2}$. Then, for every $\varepsilon > 0$ the positive element $(x^*x)^{1/2} + \varepsilon 1$ is invertible, and so is $u \{(x^*x)^{1/2} + \varepsilon 1\}$. The result then follows from the obvious equality $\|x - u \{(x^*x)^{1/2} + \varepsilon 1\}\| = \varepsilon$.

3. Notation. For a C*-algebra $A$ and an integer $n \geq 2$, we denote by $M_n(A)$ the C*-algebra of $n \times n$ matrices with coefficients in $A$. 
The key technical result in this paper is the following.

4. Lemma. Suppose $A$ is a unital $C^*$-algebra and $x \in GL(A)$. Then there exist a unitary element $U \in M_2(A)$ and elements $y, z \in A$, such that

$$(1) \quad U^* \begin{bmatrix} 2 & x \\ 0 & 0 \end{bmatrix} U = \begin{bmatrix} 1 & y \\ z & 1 \end{bmatrix}.$$  

Proof. Consider the function defined by

$$f(t) = \frac{(1-t)^2}{t}, \quad 0 < t < 1.$$  

It is obvious that $f : (0, 1) \to (0, \infty)$ is a homeomorphism. Since $x$ is invertible, the spectrum of $xx^*$ is contained in $(0, \infty)$, so by functional calculus there exists an invertible positive element $w \in A$, with $\|w\| < 1$, such that $f(w) = x x^*$, which means

$$(2) \quad x x^* = (1 - w)^2 w^{-1}.$$  

Define the elements

$$a = (1 + w)^{-1/2} w^{1/2}, \quad b = (1 + w)^{-1/2},$$  

$$c = x^{-1}(1 + w)^{-1/2} w^{-1/2}(1 - w), \quad d = -x^{-1}(1 + w)^{-1/2}(1 - w).$$  

First, we have

$$(3) \quad a a^* + b b^* = (1 + w)^{-1/2}[w + 1](1 + w)^{-1/2} = 1,$$  

and using (2) we also have

$$(4) \quad c c^* + d d^* = x^{-1}(1 + w)^{-1/2}(1 - w)[w^{-1} + 1](1 - w)(1 + w)^{-1/2}(x^*)^{-1}$$  

$$= x^{-1}(1 - w)^2 w^{-1}(x^*)^{-1} = x^{-1}(x x^*)(x^*)^{-1} = 1.$$  

Secondly, since by taking inverses, (2) yields

$$(5) \quad (x^*)^{-1} x^{-1} = w(1 - w)^{-2},$$  

so we also get

$$(6) \quad a^* a + c^* c = w^{1/2}(1 + w)^{-1/2}[1 + (1 - w)(x^*)^{-1} x^{-1}(1 - w)](1 + w)^{-1/2} w^{1/2}$$  

$$= w^{1/2}(1 + w)^{-1/2}[1 + w^{-1}](1 + w)^{-1/2} w^{1/2} = 1,$$  

$$(7) \quad b^* b + d^* d = (1 + w)^{-1/2}[1 + (1 - w)(x^*)^{-1} x^{-1}(1 - w)](1 + w)^{-1/2}$$  

$$= (1 + w)^{-1/2}[1 + w](1 + w)^{-1/2}.$$  

Finally, we notice that

$$(8) \quad a c^* + b d^* = (1 + w)^{-1/2}[w^{1/2} w^{-1/2} - 1](1 - w)(1 + w)^{-1/2}(x^*)^{-1} = 0,$$  

and using (5) we also have

$$(9) \quad a^* b + c^* d = (1 + w)^{-1/2}[w^{1/2} - w^{-1/2}(1 - w)(x^*)^{-1} x(1 - w)](1 + w)^{-1/2}$$  

$$= (1 + w)^{-1/2}[w^{1/2} - w^{-1/2} w](1 + w)^{-1/2} = 0.$$  

If we define the matrix $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then (3), (4), and (8) give $UU^* = I$, while (6), (7), and (9) give $U^*U = I$, so $U$ is indeed unitary. (Here $I$ denotes the unit in $M_2(A).$)
Let us now observe that

