

## A NOTE ON IDEMPOTENTS IN FINITE AW\*-FACTORS

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ABSTRACT. We prove that the value of the quasi-trace on an idempotent element in an AW\*-factor of type II<sub>1</sub> is the same as the dimension of its left (or right) support.

It is a long-standing open problem (due to Kaplansky) to prove that *an AW\*-factor of type II<sub>1</sub> is in fact a von Neumann algebra*. A remarkable answer, in the affirmative, was found by Haagerup ([Ha]), who proved that if an AW\*-factor  $A$  is generated by an exact C\*-algebra, then  $A$  is indeed a von Neumann algebra.

The main object, that was investigated in connection with Kaplansky's problem, is the *quasi-trace*, whose construction we briefly recall below.

One starts with an AW\*-factor of type II<sub>1</sub>, say  $A$ . Denote by  $\mathcal{P}(A)$  the collection of projections in  $A$ , that is

$$\mathcal{P}(A) = \{p \in A : p = p^* = p^2\}.$$

A key fact is then the existence of a (unique) *dimension function*  $D : \mathcal{P}(A) \rightarrow [0, 1]$  with the following properties:

- $D(p) = D(q) \iff p \sim q$ ;
- if  $p \perp q$ , then  $D(p + q) = D(p) + D(q)$ ;
- $D(1) = 1$ .

The symbol “ $\sim$ ” denotes the Murray-von Neumann equivalence relation ( $p \sim q \iff \exists x \in A$  with  $p = x^*x$  and  $q = xx^*$ ), while “ $\perp$ ” denotes the orthogonality relation ( $p \perp q \iff pq = 0$ ; this implies that  $p + q$  is again a projection).

Once the dimension function is defined, it is extended to self-adjoint elements with finite spectrum. More explicitly, if  $a \in A$  is self-adjoint with finite spectrum, then there are (real) numbers  $\alpha_1, \dots, \alpha_n$  and pairwise orthogonal projections  $p_1, \dots, p_n$ , such that  $a = \sum_{k=1}^n \alpha_k p_k$ . We then define  $d(a) = \sum_{k=1}^n \alpha_k D(p_k)$ .

For an arbitrary self-adjoint element  $a \in A$ , one can approximate uniformly  $a$  with a sequence  $(a_n)_{n \geq 1} \in \{a\}''$  of elements with finite spectrum. (Here  $\{a\}''$  stands for the AW\*-subalgebra generated by  $a$  and 1.) It turns out that the limit  $q(a) = \lim_{n \rightarrow \infty} d(a_n)$  is independent of the particular choice of  $(a_n)_{n \geq 1}$ .

Finally, for an arbitrary element  $x \in A$ , one defines  $Q(x) = q(\operatorname{Re} x) + iq(\operatorname{Im} x)$ , where  $\operatorname{Re} x = \frac{1}{2}(x + x^*)$  and  $\operatorname{Im} x = \frac{1}{2i}(x - x^*)$ .

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The map  $Q : A \rightarrow \mathbb{C}$ , defined this way, is the unique one with the properties:

- (i)  $Q$  is *linear*, when restricted to *abelian*  $C^*$ -subalgebras of  $A$ ;
- (ii)  $Q(x^*x) = Q(xx^*) \geq 0$ , for all  $x \in A$ ;
- (iii)  $Q(x) = Q(\operatorname{Re} x) + iQ(\operatorname{Im} x)$ , for all  $x \in A$ ;
- (iv)  $Q(1) = 1$ .

It is obvious that  $Q|_{\mathcal{P}(A)} = D$ . The map  $Q$  is called *the quasi-trace of  $A$* .

It is well known that an  $AW^*$ -factor of type  $II_1$  is a von Neumann algebra if and only if its quasi-trace is *linear*. Haagerup's solution for Kaplansky's problem goes through the proof of the linearity of the quasi-trace.