\begin{align}
2a + xc &= 2(1 + w)^{-1/2}w^{1/2} + (1 + w)^{1/2}(1 - w)w^{-1/2} \\
&= (1 + w)^{-1/2}(w^{1/2} + w^{-1/2}) = a + bw^{-1/2}, \\
2b + xd &= 2(1 + w)^{-1/2} - (1 + w)^{-1/2}(1 - w) \\
&= (1 + w)^{-1/2}(w + 1) = aw^{1/2} + b, \\
c + dw^{-1/2} &= x^{-1}(1 + w)^{-1/2}[(1 - w)w^{1/2} - (1 - w)w^{-1/2}] = 0, \\
cw^{1/2} + d &= (c + dw^{-1/2})w^{1/2} = 0.
\end{align}

These equalities prove exactly that

\begin{equation}
[2 \ x] \\
0 \ 0 \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & w^{1/2} \\ w^{-1/2} & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}.
\end{equation}

\end{proof}

The next result is a particular case of the main result.

**5. Lemma.** Let \( A \) be an AW*-factor of type \( II_1 \), and let \( e \in A \) be an idempotent with \( D(L(e)) = \frac{1}{2} \). Then, for any \( \lambda \in \mathbb{C} \), we have

\begin{equation}
Q(\lambda e) = \lambda/2.
\end{equation}

**Proof.** Denote, for simplicity, the projection \( p(x) \) by \( p \). The assumption is that \( p \sim 1 - p \). Then we have a *-isomorphism \( \Phi: A \to M_2(pAp) \), such that \( \Phi(p) = [\frac{1}{0} \ 0] \). It is obvious that, since \( e = pe \), there exists an element \( x \in pAp \) such that \( \Phi(e) = [\frac{1}{x} \ 0] \). By Remark 2, we can find a sequence \( (x_n)_{n \geq 1} \) of invertible elements in \( pAp \) (the unit in \( pAp \)) with \( \lim_{n \to \infty} \|x_n - x\| = 0 \). Define the sequence

\[ E_n = \begin{bmatrix} 1 & x_n \\ 0 & 0 \end{bmatrix} \in M_2(pAp), \ n \geq 1. \]

By Lemma 4, one can find two sequences \( (y_n)_{n \geq 1} \) and \( (z_n)_{n \geq 1} \) in \( pAp \), and a sequence of unitaries \( (U_n)_{n \geq 1} \subset M_2(pAp) \), such that

\begin{equation}
E_n = U_n \begin{bmatrix} \frac{1}{2} & y_n \\ z_n & \frac{1}{2} \end{bmatrix} U_n^*, \text{ for all } n \geq 1.
\end{equation}

Define \( e_n = \Phi^{-1}(E_n) \), \( u_n = \Phi^{-1}(U_n) \), and \( a_n = u_n^*e_nu_n - \frac{1}{2}1 \), \( n \geq 1 \), so that we have

\begin{equation}
\Phi(a_n) = \begin{bmatrix} 0 & y_n \\ z_n & 0 \end{bmatrix}, \text{ for all } n \geq 1.
\end{equation}

Now fix a complex number \( \lambda \). On the one hand, we have

\[ \lambda e_n = u_n(\frac{\lambda}{2} + a_n)u_n^* = \frac{\lambda}{2}1 + u_n a_n u_n^*, \text{ for all } n \geq 1. \]

This gives, for every \( n \geq 1 \), the equalities

\begin{align}
\text{Re}(\lambda e_n) &= (\text{Re}\frac{\lambda}{2})1 + \text{Re}(\lambda u_n a_n u_n^*), \\
\text{Im}(\lambda e_n) &= (\text{Im}\frac{\lambda}{2})1 + \text{Im}(\lambda u_n a_n u_n^*).
\end{align}
Notice however that, using the unitary invariance (property (S) for $s$ unitary), together with (17) and (18), gives, for every $n \geq 1$, the equalities
\[
Q(\lambda e_n) = Q\left(\left(\text{Re} \frac{1}{2}\lambda \right) + \text{Re}(\lambda u_n a_n u_n^*) + iQ\left(\left(\text{Im} \frac{1}{2}\lambda \right) + \text{Im}(\lambda u_n a_n u_n^*)\right)\right)
\]
\[
= \text{Re} \frac{1}{2} + Q(\text{Re}(\lambda u_n a_n u_n^*)) + iQ(\text{Im}(\lambda u_n a_n u_n^*))
\]
\[
= \frac{1}{2} + Q(\lambda u_n a_n u_n^*) = \frac{1}{2} + Q(u_n(\lambda a_n)u_n^*) = \frac{1}{2} + Q(\lambda a_n).
\]