On the one hand, one can easily see that the linearity of the quasi-trace is equivalent to its scalar homogeneity (compare with (i) above):

$$(H) \quad Q(\alpha x) = \alpha Q(x), \text{ for all } x \in A, \alpha \in \mathbb{C}.$$

Notice that (H) holds when either  $\alpha \in \mathbb{R}$  or when  $x$  is *normal*. On the other hand, it is again easy to note that the linearity of the quasi-trace is equivalent to the similarity invariance property

$$(S) \quad Q(sxs^{-1}) = Q(x), \text{ for all } x \in A, s \in GL(A).$$

(Here  $GL(A)$  denotes the group of invertible elements in  $A$ .) Notice that (S) is true if  $s$  is *unitary*.

The purpose of this note is to prove that *both (H) and (S) hold if  $x \in A$  is an idempotent (i.e.  $x^2 = x$ )*.

1. *Notations.* If  $A$  is an  $AW^*$ -algebra, for an element  $x \in A$ , we denote by  $\mathbf{L}(x)$  (resp.  $\mathbf{R}(x)$ ) the *left* (resp. *right support*) of  $x$ . Recall that both  $\mathbf{L}(x)$  and  $\mathbf{R}(x)$  are projections, and moreover we have

$$\mathbf{L}(x) \sim \mathbf{R}(x), \text{ for all } x \in A.$$

It is known (see [Ka]) that, for every  $x \in A$ , there exists a unique partial isometry  $v$  such that

- (i)  $x = v(x^*x)^{1/2}$ ;
- (ii)  $vv^* = \mathbf{L}(x)$ ;
- (iii)  $v^*v = \mathbf{R}(x)$ .

(This property is referred to as the *Polar Decomposition*.)

2. *Remark.* If  $A$  is a finite  $AW^*$ -algebra, then the group  $GL(A)$ , of invertible elements, is dense in  $A$  in the norm topology.

Indeed, on the one hand, since  $\mathbf{L}(x) \sim \mathbf{R}(x)$ , by the finiteness assumption we also have  $1 - \mathbf{L}(x) \sim 1 - \mathbf{R}(x)$ . In particular, there exists a partial isometry  $w \in A$  such that  $1 - \mathbf{L}(x) = ww^*$  and  $1 - \mathbf{R}(x) = w^*w$ . On the other hand, if we take  $x = v(x^*x)^{1/2}$  to be the polar decomposition described above, then we obviously have  $w^*v = 0$ , so the element  $u = v + w$  is *unitary*, and we still have  $x = u(x^*x)^{1/2}$ . Then, for every  $\varepsilon > 0$  the positive element  $(x^*x)^{1/2} + \varepsilon 1$  is invertible, and so is  $u\{(x^*x)^{1/2} + \varepsilon 1\}$ . The result then follows from the obvious equality  $\|x - u\{(x^*x)^{1/2} + \varepsilon 1\}\| = \varepsilon$ .

3. *Notation.* For a  $C^*$ -algebra  $A$  and an integer  $n \geq 2$ , we denote by  $M_n(A)$  the  $C^*$ -algebra of  $n \times n$  matrices with coefficients in  $A$ .

The key technical result in this paper is the following.

**4. Lemma.** *Suppose  $A$  is a unital  $C^*$ -algebra and  $x \in GL(A)$ . Then there exist a unitary element  $U \in M_2(A)$  and elements  $y, z \in A$ , such that*

$$(1) \quad U^* \begin{bmatrix} 2 & x \\ 0 & 0 \end{bmatrix} U = \begin{bmatrix} 1 & y \\ z & 1 \end{bmatrix}.$$

*Proof.* Consider the function defined by

$$f(t) = \frac{(1-t)^2}{t}, \quad 0 < t < 1.$$

It is obvious that  $f : (0, 1) \rightarrow (0, \infty)$  is a homeomorphism. Since  $x$  is invertible, the spectrum of  $xx^*$  is contained in  $(0, \infty)$ , so by functional calculus there exists an invertible positive element  $w \in A$ , with  $\|w\| < 1$ , such that  $f(w) = xx^*$ , which means

$$(2) \quad xx^* = (1-w)^2 w^{-1}.$$

Define the elements

$$\begin{aligned} a &= (1+w)^{-1/2} w^{1/2}, & b &= (1+w)^{-1/2}, \\ c &= x^{-1}(1+w)^{-1/2} w^{-1/2}(1-w), & d &= -x^{-1}(1+w)^{-1/2}(1-w). \end{aligned}$$

First, we have

$$(3) \quad aa^* + bb^* = (1+w)^{-1/2}[w+1](1+w)^{-1/2} = 1,$$

and using (2) we also have

$$(4) \quad \begin{aligned} cc^* + dd^* &= x^{-1}(1+w)^{-1/2}(1-w)[w^{-1}+1](1-w)(1+w)^{-1/2}(x^*)^{-1} \\ &= x^{-1}(1-w)^2 w^{-1}(x^*)^{-1} = x^{-1}(xx^*)(x^*)^{-1} = 1. \end{aligned}$$

Secondly, since by taking inverses, (2) yields

$$(5) \quad (x^*)^{-1} x^{-1} = w(1-w)^{-2},$$

so we also get

$$(6) \quad \begin{aligned} a^* a + c^* c &= w^{1/2}(1+w)^{-1/2}[1+(1-w)(x^*)^{-1}x^{-1}(1-w)](1+w)^{-1/2} w^{1/2} \\ &= w^{1/2}(1+w)^{-1/2}[1+w^{-1}](1+w)^{-1/2} w^{1/2} = 1, \end{aligned}$$

$$(7) \quad \begin{aligned} b^* b + d^* d &= (1+w)^{-1/2}[1+(1-w)(x^*)^{-1}x^{-1}(1-w)](1+w)^{-1/2} \\ &= (1+w)^{-1/2}[1+w](1+w)^{-1/2} \mathbf{1}. \end{aligned}$$

Finally, we notice that

$$(8) \quad ac^* + bd^* = (1+w)^{-1/2}[w^{1/2}w^{-1/2} - 1](1-w)(1+w)^{-1/2}(x^*)^{-1} = 0,$$

and using (5) we also have

$$(9) \quad \begin{aligned} a^* b + c^* d &= (1+w)^{-1/2}[w^{1/2} - w^{-1/2}(1-w)(x^*)^{-1}x(1-w)](1+w)^{-1/2} \\ &= (1+w)^{-1/2}[w^{1/2} - w^{-1/2}w](1+w)^{-1/2} = 0. \end{aligned}$$

If we define the matrix  $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then (3), (4), and (8) give  $UU^* = I$ , while (6), (7), and (9) give  $U^*U = I$ , so  $U$  is indeed unitary. (Here  $I$  denotes the unit in  $M_2(A)$ .)

Let us now observe that

$$(10) \quad \begin{aligned} 2a + xc &= 2(1 + w)^{-1/2}w^{1/2} + (1 + w)^{1/2}(1 - w)w^{-1/2} \\ &= (1 + w)^{-1/2}(w^{1/2} + w^{-1/2}) = a + bw^{-1/2}, \end{aligned}$$

$$(11) \quad \begin{aligned} 2b + xd &= 2(1 + w)^{-1/2} - (1 + w)^{-1/2}(1 - w) \\ &= (1 + w)^{-1/2}(w + 1) = aw^{1/2} + b, \end{aligned}$$

$$(12) \quad c + dw^{-1/2} = x^{-1}(1 + w)^{-1/2}[(1 - w)w^{-1/2} - (1 - w)w^{-1/2}] = 0,$$

$$(13) \quad cw^{1/2} + d = (c + dw^{-1/2})w^{1/2} = 0.$$

These equalities prove exactly that

$$\begin{bmatrix} 2 & x \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & w^{1/2} \\ w^{-1/2} & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

□

The next result is a particular case of the main result.