On the other hand, if we define $v = 1 - 2p$ (which is obviously a unitary in $A$), then for all $n \geq 1$ we have
\[
\Phi(v(\lambda a_n)v^*) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -\lambda y_n \\ \lambda y_n \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -\lambda y_n \\ -\lambda y_n & 0 \end{bmatrix} = \Phi(-\lambda a_n).
\]
This gives
\[
(20) \quad v(\lambda a_n)v^* = -\lambda a_n, \text{ for all } n \geq 1.
\]
Using the unitary invariance, combined with real scalar homogeneity (property (H) with $\alpha \in \mathbb{R}$), equality (20) gives
\[
Q(\lambda a_n) = Q(v(\lambda a_n)v^*) = Q(-\lambda a_n) = -Q(\lambda a_n),
\]
which forces
\[
(21) \quad Q(\lambda a_n) = 0, \text{ for all } n \geq 1.
\]
Combining (21) with (19) gives
\[
(22) \quad Q(\lambda e_n) = \lambda/2, \text{ for all } n \geq 1.
\]
It is obvious that, by construction, we have $\lim_{n \to \infty} \| E_n - \Phi(e) \| = 0$, which means that $\lim_{n \to \infty} \| e_n - e \| = 0$. Using the norm continuity of the quasi-trace (see [BH]), combined with (22), gives the desired result.

We are now ready to prove the main result.

6. Theorem. Let $A$ be an AW*-factor of type $II_1$, and let $e \in A$ be an idempotent element. Then, for any $\lambda \in \mathbb{C}$, one has the equality
\[
(23) \quad Q(\lambda e) = \lambda D(L(e)).
\]
Proof. The proof will be carried out in two steps.

Particular Case. Assume $D(L(e)) \leq \frac{1}{e}$. Denote, for simplicity $L(e)$ by $p$, and $R(e)$ by $q$. By the Paralleglogram Law (see [Ka]) we have
\[
p \lor q - p \sim q - p \land q.
\]
Since $p \sim q$, we get
\[
D(p \lor q - p) = D(q - p \land q) = D(q) - D(p \land q) \leq D(q) = D(p).
\]
Using the intermediate value property for $D$, we can find a projection $r \leq 1 - p \lor q$ such that $D(r) + D(p \lor q - p) = D(p)$. Put $q_0 = r + p \lor q - p$. We have $p \perp q_0$ and $p + q_0 \geq p \lor q$. Let us work in the AW*-algebra $A_0 = (p + q_0)A(p + q_0)$. Obviously $A_0$ is again an AW*-factor of type $II_1$, so it carries its quasitrace $Q_0$. By the uniqueness of the quasi-trace, it is obvious that
\[
(24) \quad Q_0(x) = \frac{Q(x)}{D(p + q_0)}, \text{ for all } x \in A_0.
\]
Notice that $e \in (p \lor q)A(p \lor q)$, so in particular $e$ belongs to $A_0$. In $A_0$, we have $L(e) = p$, and $p \sim q_0 = 1_{A_0} - p$, which means that $D_0(p) = \frac{1}{2}$. (Here $D_0$ denotes the dimension function in $A_0$). By Lemma 5, we get

$$Q_0(\lambda e) = \lambda/2,$$

which combined with (24) yields

$$\frac{\lambda}{2} = \frac{Q(\lambda e)}{D(p + q_0)} = \frac{Q(\lambda e)}{2D(p)},$$

which obviously proves (23).

**General Case.** One knows (see [15]) that $M_2(A)$ is also an AW*-factor of type II$_1$. Moreover, if we denote by $Q^{(2)}$ the quasi-trace of $M_2(A)$, we have (as above)

(25)

$$Q(x) = 2Q^{(2)}(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}),$$

for all $x \in A$.