**5. Lemma.** *Let  $A$  be an  $AW^*$ -factor of type  $II_1$ , and let  $e \in A$  be an idempotent with  $D(\mathbf{L}(e)) = \frac{1}{2}$ . Then, for any  $\lambda \in \mathbb{C}$ , we have*

$$(14) \quad Q(\lambda e) = \lambda/2.$$

*Proof.* Denote, for simplicity, the projection  $\mathbf{L}(x)$  by  $p$ . The assumption is that  $p \sim 1 - p$ . Then we have a  $*$ -isomorphism  $\Phi : A \rightarrow M_2(pAp)$ , such that  $\Phi(p) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ . It is obvious that, since  $e = pe$ , there exists an element  $x \in pAp$  such that  $\Phi(e) = \begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix}$ . By Remark 2, we can find a sequence  $(x_n)_{n \geq 1}$  of invertible elements in  $pAp$  (the unit in  $pAp$  is  $p$ ) with  $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ . Define the sequence

$$E_n = \begin{bmatrix} 1 & x_n \\ 0 & 0 \end{bmatrix} \in M_2(pAp), \quad n \geq 1.$$

By Lemma 4, one can find two sequences  $(y_n)_{n \geq 1}$  and  $(z_n)_{n \geq 1}$  in  $pAp$ , and a sequence of unitaries  $(U_n)_{n \geq 1} \subset M_2(pAp)$ , such that

$$(15) \quad E_n = U_n \begin{bmatrix} \frac{1}{2} & y_n \\ z_n & \frac{1}{2} \end{bmatrix} U_n^*, \quad \text{for all } n \geq 1.$$

Define  $e_n = \Phi^{-1}(E_n)$ ,  $u_n = \Phi^{-1}(U_n)$ , and  $a_n = u_n^* e_n u_n - \frac{1}{2}1$ ,  $n \geq 1$ , so that we have

$$(16) \quad \Phi(a_n) = \begin{bmatrix} 0 & y_n \\ z_n & 0 \end{bmatrix}, \quad \text{for all } n \geq 1.$$

Now fix a complex number  $\lambda$ . On the one hand, we have

$$\lambda e_n = u_n \left( \frac{\lambda}{2} + a_n \right) u_n^* = \frac{\lambda}{2}1 + u_n a_n u_n^*, \quad \text{for all } n \geq 1.$$

This gives, for every  $n \geq 1$ , the equalities

$$(17) \quad \operatorname{Re}(\lambda e_n) = \left( \operatorname{Re} \frac{\lambda}{2} \right) 1 + \operatorname{Re}(\lambda u_n a_n u_n^*),$$

$$(18) \quad \operatorname{Im}(\lambda e_n) = \left( \operatorname{Im} \frac{\lambda}{2} \right) 1 + \operatorname{Im}(\lambda u_n a_n u_n^*).$$

Notice however that, using the *unitary invariance* (property (S) for  $s$  unitary), together with (17) and (18), gives, for every  $n \geq 1$ , the equalities

$$\begin{aligned}
 Q(\lambda e_n) &= Q\left(\left(\operatorname{Re}\frac{\lambda}{2}\right)1 + \operatorname{Re}(\lambda u_n a_n u_n^*)\right) + iQ\left(\left(\operatorname{Im}\frac{\lambda}{2}\right)1 + \operatorname{Im}(\lambda u_n a_n u_n^*)\right) \\
 (19) \quad &= \operatorname{Re}\frac{\lambda}{2} + Q\left(\operatorname{Re}(\lambda u_n a_n u_n^*)\right) + i\operatorname{Im}\frac{\lambda}{2} + iQ\left(\operatorname{Im}(\lambda u_n a_n u_n^*)\right) \\
 &= \frac{\lambda}{2} + Q(\lambda u_n a_n u_n^*) = \frac{\lambda}{2} + Q(u_n(\lambda a_n)u_n^*) = \frac{\lambda}{2} + Q(\lambda a_n).
 \end{aligned}$$

On the other hand, if we define  $v = 1 - 2p$  (which is obviously a unitary in  $A$ ), then for all  $n \geq 1$  we have

$$\Phi(v(\lambda a_n)v^*) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & \lambda y_n \\ \lambda z_n & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -\lambda y_n \\ -\lambda z_n & 0 \end{bmatrix} = \Phi(-\lambda a_n).$$

This gives

$$(20) \quad v(\lambda a_n)v^* = -\lambda a_n, \text{ for all } n \geq 1.$$

Using the unitary invariance, combined with *real scalar homogeneity* (property (H) with  $\alpha \in \mathbb{R}$ ), equality (20) gives

$$Q(\lambda a_n) = Q(v(\lambda a_n)v^*) = Q(-\lambda a_n) = -Q(\lambda a_n),$$

which forces

$$(21) \quad Q(\lambda a_n) = 0, \text{ for all } n \geq 1.$$

Combining (21) with (19) gives

$$(22) \quad Q(\lambda e_n) = \lambda/2, \text{ for all } n \geq 1.$$

It is obvious that, by construction, we have  $\lim_{n \rightarrow \infty} \|E_n - \Phi(e)\| = 0$ , which means that  $\lim_{n \rightarrow \infty} \|e_n - e\| = 0$ . Using the norm continuity of the quasi-trace (see [BH]), combined with (22), gives the desired result.  $\square$

We are now ready to prove the main result.

**6. Theorem.** *Let  $A$  be an AW\*-factor of type  $II_1$ , and let  $e \in A$  be an idempotent element. Then, for any  $\lambda \in \mathbb{C}$ , one has the equality*

$$(23) \quad Q(\lambda e) = \lambda D(\mathbf{L}(e)).$$

*Proof.* The proof will be carried out in two steps.

PARTICULAR CASE: Assume  $D(\mathbf{L}(e)) \leq \frac{1}{2}$ .

Denote, for simplicity  $\mathbf{L}(e)$  by  $p$ , and  $\mathbf{R}(e)$  by  $q$ . By the *Parallelogram Law* (see [Ka]) we have

$$p \vee q - p \sim q - p \wedge q.$$

Since  $p \sim q$ , we get

$$D(p \vee q - p) = D(q - p \wedge q) = D(q) - D(p \wedge q) \leq D(q) = D(p).$$

Using the intermediate value property for  $D$ , we can find a projection  $r \leq 1 - p \vee q$  such that  $D(r) + D(p \vee q - p) = D(p)$ . Put  $q_0 = r + p \vee q - p$ . We have  $p \perp q_0$  and  $p + q_0 \geq p \vee q$ . Let us work in the AW\*-algebra  $A_0 = (p + q_0)A(p + q_0)$ . Obviously  $A_0$  is again an AW\*-factor of type  $II_1$ , so it carries its quasi-trace  $Q_0$ . By the uniqueness of the quasi-trace, it is obvious that

$$(24) \quad Q_0(x) = \frac{Q(x)}{D(p + q_0)}, \text{ for all } x \in A_0.$$

Notice that  $e \in (p \vee q)A(p \vee q)$ , so in particular  $e$  belongs to  $A_0$ . In  $A_0$ , we have  $\mathbf{L}(e) = p$ , and  $p \sim q_0 = 1_{A_0} - p$ , which means that  $D_0(p) = \frac{1}{2}$ . (Here  $D_0$  denotes the dimension function in  $A_0$ ). By Lemma 5, we get

$$Q_0(\lambda e) = \lambda/2,$$

which combined with (24) yields

$$\frac{\lambda}{2} = \frac{Q(\lambda e)}{D(p + q_0)} = \frac{Q(\lambda e)}{2D(p)},$$

which obviously proves (23).