Define the idempotent $E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in M_2(A)$, and the projection $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(A)$. Using (25) we have $Q(e) = 2Q^{(2)}(E)$. If we denote by $D^{(2)}$ the dimension function of $M_2(A)$, then by (25) we also have $D(p) = 2D^{(2)}(P)$, which gives $D^{(2)}(P) \leq \frac{1}{2}$. Using the obvious equality $L(E) = P$, by the particular case above applied to $M_2(A)$, we get

$$Q(\lambda e) = 2Q^{(2)}(\lambda E) = 2\lambda D^{(2)}(P) = \lambda D(p).$$

\[\square\]

**7. Corollary.** Let $A$ be an AW*-factor of type II$_1$, and let $e \in A$ be an idempotent. Then, for every $s \in GL(A)$, we have

(26)

$$Q(ese^{-1}) = Q(e).$$

**Proof.** Put $p = L(e)$. If we define $t = 1 - e(1 - p)$, then $t$ is invertible, and $e = tpt^{-1}$. This computation shows that it is enough to prove (26) in the case when $e = e^*$. Let $q = L(ese^{-1})$. Arguing as above, there exists some $x \in GL(A)$ such that $ses^{-1} = xqe^{-1}$. So if we put $y = x^{-1}s$, we have $yey^{-1} = q$, with both $e$ and $q$ self-adjoint idempotents. Since this obviously forces $e \sim q$, we get $D(e) = D(q) = Q(ese^{-1})$. \[\square\]

**8. Corollary.** Let $A$ be an AW*-factor of type II$_1$ and let $e_1, e_2 \in A$ be idempotents, such that $e_1e_2 = e_2e_1 = 0$. (This implies that $e_1 + e_2$ is again an idempotent.) Then

$$Q(e_1 + e_2) = Q(e_1) + Q(e_2).$$

**Proof.** Let $p = L(e_1 + e_2)$. As in the proof of the preceding corollary, there exists an element $s \in GL(A)$, such that $p = s(e_1 + e_2)s^{-1}$. Put $f_k = ses^{-1}$, $k = 1, 2$, so that $f_1 + f_2 = p$. But now we also have $pf_k = f_kp = f_k$, $k = 1, 2$, which means in particular that $f_1, f_2 \in pAp$. So, if we work in the AW*-factor (again of type II$_1$) $A_0 = pAp$, we will have $f_1 + f_2 = 1_{A_0}$. On the one hand, using the notations from the proof of Theorem 6, we have

$$Q_0(f_1) + Q_0(f_2) = 1.$$

On the other hand, we have

$$Q_0(f_k) = \frac{Q(f_k)}{D(p)}, \quad k = 1, 2,$$
so we get

\[ D(p) = Q(f_1) + Q(f_2). \]

Finally, using the preceding corollary, we get

\[ Q(e_1 + e_2) = D(p) = Q(f_1) + Q(f_2) = Q(se_1 s^{-1}) + Q(se_2 s^{-1}) = Q(e_1) + Q(e_2). \]

9. Comment. A. Theorem 6 can be analyzed from a different point of view, as follows. In principle, one can extend the dimension function \( D \) to the collection of all idempotents by \( D(e) = D(L(e)) \). One can easily prove that this extended dimension function will have the same properties as the usual dimension function (when Murray-von Neumann equivalence is extended to idempotents). The point of Theorem 6 is then the fact that \( D = Q \).

B. It is interesting to note that the linearity of the quasi-trace is equivalent to the following condition:

\[ (*) \text{ For any family of idempotents } e_1, \ldots, e_n \in A, \text{ such that } e_j e_k = 0 \text{ for all } j, k \in \{1, \ldots, n\} \text{ with } j \neq k, \text{ and any family of numbers } \alpha_1, \ldots, \alpha_n \in \mathbb{R}, \text{ one has:} \]

\[ Q(\alpha_1 e_1 + \cdots + \alpha_n e_n) = \alpha_1 Q(e_1) + \cdots + \alpha_n Q(e_n). \]

This will be discussed in a future paper.

References


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