GENERAL CASE. One knows (see [Be]) that  $M_2(A)$  is also an AW\*-factor of type  $II_1$ . Moreover, if we denote by  $Q^{(2)}$  the quasi-trace of  $M_2(A)$ , we have (as above)

$$(25) \quad Q(x) = 2Q^{(2)}\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right), \text{ for all } x \in A.$$

Define the idempotent  $E = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \in M_2(A)$ , and the projection  $P = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \in M_2(A)$ . Using (25) we have  $Q(e) = 2Q^{(2)}(E)$ . If we denote by  $D^{(2)}$  the dimension function of  $M_2(A)$ , then by (25) we also have  $D(p) = 2D^{(2)}(P)$ , which gives  $D^{(2)}(P) \leq \frac{1}{2}$ . Using the obvious equality  $\mathbf{L}(E) = P$ , by the particular case above applied to  $M_2(A)$ , we get

$$Q(\lambda e) = 2Q^{(2)}(\lambda E) = 2\lambda D^{(2)}(P) = \lambda D(p).$$

□

**7. Corollary.** *Let  $A$  be an AW\*-factor of type  $II_1$ , and let  $e \in A$  be an idempotent. Then, for every  $s \in GL(A)$ , we have*

$$(26) \quad Q(ses^{-1}) = Q(e).$$

*Proof.* Put  $p = \mathbf{L}(e)$ . If we define  $t = 1 - e(1 - p)$ , then  $t$  is invertible, and  $e = tpt^{-1}$ . This computation shows that it is enough to prove (26) in the case when  $e = e^*$ . Let  $q = \mathbf{L}(ses^{-1})$ . Arguing as above, there exists some  $x \in GL(A)$  such that  $ses^{-1} = xqx^{-1}$ . So if we put  $y = x^{-1}s$ , we have  $yey^{-1} = q$ , with both  $e$  and  $q$  self-adjoint idempotents. Since this obviously forces  $e \sim q$ , we get  $D(e) = D(q) = Q(ses^{-1})$ . □

**8. Corollary.** *Let  $A$  be an AW\*-factor of type  $II_1$  and let  $e_1, e_2 \in A$  be idempotents, such that  $e_1e_2 = e_2e_1 = 0$ . (This implies that  $e_1 + e_2$  is again an idempotent.) Then*

$$Q(e_1 + e_2) = Q(e_1) + Q(e_2).$$

*Proof.* Let  $p = \mathbf{L}(e_1 + e_2)$ . As in the proof of the preceding corollary, there exists an element  $s \in GL(A)$ , such that  $p = s(e_1 + e_2)s^{-1}$ . Put  $f_k = se_k s^{-1}$ ,  $k = 1, 2$ , so that  $f_1 + f_2 = p$ . But now we also have  $pf_k = f_k p = f_k$ ,  $k = 1, 2$ , which means in particular that  $f_1, f_2 \in pAp$ . So, if we work in the AW\*-factor (again of type  $II_1$ )  $A_0 = pAp$ , we will have  $f_1 + f_2 = 1_{A_0}$ . On the one hand, using the notations from the proof of Theorem 6, we have

$$Q_0(f_1) + Q_0(f_2) = 1.$$

On the other hand, we have

$$Q_0(f_k) = \frac{Q(f_k)}{D(p)}, \quad k = 1, 2,$$

so we get

$$D(p) = Q(f_1) + Q(f_2).$$

Finally, using the preceding corollary, we get

$$Q(e_1 + e_2) = D(p) = Q(f_1) + Q(f_2) = Q(se_1s^{-1}) + Q(se_2s^{-1}) = Q(e_1) + Q(e_2).$$

□

**9. Comment.** A. Theorem 6 can be analyzed from a different point of view, as follows. In principle, one can extend the dimension function  $D$  to the collection of all idempotents by  $\tilde{D}(e) = D(\mathbf{L}(e))$ . One can easily prove that this extended dimension function will have the same properties as the usual dimension function (when Murray-von Neumann equivalence is extended to idempotents). The point of Theorem 6 is then the fact that  $\tilde{D} = Q$ .

B. It is interesting to note that the linearity of the quasi-trace is equivalent to the following condition:

(\*) For any family of idempotents  $e_1, \dots, e_n \in A$ , such that  $e_j e_k = 0$  for all  $j, k \in \{1, \dots, n\}$  with  $j \neq k$ , and any family of numbers  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ , one has:

$$Q(\alpha_1 e_1 + \dots + \alpha_n e_n) = \alpha_1 Q(e_1) + \dots + \alpha_n Q(e_n).$$

This will be discussed in a future paper.

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